The general form of the coupled Horndeski Lagrangian that allows cosmological scaling solutions

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1 Introduction

- Models based on scalar fields to explain the accelerated expansion of the Universe.

- The main goal: suitable solutions to the background and perturbation equations of motion and to study their stability properties and their degree of independence of the initial conditions.

- Scalar field Lagrangian expanded by including terms coupled to gravity and terms that are general functions of the kinetic energy.

- The most general 4-dimensional, Lorentz covariant, local scalar-tensor theories keeping the equations of motion at second order are described by the Horndeski Lagrangian (Deffayet, Deser and Esposito-Farese, 2009)

The Horndeski Lagrangian

Horndeski (1975), Deffayet et al (2011).

No ghosts, No classical instabilities

$$\mathcal{L}_H = \sum_{i=2}^{5} \mathcal{L}_i \tag{1}$$

$$\begin{aligned}
\mathcal{L}_2 &= K(\phi, X), &(2)\\
\mathcal{L}_3 &= -G_3(\phi, X)\Box\phi, &(3)\\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)], &(4)\\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu,\nu}(\nabla^\mu\nabla^\nu\phi) &(5)\\
&\quad -\frac{1}{6}G_{5,X}[(\Box\phi)^3 - 3(\Box\phi)((\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)]. &(6)
\end{aligned}$$

where $X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$.

Application to specific theories (De Felice, Kobayashi, Tsujikawa, PLB 2011)

$f(R)$ **theories -** $\mathcal{L} = \frac{M_{pl}^2}{2} f(R)$ (J. O'Hanlon (PRL 1972), T. Chiba (PLB 2003))

$$K = -\frac{M_{pl}^2}{2}(Rf_R - f), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2} M_{pl}\phi, \quad \phi = M_{pl} f_R. \tag{7}$$

Brans-Dicke theories with potential $V(\phi)$

$$K = \frac{M_{pl}\omega_{BD} X}{\phi} - V(\phi), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2} M_{pl}\phi. \tag{8}$$

Covariant Galileon without the field potential (C. Deffayet et al, PRD2009).

$$K = -c_2 X, \quad G_3 = \frac{c^3}{M^3} X, \quad G_4 = \frac{1}{2} M_{pl} - \frac{c^4}{M^6} X^2, \quad G_5 = \frac{3c_5}{M^9} X^2 \tag{9}$$

Kinetic gravity braidings (C. Deffayet, O. Pujolas, I. Sawicki, A. Vikman, JCAP 1010 (2010) 026).

$$K = K(\phi, X), \quad G_3 = G_3(\phi, X), \quad G_4 = \frac{1}{2} M_{pl}\phi, \quad G_5 = 0 \tag{10}$$

Purely kinetic coupled gravity (Gubitosi and Linder, PLB 2011)

$$K = X, \quad G_4 = \frac{1}{2} M_{pl}, \quad G_5 = -\lambda \frac{\phi}{M_{pl}^2}. \tag{11}$$

- An exhaustive study of the Horndeski model is very difficult due to the number of free functions.

- It is therefore interesting to ask whether one can find some general property without solving the equations of motion.

- An important class of cosmological solutions that has been studied for several models are the so-called scaling solutions, defined by the ratio of energy density Ω_m/Ω_ϕ constant.

- A second condition that has also been often employed to simplify the treatment is that the field equation of state ω_ϕ remains constant.

- Scaling solutions are particularly interesting because one can hope to employ them to avoid the problem of the coincidence between the present matter and dark energy densities, i.e. the fact that today the two density fractions Ω_m, Ω_ϕ are very similar.

- We consider that there is only one type of matter of energy density $\rho_m = -T^0_0$, in the Einstein frame, where the energy-momentum tensor is defined by (we neglect the small baryon component)

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{12}$$

In this frame matter is directly coupled to the scalar field through the function $Q(\phi)$, where

$$Q = -\frac{1}{\rho_m\sqrt{-g}}\frac{\delta S_m}{\delta\phi}. \tag{13}$$

- F. Piazza and S. Tsujikawa (JCAP, 2004): the most general action of the scalar quadratic kinetic term and of the field itself that contains scaling solutions must have the form

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + K(\phi, X)\right] + S_m(\phi, \psi_i, g_{\mu\nu}) \tag{14}$$

with

$$K(\phi, X) = Xg(Xe^{\lambda\phi}), \tag{15}$$

where g an arbitrary function and λ a constant related to the coupling between matter and scalar field.

