Starobinsky inflation and the order reduction technique

Waleska Priscylla Florencio de Medeiros
Supervisor: Prof. Dr. Oliver Fabio Piattella
Co-supervisor: Prof. Dr. Júlio César Fabris

VI José Plínio Baptista School of Cosmology Friedmann and the LCDM model

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Abstract

- The order reduction technique (ORT) is an iterative method of solution of higher order differential equations.
- The intention is to show a more careful analysis of the conditions for the validity of the method.
- With this in mind, a few simple situations in which the iterative order reduction converges analytically to the exact solutions are presented to the following cases:

  1. The study of the dynamics of the motion of a charged particle.
  2. The harmonic oscillator.
  3. The inflationary paradigm of Starobinsky.

  The main results discussed are based on the work[1].

[1] W. P. F. De Medeiros and D. Müller. On the order reduction. Eur. Phys. J. C 81 231 (2021). arXiv:2009.00629.

Ostrogradsky's instability (1850)

- The theorem of Ostrogradsky[2]: for a nondegenerate Lagrangian, where $\partial L/\partial q^{(N)}$ depends on $q^{(N)}$, and also that depends on time derivatives greater than first order as $L(q, \dot{q}, ..., q^{(N)}, t)$, the Hamiltonian becomes linear in the canonical momentum P_1, P_2, ..., P_{N-1}.
  $\Rightarrow$ Unstable, the Hamiltonian becomes unbounded.
- A dynamical variable that suffers from Ostrogradsky's instability will carry kinetic terms with opposite signs, which through couplings can result in infinite energy transfers between degrees of freedom, while the total energy of the system is conserved.
- This type of instability is known today as ghosts.

[2] M. Ostrogradsky, Mem. Ac. St. Petersbourg VI 4, 385, 1850.

The equation of Lorentz-Abraham-Dirac (LAD)

- The equation of motion of a charged and accelerated particle become known as the **Lorentz-Abraham-Dirac** (LAD) equation[3]

$$a^\mu = \frac{q}{m} F^\mu{}_\nu u^\nu + \frac{2}{3}\frac{q^2}{mc^3}\left(\delta^\mu{}_\nu + u^\mu u_\nu\right)\dot{a}^\nu \tag{1}$$

  in absence of external gravitational fields.

- The second term is the force acting back on the particle, called electromagnetic self-force.
- This model allows non-physical solutions, Dirac (1938). Ex.: The **runaway solution**.

[3] H. A. Lorentz, Encykl. Mathe. Wiss. 2, 145–280, 1904. M. Abraham, Theorie der Elektrizität: Elektromagnetische Theorie der Strahlung, vol. 2. Teubner, 1905. P. A. M. Dirac, Proc. Roy. Soc. A167, 148, 1938.

Landau and Lifshitz (1951)

- Consider the non-relativistic version of the LAD equation,

$$\boxed{m\dot{\mathbf{v}} = e\mathbf{E} + \frac{e}{c}\mathbf{v}\times\mathbf{H} + \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}},} \tag{2}$$

  in the reference system that the charge is momentarily at rest.
- On the Landau and Lifshitz method the self-force term is considered as a perturbation[4]. The second derivative of velocity can be written as

$$\ddot{\mathbf{v}} = \frac{e}{m}\dot{\mathbf{E}} + \frac{e}{mc}\dot{\mathbf{v}}\times\mathbf{H}. \tag{3}$$

- Then, substituting $\dot{\mathbf{v}} = e\mathbf{E}/m$ in the second term, the self-force term of the equation (2) results in

$$\boxed{\mathbf{f}_{self} = \frac{2e^3}{3mc^3}\dot{\mathbf{E}} + \frac{2e^4}{3m^2c^4}\mathbf{E}\times\mathbf{H}.} \tag{4}$$

[4] L.D. Landau, E.M. Lifshitz, The Classical Theory of Fields. Pergamon, Oxford, 1951.

Landau and Lifshitz (1951)

- Through the condition imposed by the method, $\mathbf{f}_{self} \ll e\mathbf{E}$, and let ω the frequency of motion, we have $\dot{\mathbf{E}} \propto \omega \mathbf{E}$, so from the terms of the self-force (4),

$$\mathbf{f}_{self} = \frac{2e^3}{3mc^3}\dot{\mathbf{E}} + \frac{2e^4}{3m^2c^4}\mathbf{E} \times \mathbf{H},$$

  the follow conditions are obtained;

$$\frac{e^2}{mc^3}\omega \ll 1, \qquad \mathbf{H} \ll \frac{m^2c^4}{e^3}. \tag{5}$$

- Introducing the wavelength $\lambda \sim c/\omega$, we have from the first condition (5),

$$\lambda \gg \frac{e^2}{mc^2}. \tag{6}$$

  Here e^2/mc^2 is the electric charge "radius".
- On the other hand, the external frequency is lower than the natural frequency of the system.

