Quantum fluctuations in contracting cosmologies

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Outline

Motivation

Stochastic Formalism

- Allow us to compute non-linear primordial density perturbations beyond usual perturbative approach!

    1. Correlations functions in stochastic inflation (V.Vennin and A. Starobinsky, arXiv:1506.04732);
    2. Quantum diffusion during inflation and primordial black holes (Chris Pattison et al., arXiv:1707.00537);

Basic idea of inflation

From Einstein equations, the scale factor satisfies

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3w)\rho \quad \text{where} \quad w = \frac{P}{\rho} \; . \tag{1}$$

If $w < -1/3$, we have an accelerated expansion

$$\ddot{a} > 0 \; . \tag{2}$$

The amount of inflation is quantified by the number of e-folds

$$N = \log\left(\frac{a}{a_i}\right) \; . \tag{3}$$

We require $N \gtrsim 60$ to solve flatness and horizon problems!

Rolling models of inflation

- Equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V'(\phi) = 0$$

- Flat region
  - $V(\phi)$ almost constant
  - ρ_{vac} dominates energy density
  - $a \approx a_i e^{Ht}$
- Decay of ϕ
  - Particle production
  - Reheating

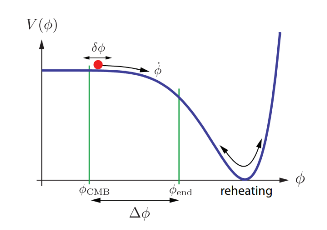


Figure 1: Example of an inflaton potential. **TASI Lectures on Inflation** [arXiv:0907.5424].

Inflationary models and Observation

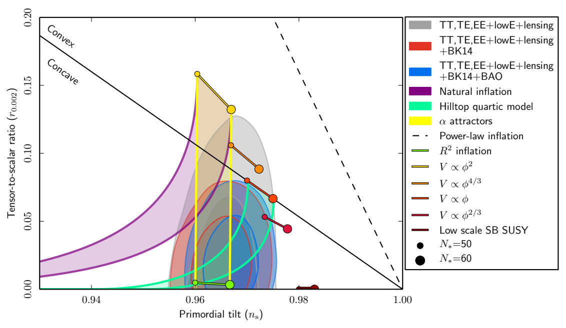


Figure 2: Marginalized joint 68% and 95% CL regions for n_s and r at $k = 0.002\mathrm{Mpc}^{-1}$ from Planck alone and in combination with BK14 or BK14 plus BAO data, compared to the theoretical predictions of selected inflationary models. **Planck 2018 results. X. Constraints on inflation** [arXiv:1807.06211].

To summarise

Modern view of cosmology

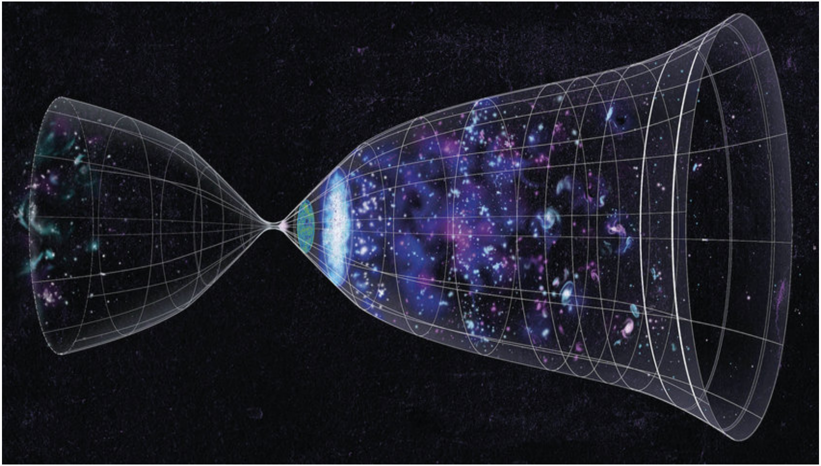
- Origin of large-scale structures from quantum vacuum fluctuations;
- Small-scale initial perturbations stretched by accelerated expansion;
- Classical inflation: slowly-rolling, self-interacting scalar field, almost scale-invariant spectrum. Very successful paradigm!