- L. Amendola, M. Quartin, S. Tsujikawa, I. Waga (PRD, 2006): the former result has been extended for the case of variable coupling

- A. R. Gomes and L. Amendola (JCAP, 2014): inclusion of $G_3(\phi, X)\nabla_\mu\nabla^\mu\phi$ in the Lagrangian, with variable coupling

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + K(\phi, X) - G_3(\phi, X)\nabla_\mu\nabla^\mu\phi\right] + S_m(\phi, \psi_i, g_{\mu\nu}) \tag{16}$$

with solution

$$\mathcal{L}(X, \Box\phi, \phi) = Xg(Y) - (aY^2 + rY)\Box\phi. \tag{17}$$

where

$$Y = Xe^{\lambda\phi} \tag{18}$$

and

$$\lambda = Q\left(\frac{1 + w_{eff}}{\Omega_\phi(w_m - w_\phi)}\right). \tag{19}$$

2 General Horndeski Lagrangian and equations of motion

- We consider a FLRW flat metric with $ds^2 = -dt^2 + \mathcal{A}^2(t)d\mathbf{x}^2$, where $\mathcal{A}(t)$ is the scale factor and $H = \dot{\mathcal{A}}/\mathcal{A}$. We consider pressureless matter.

  Varying the action with respect to $g_{\mu\nu}$, and defining $H = \dot{\mathcal{A}}/\mathcal{A}$, one gets (De Felice and Tsujikawa, JCAP 2012)

$$\sum_{i=2}^{5} \mathcal{E}_i = -\rho_m, \tag{20}$$

$$\sum_{i=2}^{5} \mathcal{P}_i = 0, \tag{21}$$

  where

$$\mathcal{E}_2 = 2XK_{,X} - K, \tag{22}$$

$$\mathcal{E}_3 = 6X\dot{\phi}HG_{3,X} - 2XG_{3,\phi}, \tag{23}$$

$$\mathcal{E}_4 = -6H^2G_4 + 24H^2X(G_{4,X} + XG_{4,XX}) - 12HX\dot{\phi}G_{4,\phi X} - 6H\dot{\phi}G_{4,\phi}, \tag{24}$$

$$\mathcal{E}_5 = 2H^3X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}) - 6H^2X(3G_{5,\phi} + 2XG_{5,\phi,X}). \tag{25}$$

and

$$\mathcal{P}_2 = K, \tag{26}$$
$$\mathcal{P}_3 = -2X(G_{3,\phi} + \ddot{\phi}G_{3,X}), \tag{27}$$
$$\begin{aligned}\mathcal{P}_4 = {} & 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4,X} - 4H\dot{X}G_{4,X} - 8\dot{H}XG_{4,X} - 8HX\dot{X}G_{4,XX} \\ & +2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} + 4XG_{4,\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X},\end{aligned} \tag{28}$$
$$\begin{aligned}\mathcal{P}_5 = {} & -2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5,X} - 4H^2X^2\ddot{\phi}G_{5,XX} \\ & +4HX(\dot{X} - HX)G_{5,\phi X} + 2[2(\dot{H}X + H\dot{X}) + 3H^2X]G_{5,\phi} + 4HX\dot{\phi}G_{5,\phi\phi}.\end{aligned} \tag{29}$$

- Now, in order to confront these models with SNIa observations, we isolate from the complete action a term corresponding to the Einstein-Hilbert one. Then the action is rewritten as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \mathcal{L}\right) + S_m(\phi, \psi_i, g_{\mu\nu}) \tag{30}$$

with

$$
\begin{aligned}
\mathcal{L} \;=\;& K(\phi, X) - G_3(\phi, X)\Box\phi + \left(G_4(\phi, X) - \frac{1}{2}\right)R + G_{4,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)] \\
& + G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) \\
& + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi) \qquad (31)
\end{aligned}
$$

and we write the Einstein equations in the usual form (Tsujikawa, 2011)

$$H^2 = \frac{1}{3}(\rho_\phi + \rho_m) \qquad (32)$$

and

$$-2\dot{H} = \rho_m + \rho_\phi + p. \qquad (33)$$

Comparing Eqs. (32), (33) with Eqs. (20), (21), we arrive to useful definitions for the energy density (ρ_ϕ) and pressure (p) of dark energy:

$$\rho_\phi \equiv \sum_{i=2}^{5} \mathcal{E}_i + 3H^2, \tag{34}$$

$$p \equiv \sum_{i=2}^{5} \mathcal{P}_i - (3H^2 + 2\dot{H}), \tag{35}$$

or

$$\begin{aligned}
\rho_\phi \equiv\ & 2XK_{,X} - K + 6X\dot{\phi}HG_{3,X} - 2XG_{3,\phi} + 3H^2(1-2G_4) \\
& +24H^2X(G_{4,X} + XG_{4,XX}) - 12HX\dot{\phi}G_{4,\phi X} - 6H\dot{\phi}G_{4,\phi} \\
& +2H^3X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}) - 6H^2X(3G_{5,\phi} + 2XG_{5,\phi,X}),
\end{aligned} \tag{36}$$

$$\begin{aligned}
p \equiv\ & K - 2X(G_{3,\phi} + \ddot{\phi}G_{3,X}) - (3H^2 + 2\dot{H})(1-2G_4) - 12H^2XG_{4,X} \\
& -4H\dot{X}G_{4,X} - 8\dot{H}XG_{4,X} - 8HX\dot{X}G_{4,XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} \\
& +4XG_{4,\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X} \\
& -2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5,X} - 4H^2X^2\ddot{\phi}G_{5,XX} \\
& +4HX(\dot{X} - HX)G_{5,\phi X} + 2[2(\dot{H}X + H\dot{X}) + 3H^2X]G_{5,\phi} + 4HX\dot{\phi}G_{5,\phi\phi}.
\end{aligned} \tag{37}$$

Defining

$$\Omega_\phi = \frac{\rho_\phi}{3H^2}, \quad \Omega_m = \frac{\rho_m}{3H^2} \tag{38}$$

we can rewrite the Friedman equation (Eq. (32)) as

$$\Omega_\phi + \Omega_m = 1. \tag{39}$$

Now we introduce the e-folding time $N = \log a$, so that $d/dt = Hd/dN$. Then the equations of motion for the scalar field ϕ and matter can be written as

$$\frac{d\rho_\phi}{dN} + 3(1 + w_\phi)\rho_\phi = -\rho_m Q \frac{d\phi}{dN} \tag{40}$$

$$\frac{d\rho_m}{dN} + 3\rho_m = \rho_m Q \frac{d\phi}{dN}, \tag{41}$$

where $w_\phi = p/\rho_\phi$.

3 Scaling condition and master equation

The condition Ω_ϕ/Ω_m constant defines scaling solutions. This is equivalent to ρ_ϕ/ρ_m constant, or to

$$\frac{d\log\rho_\phi}{dN} = \frac{d\log\rho_m}{dN} \tag{42}$$

Also, from Eq. (39) we get that Ω_ϕ is a constant. We also assume that on scaling solutions the equation of state parameter w_ϕ is a constant (Tsujikawa and Sami, PLB 2004). Then we get

$$\begin{aligned} \frac{d\phi}{dN} &= -\frac{3\Omega_\phi w_\phi}{Q} \propto \frac{1}{Q}, && (43) \\ \frac{d\log\rho_\phi}{dN} &= \frac{d\log\rho_m}{dN} = -3(1+w_{eff}), && (44) \\ \frac{d\log p}{dN} &= -3(1+w_{eff}), && (45) \end{aligned}$$

where $w_{eff} = w_\phi\Omega_\phi$.

- We proved that, for the FRW metric, the pressure is equivalent to the original Lagrangian density: $p = \mathcal{L}$

For FRW metric we also get

$$R = \left(\frac{1}{w_\phi \Omega_\phi} - 3\right) p. \tag{46}$$

Then we can write the pressure of dark energy as

$$\begin{aligned} p &= \frac{1}{f(\phi, X)} \Big[K(\phi, X) - G_3(\phi, X) \Box\phi + G_{4,X} [(\Box\phi)^2 - \boxtimes\phi] \\ &\quad + G_5(\phi, X)(\ominus\phi) - \frac{1}{6} G_{5,X} [(\Box\phi)^3 - 3(\Box\phi) \boxtimes \phi + 2(\boxdot\phi)\Big] \end{aligned} \tag{47}$$

That is, $p = p(X, \Box\phi, \boxtimes\phi, \ominus\phi, \boxdot\phi, \phi)$

with

$$f(\phi, X) \equiv \left[1 + \left(\frac{1}{2} - G_4(\phi, X)\right) \tilde{c}\right]. \tag{48}$$

and

$$
\begin{aligned}
\boxtimes\phi &\equiv (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) &(49)\\
\ominus\phi &\equiv G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) &(50)\\
\boxdot\phi &\equiv (\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla^\mu\phi) &(51)
\end{aligned}
$$