Lorentz-Abraham-Dirac equation

- Consider the relativistic LAD equation without the presence of gravitational fields, Eq. (1),

$$\boxed{a^\mu = \frac{q}{m} F^\mu{}_\nu u^\nu + \frac{2}{3}\frac{q^2}{mc^3}\left(\delta^\mu{}_\nu + u^\mu u_\nu\right)\dot{a}^\nu}$$

- Let's proceed with the order reduction applied to the LAD equation for a constant electric field E in the x direction.
- Of course, this fulfills its convergence requirements.
- To a lowest approximation, we have

$$a_0^\mu = \frac{q}{m} F^\mu{}_\nu u_0^\nu. \tag{7}$$

Lorentz-Abraham-Dirac equation

- Consider a time-like $u^\mu = (u^0, u^1, 0, 0)$ with $u_\mu u^\mu = -1$. Then, the zeroth-order solution (7) is

$$u_0^\mu = (\cosh(qEt/m), \sinh(qEt/m), 0, 0). \tag{8}$$

- The derivative of (7) substituted into the second term in the right-hand side of (7) vanishes, showing that perturbatively the exact result is consistently obtained with the order reduction.
- This is not anything new, since this exact solution was found by Dirac (1938)[5], see also[6]. Similar applications have already been made by Eliezer and Peierls[7].

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[5] P. A. M. Dirac, Proc. Roy. Soc. A167, 148, 1938.

[6] F. Rohrlich, Classical Charged Particles, Addison-Wesley, Reading, Mass., 1965.

[7] C. J. Eliezer and R. E. Peierls. On the classical theory of particles. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 194, 543 (1948)

Harmonic oscillator

- Consider the harmonic oscillator,

$$\boxed{\epsilon x'' + x' + \frac{\omega^2}{\gamma^2} x = \frac{f_0}{\gamma^2} e^{i(\Omega/\gamma)\tau}.} \tag{9}$$

  The derivatives are with respect to the dimensionless time $\tau \equiv \gamma t$.

- As usual, $0 \leq \epsilon \leq 1$ is a dimensionless perturbative parameter and at the end is made $\epsilon = 1$ to return to the original equation.
- In this case, the technique of the order reduction consists in considering $|x''| < |x'| < |x|$, which results in the recurrence relation

$$\epsilon x_n'' + x_{n+1}' + \frac{\omega^2}{\gamma^2} x_{n+1} = \frac{f_0}{\gamma^2} e^{i(\Omega/\gamma)\tau}. \tag{10}$$

Harmonic oscillator

- To lowest order

$$x_0' + \frac{\omega^2}{\gamma^2} x_0 = \frac{f_0}{\gamma^2} e^{i(\Omega/\gamma)\tau}, \tag{11}$$

which can be easily solved assuming $x_0 = c_0 e^{i(\Omega/\gamma)\tau}$, when c_0 is

$$c_0 = \frac{f_0}{\omega^2 + i\gamma\Omega}. \tag{12}$$

- To first order,

$$\epsilon x_0'' + x_1' + \frac{\omega^2}{\gamma^2} x_1 = \frac{f_0}{\gamma^2} e^{i(\Omega/\gamma)\tau}, \tag{13}$$

assuming $x_1 = c_1 e^{i(\Omega/\gamma)\tau}$ and using (12), c_1 results in

$$c_1 = \frac{f_0 + \Omega^2 \epsilon c_0}{\omega^2 + i\gamma\Omega}. \tag{14}$$

Harmonic oscillator

- Successively,

$$c_{n+1} = \frac{f_0 + \Omega^2 \epsilon c_n}{\omega^2 + i\gamma\Omega}, \tag{15}$$

and therefore, as long as $\Omega/\omega < 1$, the order reduction, in this case, converges to the exact particular solution

$$\boxed{x_{n\to\infty} = \frac{f_0 e^{i(\Omega/\gamma)\tau}}{\omega^2 + i\gamma\Omega - \Omega^2\epsilon}} \tag{16}$$

for the non homogeneous equation (9).

- This is the situation in which it is very well known the convergence of the order reduction.

Harmonic oscillator

- This system has 3 regimes. The underdamped, overdamped, and critically damped.
- When the system is underdamped $|x''|$ is of the same order of $|x|(\omega/\gamma)^2$. Thus, the order reduction technique does not apply to the underdamped regime.
- To apply the order reduction in the homogeneous equation version of (9),

$$\boxed{x_n' + \frac{\omega^2}{\gamma^2} x_n = -\epsilon x_{n-1}'',} \tag{17}$$

it is also necessary that $|x''| < |x'| < |x|$.