But inflation is not a complete theory

- Ignores initial singularity
- Trans-Planckian modes
- Fine-tuning of the potential, etc...

Bouncing models can resolve some inflation problems.
Need for: contracting phase + bounce mechanism + expanding phase

How a bouncing universe could look like



Credits: https://www.aei.mpg.de/gravitation-and-cosmology

Collapsing models

We do not have a unique prescription for the initial conditions of our Universe, so there is no reason to avoid considering other mechanisms beyond inflation.

Collapse scenario depends on potential

- Non-stiff collapse: $P < \rho$ with $V > 0$; (including scale-invariant collapse)
- Pre-Big Bang collapse: $P = \rho$ with $V = 0$; (blue tilted)
- Ekpyrotic collapse: $P \gg \rho$ with $V < 0$; (ultra-stiff fast-roll collapse)

Classical stability well-known (I. Heard and D.Wands, arXiv:gr-qc/0206085).

Objective of this work

Study collapse scenarios with quantum fluctuations.

FRW Collapse

$$L = \sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right] \quad \text{and} \quad ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \ .$$

Scalar field with energy density and pressure

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\ , \quad P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)\ , \tag{4}$$

Equation of State

$$P = w\rho\ . \tag{5}$$

We choose $V(\varphi) = V_0 e^{-\kappa\lambda\varphi} \implies$ scaling solution with

$$a \propto |t|^p \quad \text{where} \quad p = \frac{2}{\lambda^2} \quad \text{and} \quad \lambda^2 = 3(1+w)\ . \tag{6}$$

Dynamical system

Reducing dynamics to a one-dimensional problem

Klein-Gordon equation + Friedmann constraint

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0 \qquad ; \quad H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V\right) . \tag{7}$$

Changing to dimensionless variables

$$x = \frac{\kappa\dot{\varphi}}{\sqrt{6}H} , \quad y = \frac{\kappa\sqrt{\pm V}}{\sqrt{3}H} , \tag{8}$$

The Friedmann constraint becomes

$$x^2 \pm y^2 = 1 , \tag{9}$$

Dynamical system (prime: $N = \ln(a)$)

$$x' = -3x(1 - x^2) \pm \lambda\sqrt{3/2}y^2 , \tag{10}$$

$$y' = xy(3x - \lambda\sqrt{3/2}) . \tag{11}$$

Stability analysis

Equation of state

$$w = \frac{x^2 \mp y^2}{x^2 \pm y^2} \, . \qquad (12)$$

Critical points

$$(A_\pm)\; x_{A_\pm} = \pm 1 \, , \quad y_A = 0 \, ; \quad (13)$$

$$(B)\; x_B = \frac{\lambda}{\sqrt{6}} \, , \quad y_B = \sqrt{1 - \frac{\lambda^2}{6}} \, ; \qquad (14)$$

the solution (B) exists for $\pm(6 - \lambda^2) > 0$.

- $\lambda^2 < 6$: flat positive
- $\lambda^2 > 6$: steep negative

Linear perturbations around x_B

$$x' = \frac{(\lambda^2 - 6)}{3}(x - x_B) \, . \qquad (15)$$

What if we add noise to x?

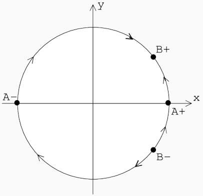


Figure 3: Phase-space for flat positive potentials, $\lambda^2 < 6$. Friedmann constraint $x^2 + y^2 = 1$. Arrows indicate evolution in cosmic time, t.

1D Phase-space

In Summary

- Expanding universe ($N \to +\infty$):
  - Scaling solution stable for positive, flat potential $\lambda^2 < 6$ (including inflation, $\lambda^2 \ll 1$).
  - Scaling solution unstable for negative, steep potential $\lambda^2 > 6$.

- Contracting universe ($N \to -\infty$):
  - Scaling solution stable for negative steep potential $\lambda^2 > 6$ (including ekpyrosis, $\lambda^2 \gg 6$).
  - Scaling solution unstable for positive flat potential $\lambda^2 < 6$ (including matter collapse, $\lambda^2 = 3$).