A generalized "master equation" for $p = p(X, \Box\phi, \boxtimes\phi, \boxbar\phi, \boxdot\phi, \phi)$is found to be

$$
\begin{aligned}
&\frac{\partial\log p}{\partial\phi}\frac{d\phi}{dN} + \frac{\partial\log p}{\partial\log X}\frac{d\log X}{dN} + \frac{\partial\log p}{\partial\log\Box\phi}\frac{d\log\Box\phi}{dN} + \frac{\partial\log p}{\partial\log\boxtimes\phi}\frac{d\log\boxtimes\phi}{dN}\\
&+\frac{\partial\log p}{\partial\log\ominus\phi}\frac{d\log\ominus\phi}{dN} + \frac{\partial\log p}{\partial\log\boxdot\phi}\frac{d\log\boxdot\phi}{dN} = -3(1+w_{eff}). \qquad (52)
\end{aligned}
$$

Each partial derivative was obtained, restricting the coupling to $\frac{1}{Q^2}\frac{dQ}{d\phi} = c$ constant, leading to

$$
\begin{aligned}
&\left(1+\frac{2}{\lambda Q^2}\frac{dQ}{d\phi}\right)\frac{\partial\log p}{\partial\log X} + \left(1+\frac{1}{\lambda Q^2}\frac{dQ}{d\phi}\right)\frac{\partial\log p}{\partial\log\Box\phi} + \left(2+\frac{2}{\lambda Q^2}\frac{dQ}{d\phi}\right)\frac{\partial\log p}{\partial\log\boxtimes\phi}\\
&+\left(2+\frac{1}{\lambda Q^2}\frac{dQ}{d\phi}\right)\frac{\partial\log p}{\partial\log\ominus\phi} + \left(3+\frac{3}{\lambda Q^2}\frac{dQ}{d\phi}\right)\frac{\partial\log p}{\partial\log\boxdot\phi} - \frac{1}{\lambda Q}\frac{\partial\log p}{\partial\phi} = 1. \qquad (53)
\end{aligned}
$$

4 Solutions for the master equation

We start with Eq. (53) rewritten as

$$\left(1+\frac{2}{Q}\frac{dQ}{d\psi}\right)\frac{\partial \log p}{\partial \log X}+\left(1+\frac{1}{Q}\frac{dQ}{d\psi}\right)\frac{\partial \log p}{\partial \log \Box\phi}+\left(2+\frac{2}{Q}\frac{dQ}{d\psi}\right)\frac{\partial \log p}{\partial \log \boxtimes\phi}$$
$$+\left(2+\frac{1}{Q}\frac{dQ}{d\psi}\right)\frac{\partial \log p}{\partial \log \ominus\phi}+\left(3+\frac{3}{Q}\frac{dQ}{d\psi}\right)\frac{\partial \log p}{\partial \log \boxdot\phi}-\frac{\partial \log p}{\partial\psi}=1, \quad (54)$$

where

$$\psi=\int_{\phi} du[\lambda Q(u)]. \quad (55)$$

Now set

$$p=XQ^{2}(\phi)\tilde{G}(X,\Box\phi,\boxtimes\phi,\ominus\phi,\boxdot\phi,\phi). \quad (56)$$

where $\tilde{G}$ is an arbitrary function of its argument. Then for $\tilde{G} \neq 0$ we obtain

$$\left(1+\frac{2}{Q}\frac{dQ}{d\psi}\right)X\frac{\partial\tilde{G}}{\partial X}+\left(1+\frac{1}{Q}\frac{dQ}{d\psi}\right)\Box\phi\frac{\partial\tilde{G}}{\partial\Box\phi}+\left(2+\frac{2}{Q}\frac{dQ}{d\psi}\right)(\boxtimes\phi)\frac{\partial\tilde{G}}{\partial(\boxtimes\phi)}$$
$$+\left(2+\frac{1}{Q}\frac{dQ}{d\psi}\right)(\ominus\phi)\frac{\partial\tilde{G}}{\partial(\ominus\phi)}+\left(3+\frac{3}{Q}\frac{dQ}{d\psi}\right)(\boxdot\phi)\frac{\partial\tilde{G}}{\partial(\boxdot\phi)}-\frac{\partial\tilde{G}}{\partial\psi}=0. \tag{57}$$

Here, inspired by the expression given by Eq. (47) for p, we look for solutions of $\tilde{G}$ of the form

$$\tilde{G}(X,\Box\phi,\boxtimes\phi,\ominus\phi,\boxdot\phi,\phi)=\frac{1}{f(\phi,X)}\tilde{g}(X,\Box\phi,\boxtimes\phi,\boxbar\phi,\boxdot\phi,\phi). \tag{58}$$