Field perturbations

During accelerated expansion or collapse

- $|aH|$ increases $\rightarrow$ modes starting on sub-Hubble scales ($k^2 > a^2H^2$) stretched up to super-Hubble scales ($k^2 < a^2H^2$).

Result

- Quantum vacuum fluctuations $k^2/a^2 \gg H^2$ at early times[1] $\rightarrow$ well-defined predictions for the power spectrum of perturbations on super-Hubble scales.

[1] which means $\delta\varphi \simeq \frac{e^{-ikt/a}}{a\sqrt{2k}} \quad \text{for } k^2/a^2 \gg H^2$.

Field perturbations

Perturbations evolution

Evolution of the perturbed scalar field ($v = a\delta\varphi$)

$$\frac{d^2 v}{d\eta^2} + \left(k^2 - \frac{\nu^2 - 1/4}{\eta^2}\right) v = 0 \; . \tag{16}$$

In power-law cosmology

$$a \propto |t|^p \quad \text{where} \quad \nu = \frac{3}{2} + \frac{1}{p-1} \; . \tag{17}$$

The growing mode solution of quantum fluctuations for a given k is

$$\delta\varphi_k = \frac{i}{a}\sqrt{\frac{1}{4\pi k}}\frac{\Gamma(|\nu|)2^{|\nu|}}{|k\eta|^{|\nu|-1/2}} \; . \tag{18}$$

Power-law collapse

Predictions

- Power spectrum on super-Hubble scales as $\eta \to 0$

$$\mathcal{P}_{\delta\varphi} = \left[\frac{\Gamma(|\nu|)2^{|\nu|}}{(\nu - 1/2)2^{3/2}\Gamma(3/2)}\right]^2 \left(\frac{H}{2\pi}\right)^2 |k\eta|^{3-2|\nu|} \, . \qquad (19)$$

- Power-law collapse $\implies$ power-law spectrum

$$\Delta n_{\delta\varphi} = \frac{d \ln \mathcal{P}_{\delta\varphi}}{d \ln k} = 3 - 2|\nu| \, . \qquad (20)$$

- $\Delta n_{\delta\varphi} = 0$ for
  - Slow-roll inflation ($w = -1$ and $\nu = 3/2$);
  - Pressureless collapse ($w = 0$ and $\nu = -3/2$);

Two approaches

Standard perturbation theory

- $\varphi = \varphi_0 + \delta\varphi$;

Stochastic formalism

Introduces a coarse-graining scale $k_\sigma = \sigma a H$

$$\begin{aligned}\varphi =& \varphi_{long} + \varphi_{short} \\ &= \int_0^{k_\sigma} d^3k \varphi_k e^{ikx} + \int_{k_\sigma}^{\infty} d^3k \varphi_k e^{ikx};\end{aligned}$$

- quantise small scales (sub-Hubble) fluctuations;
- large scale squeezed state, effectively classical;
- absorb into local stochastic FLRW background;

Quantifying how quantum noises modify the long-wavelength (or coarse-grained) field

Coarse-grained field and momentum (**J.Grain** and **V.Vennin**, JCAP 05(2017)045)

$$\dot{\overline{\varphi}} = a^{-3}\overline{\pi}_{\varphi} + \xi_{\varphi} \; , \quad \dot{\overline{\pi}} = -a^{3} V_{,\overline{\varphi}} + \xi_{\pi} \; . \tag{21}$$

Time-dependent cut-off scale (coarse-graining scale)

$$k_{\sigma} = \sigma a H \; . \tag{22}$$

Noises (small-wavelength part) described by two-points correlation matrix $\Xi_{f,g}$

$$\Xi_{f,g} = \langle 0 | \xi_f \xi_g | 0 \rangle = \frac{1}{6\pi^2} \frac{dk_{\sigma}^3(N)}{dN} f_k(N) g_k^{\star}(N) \; . \tag{23}$$

Noise growth in a collapsing universe?

Quantum noise

Perturbing EOS (note the relation $\delta w = 4x_B\delta x$)

$$\delta x = \frac{\kappa}{\sqrt{6}H}\left(\delta\dot{\varphi} - A\dot{\varphi} - \frac{\dot{\varphi}}{H}\delta H\right) . \tag{24}$$

Correlation matrix of the noise at critical point (B)

$$\Xi_{x,x} = g(\nu,\lambda)\frac{\left(|\nu| - \nu\right)^2}{\sigma^{2|\nu|-3}}\kappa^2 H_\star^2 \exp\left[-\frac{3-2\nu}{\nu-1/2}(N_\star - N)\right] . \tag{25}$$

No noise for $\nu > 0$: **adiabatic perturbations** (includes power-law inflation ($\nu = 3/2$) and ekpyrosis ($\nu = 1/2$)).
True at leading and next-to-leading order!

Kinetic-dominated solution (critical point A), $\lambda^2 = 6$ or $\nu = 0$
Always $\delta x = 0$ at first order!

Variance of Langevin equations

Formal solutions

Langevin equation at $x = x_B$

$$\bar{x}' = m(\bar{x} - x_B) + \hat{\xi}_x \quad \text{with} \quad m = \frac{\lambda^2 - 6}{2}\,. \tag{26}$$

Variance split into classical/quantum parts

$$\begin{aligned}\sigma_x^2(N) = \left\langle \left(\bar{x}(N) - x_B\right)^2 \right\rangle &= \sigma_{x,cl}^2(N) + \sigma_{x,qu}^2(N) \\ &= \sigma_x^2(N_\star) e^{2m(N-N_\star)} + \int_{N_\star}^{N} dS\; e^{2m(N-S)} \Xi_{x,x}(S)\end{aligned} \tag{27}$$

- For $\nu \neq -3/2$

$$\sigma_{x,qu}^2(N) = h(\nu,\lambda,\sigma)\kappa^2 H^2(N) \left\{1 - \exp\left[\frac{3+2\nu}{\nu - 1/2}\,(N_\star - N)\right]\right\} \tag{28}$$

Quantum Diffusion and Power Spectrum

- Quantum part of variance decays when

$$\frac{3+2\nu}{\nu-1/2} > 0\ . \tag{30}$$

This is the case if either $\nu > 1/2$ or $\nu < -3/2$ (ignore first case: adiabatic!)

- Shift in spectral index:

$$n_s - 1 = \frac{12w}{1+3w} = \frac{4(2\nu+3)}{3}\ . \tag{31}$$

For small positive deviation ϵ, red spectrum when $\nu = -3/2 - \epsilon$.

Spectral index

When $\nu = -3/2 - \epsilon$, where ϵ is a small positive parameter, $w < 0$ and the spectrum becomes redder.

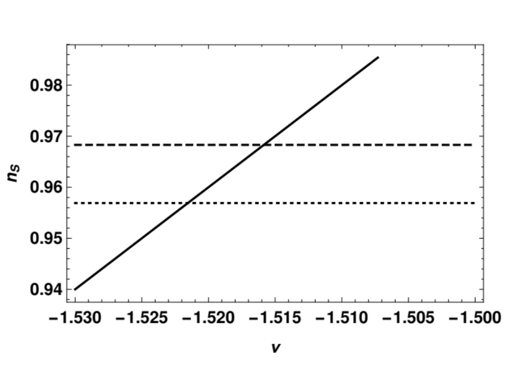


Figure 4: Evolution for $n_{\rm s}$ as function of ν. The horizontal dotted lines enclose the 68% confidence level of the values of $n_{\rm s}$ measured by Planck collaboration 2018.

Maximum lifetime of the collapse phase at the fixed point

Backreaction condition

If $\sigma^2_{x,qu} = 1 \implies$ does quantum noise change the dynamics?

- Pressureless collapse ($\nu = -3/2$)

$$|H(N)| \approx \sqrt{\frac{134}{N_\star - N}} M_{pl} \ . \tag{32}$$

  Drives away from fixed point below Planck scale if $(N_\star - N) > 134$.

- For slightly red spectrum ($\nu = -3/2 - \epsilon$, $n_s < 1$): classical perturbations grow faster
- Example from general solution: radiation-dominated collapse ($\nu = -1/2$)

$$|H(N)| \approx \frac{13}{\sigma} M_{pl} \ . \tag{33}$$

  Cannot escape fixed point since $\sigma < 1$.