Then, Eq. (57) turns into

$$\frac{1}{f}\Bigg\{\left[\left(1+\frac{2}{Q}\frac{dQ}{d\psi}\right)X\frac{\partial f}{\partial X}-\frac{\partial f}{\partial\psi}\right]\frac{\tilde{g}}{f}$$
$$-\Bigg[\left(1+\frac{2}{Q}\frac{dQ}{d\psi}\right)X\frac{\partial\tilde{g}}{\partial X}+\left(1+\frac{1}{Q}\frac{dQ}{d\psi}\right)\Box\phi\frac{\partial\tilde{g}}{\partial\Box\phi}+\left(2+\frac{2}{Q}\frac{dQ}{d\psi}\right)(\boxtimes\phi)\frac{\partial\tilde{g}}{\partial(\boxtimes\phi)}$$
$$+\left(2+\frac{1}{Q}\frac{dQ}{d\psi}\right)(\ominus\phi)\frac{\partial\tilde{g}}{\partial(\ominus\phi)}+\left(3+\frac{3}{Q}\frac{dQ}{d\psi}\right)(\boxdot\phi)\frac{\partial\tilde{g}}{\partial(\boxdot\phi)}-\frac{\partial\tilde{g}}{\partial\psi}\Bigg]\Bigg\}=0. \tag{59}$$

For $f \neq 0$ this is equivalent to $(\widehat{O}_1 f)\tilde{G}-\widehat{O}_2\tilde{g}=0$. The simplest choice is to impose that $\tilde{g}$ and f

satisfy separately the linear differential equations:

$$\hat{O}_1 f = 0, \quad (60)$$

$$\hat{O}_2 \tilde{g} = 0. \quad (61)$$

The solution of the first equation is known. For the second one, and guided by the form of p, is has the form of a superposition of general functions in terms of combinations of $X, \psi, \Box\phi, \ominus\phi, \boxdot\phi, \boxtimes\phi$:

$$\tilde{g} = g(h_b) + g_c(h_c) + g_d(h_d) + ..., \quad (62)$$

with

$$h_b(X,\psi) = f_{1b}(X) f_{3b}(\psi), \quad (63)$$

$$h_c(X,\Box\phi) = f_{1c}(X) f_{2c}(\Box\phi) \quad (64)$$

$$\vdots \quad (65)$$

$$h_q(X,\ominus\phi,\psi) = f_{1q}(X) f_{3q}(\psi) f_{5q}(\ominus\phi) \quad (66)$$

$$\vdots \quad (67)$$

Each particular differential equation is solved separately, with solutions depending on arbitrary constants. For example we get $g_q(h_q) = q\left(X(\ominus\phi)e^{3\psi}Q^3\right)$, and so on.

The general form of p impose relations between the general constants in order to get a compatible solution for the general functions.

Then, after some algebra, the general scaling Horndeski Lagrangian obtained is

$$\begin{aligned}
\mathcal{L}_H &= XQ^2g(XQ^2e^{\psi}) - [d_1XQ^3e^{\psi} + \ell_1X^2Q^5e^{2\psi}]\Box\phi \\
&+\left(h(\phi) + \frac{1}{2}d_2X^2Q^4e^{2\psi} + \frac{1}{3}\ell_2X^3e^{3\psi}Q^6\right)R \\
&+\left(d_2XQ^4e^{2\psi} + \ell_2X^2e^{3\psi}Q^6\right)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)] \\
&+qX^2e^{3\psi}Q^5G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\
&-\left(\frac{1}{6}\right)2qXe^{3\psi}Q^5[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) \\
&+2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)].
\end{aligned} \tag{68}$$

5 Conclusions

- Integrating the Lagrangian by parts, we proved that the entire Horndeski Lagrangian is equivalent to the scalar field pressure (at least in a FLRW metric).

- It is possible to show that there are two equivalent versions of the general Horndeski Lagrangian that allows for scaling solutions: in the form presented here, the general Horndeski functions K, G_3, G_4, G_5 are more evident, but the Lagrangian depends on the coupling Q; we also scaled the field to obtain the Lagrangian with a constant coupling, but with an even more intrincate structure.

- The existence of this particular class of solution is interesting since it could represent a solution of the coincidence problem. If the ratio Ω_m/Ω_ϕ depends on the fundamental constant of the theory, instead of on initial conditions, then the fact that it is close to unity would no longer be a surprising coincidence.