Hubble rate evolution

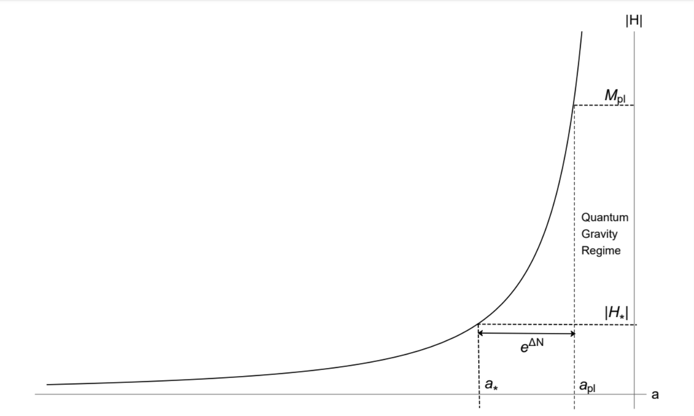


Figure 5: Evolution of the Hubble rate. To get a sensible deviation from the fixed point we start in, the initial scale must be set at low energy.

Gauge effects

Gauge transformation

The gauge parameter transforms the field perturbations $\delta\phi$ as (K. A. Malik and D. Wands, arXiv:0809.4944)

$$\tilde{\delta\phi}_{\rm UE} = \delta\phi_{\rm SF} + \alpha\frac{d\phi}{d\eta}\ . \tag{34}$$

Mechanism

This transformation parameter can be found by solving the equation

$$3\mathcal{H}\alpha' + \left(3\mathcal{H}' + \nabla^2\right)\alpha = S\,, \tag{35}$$

in which the conformal Hubble parameter $\mathcal{H}$ is

$$\mathcal{H} = aH = \left(\nu - \frac{1}{2}\right)\frac{1}{(-\eta)}\ . \tag{36}$$

The source function S is given by

$$S = \frac{Q\sqrt{2\epsilon_1}}{2M_{Pl}}{\rm sign}(\dot{\phi})\left(\mathcal{H}\frac{\epsilon_2}{2} - \frac{Q'}{Q}\right)\ . \tag{37}$$

We can solve (35) with the general solution

$$\alpha_k = \frac{1}{3\mathcal{H}}\int_{\eta_0}^{\eta} S_k(\eta')\exp\left[\frac{k^2}{3}\int_{\eta'}^{\eta}\frac{d\eta''}{\mathcal{H}(\eta'')}\right] \tag{38}$$

General case

For the large-scale limit $k\eta \ll 1$

$$\alpha_k \approx A\frac{(\nu - 3/2)^{1/2}}{(\nu - 1/2)^{5/2}}\left[-(-\eta)^{\nu-|\nu|+1} + (-\eta_0)^{\nu-|\nu|}(-\eta) + \ldots\right]. \quad (39)$$

When $\eta \to 0$, the leading term in the gauge parameter is the first inside the brackets, which starts to diverge when $\nu - |\nu| + 1 < 0$.

Let us break down the different scenarios depending on the value of ν.

Gauge correction for $\nu > 0$, $\nu \neq 1/2$

If $\nu > 0$, the previous condition never holds as $\nu = |\nu|$.

$$\alpha_k = -\frac{i\Gamma(\nu)}{3\sqrt{8\pi}M_{Pl}} \frac{(\nu - 3/2)^{1/2}}{(\nu - 1/2)^{5/2}} \left(\frac{2}{k}\right)^{\nu} \mathrm{sign}(\dot{\phi}) H_\star(-\eta_\star)^{\frac{3}{2}-\nu} \left\{\frac{1}{4(\nu-1)}k^2(-\eta_0)^2(-\eta) + \mathcal{O}[k^2(-\eta)^3]\right\}, \quad (40)$$

from which we see the gauge correction are negligible in general, and completely cancels for $\nu = 3/2$. Therefore, the usual slow-roll inflation is not affected by gauge corrections. Also verified by C. Pattison et al. arXiv:1905.06300

Gauge corrections for $\nu = 1/2$

For the ekpyrotic collapse, $\nu = 1/2$, α_k diverges, although this happens due to the choice of the scale factor.

The ekpyrotic collapse is described by the kinetic-dominated solution $x_a^2 = 1$ and $y_a^2 = 0$.

$$\delta x = \frac{\kappa}{\sqrt{6}} \left[\left(1 - x^2\right) \frac{\dot{\delta\phi}}{H} + \left(3x^4 - 3x^2 + \frac{\lambda^2}{2} y^2\right) \delta\phi \right] . \qquad (41)$$

It is clear that $\delta x = 0$ independently of the field transformation, which means that gauge corrections do not affect the classical configuration.

Gauge corrections for $\nu < 0$

For $\nu < 0$, the above condition holds if $\nu < -1/2$, which indicates that collapsing scenarios should worry about gauge corrections.

For $\nu = -3/2$ the gauge parameter (39) in this case becomes

$$\alpha_k = \frac{ik^{-3/2}\mathrm{sign}(\dot{\phi})H_\star(-\eta_\star)^3}{M_{Pl}(3\times 2^7)^{1/2}} \left\{ -\frac{1}{(-\eta)^2} - \frac{2k^2}{3} + \frac{(-\eta)}{(-\eta_0)^3} + \frac{3k^2(-\eta)}{4(-\eta_0)} - \frac{k^2(-\eta)^3}{12(-\eta_0)^3} + \mathcal{O}\left[k^4(-\eta)^2\right] \right\}, \tag{42}$$

and the leading order field contribution in α_k blows up like $(-\eta)^{-2}$ when $\eta \to 0$.

In order to quantify how the gauge effects can modify the behaviour of the classical solution for the kinetic variable x, we apply the gauge transformation into δx.

Gauge corrections for $\nu = -3/2$

For the matter collapse, the pre-factor in front of the $\delta\phi$ term is equal to zero, which leave us just with the time derivative term

$$\begin{aligned}\frac{\dot{\delta\tilde{\phi}}}{H} =& \frac{\dot{\delta\phi}}{H} + \frac{\dot{\phi}'\alpha + \phi'\dot{\alpha}}{H} \\ =& \frac{3ik^{-3/2}H_\star(-\eta_\star)^3}{2^{5/2}(-\eta)^3}\left\{1 - \frac{\operatorname{sign}(\dot{\phi})}{2} + \mathcal{O}\left[k^2(-\eta)^5\right]\right\} . \end{aligned} \tag{43}$$

Then, we see the gauge corrections in this scenario are of the form

$$\frac{\dot{\delta\tilde{\phi}}/H}{\dot{\delta\phi}/H} \approx 1 - \frac{\operatorname{sign}(\dot{\phi})}{2} + \mathcal{O}[k^2(-\eta)^5] , \tag{44}$$

which means that the gauge effects bring only a constant together with a vanishing term scaling as $(-\eta)^5$. Gauge corrections can therefore be neglected in the pressureless collapse.

Final considerations

Current results in stochastic collapse

- quantum fluctuations in a collapsing FRW cosmology behave like inflationary perturbations (Starobinsky 1979 and Wands 1999)
- non-stiff collapse ($w < 1$) classically unstable / classical lifetime depends on initial conditions
- quantum fluctuations gives a finite lifetime for w=0 (Frion, Miranda and Wands, 2019)
- quantum shear backreaction from massless test field for $w < -1/9$ (Grain and Vennin, 2020)

Summary

Inflation / Ekpyrotic collapse ($\nu > 0$)

Pressureless collapse ($\nu < 0$)

$\delta x = 0$ (adiabatic perturbation)

$\delta x \neq 0$ (non-adiabatic perturbation)

- Inflation and Ekpyrotic collapse are both classical and quantum stable;
- Pressureless collapse is quantum unstable;
- For $\nu = -3/2$, we found the quantum diffusion takes us away from the critical point if we start the collapse from very low energy scales and if it lasts more than 134 e-folds;
- Gauge effect can be safely neglected for slow-roll inflation, ekpyrotic collapse and pressureless collapse;

What's next?

- Connect these results to expanding phase (extend stochastic formalism to non-monotonic time variable)
- Bounce from stochastic geometry?

Muito obrigada!