Harmonic oscillator

- To lowest order,

$$x_0' + \frac{\omega^2}{\gamma^2} x_0 = 0. \tag{18}$$

Substituting $x_0 = c_0 e^{\lambda_0 \tau}$ into (18) gives $\lambda_0 = -\omega^2/\gamma^2$ with

$$x_0 = c_0 e^{-\omega^2 \tau/\gamma^2}. \tag{19}$$

- To obtain the next order, x_0 from (19) is replaced into

$$\epsilon x_0'' + x_1' + \frac{\omega^2}{\gamma^2} x_1 = 0, \tag{20}$$

with solution

$$x_1 = \left(c_1 - \epsilon \frac{\omega^4}{\gamma^4} c_0 \tau \right) \exp\left(-\frac{\omega^2}{\gamma^2} \tau \right), \tag{21}$$

shows an additional constant c_1.

Harmonic Oscillator

- For each higher-order perturbative approximation, there must be one additional constant.
- These additional constants are uniquely determined and made equal to c.
- Then, written up to order order 5 in ϵ the technique results in

$$x = c \exp\left[-\left(\frac{\omega^2}{\gamma^2} + \epsilon\frac{\omega^4}{\gamma^4} + 2\epsilon^2\frac{\omega^6}{\gamma^6} + 5\epsilon^3\frac{\omega^8}{\gamma^8} + 14\epsilon^4\frac{\omega^{10}}{\gamma^{10}} + 42\epsilon^5\frac{\omega^{12}}{\gamma^{12}}\right)\tau\right]. \tag{22}$$

Harmonic Oscillator

- Considering (17),

$$x_n' + \frac{\omega^2}{\gamma^2} x_n = -\epsilon x_{n-1}'',$$

as an iteration map, it is shown that this map is a contraction, and that the method converges to the exact solution

$$\boxed{x = c \exp\left[\left(-1 + \sqrt{1 - 4\epsilon\omega^2/\gamma^2}\right)\frac{\tau}{2\epsilon}\right],} \qquad (23)$$

which is a fixed point for this map.

- The other fixed point, namely

$$x = c \exp\left[\left(-1 - \sqrt{1 - 4\epsilon\omega^2/\gamma^2}\right)\frac{\tau}{2\epsilon}\right], \qquad (24)$$

it's not defined when $\epsilon \to 0$.

Harmonic Oscillator

- While when $\epsilon \to 0$; both (23) has a well defined limit $x = ce^{-\omega^2\tau/\gamma^2}$, which coincides with the exact solution $x = ce^{-\omega^2\tau/\gamma^2}$ of (17).
- This second fixed point, (24), then must be excluded and when $\epsilon \to 1$ we are left with the unique solution of (17)

$$\boxed{x = c\exp\left[\left(-1+\sqrt{1-4\omega^2/\gamma^2}\right)\frac{\tau}{2}\right],}$$

led to conclude that this iterative procedure converges to this solution.

Starobinsky inflationary model

- The main motivation for this theory is that it arises naturally as a quantum correction in a consistent model of semi-classical gravity, i.e. in a scenario where quantum matter fields are considered in a classical gravitational background. The action for this theory is:

$$\boxed{S = \int \mathrm{d}^4x\sqrt{-g}\frac{m_p^2}{2}\left\{R + \beta R^2 + \alpha\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right]\right\}} \quad (25)$$

include Starobinsky's model[8]. The value of $\beta \approx 1.305 \times 10^9$ M_{PL}^{-2} can be inferred from observation.

[8] A. A. Starobinsky. A New Type of Isotropic Cosmological Models Without Singularity. Phys. Lett. B91 99 (1980).

Starobinsky inflationary model

- For the FLRW homogeneous isotropic line element

$$g_{ab} = \mathrm{diag}[-1, a(t)^2, a(t)^2, a(t)^2],$$

with zero spatial curvature, there is the 00

$$\boxed{\frac{1}{6}H^2 + \beta\left[2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2\right] = 0,} \tag{26}$$

and the 11

$$\boxed{-\frac{1}{2}H^2 - \frac{1}{3}\dot{H} + \beta\left[-2\dddot{H} - 12\ddot{H}H - 9\dot{H}^2 - 18\dot{H}H^2\right] = 0,} \tag{27}$$

equations of motion, where $H = \dot{a}/a$ is the Hubble parameter.

Order Reduction

- Again, the order reduction is applied.
- The 00 equation of motion, results in the recurrence relation

$$\boxed{-2\epsilon\beta\frac{\ddot{H}_n}{H_n}+\epsilon\beta\frac{\dot{H}_n^2}{H_n^2}+\frac{1}{6}\left[1-36\beta\dot{H}_{n+1}\right]=0,} \qquad (28)$$

where, as usual, $0 \leq \epsilon \leq 1$ is a dimensionless perturbative parameter and at the end is made $\epsilon = 1$.
Here $H_n \neq 0$ and the equation is dimensionless in the proper time t.

- The conditions used are

$$|\beta\ddot{H}| \ll |H|, \qquad |\beta\dot{H}^2| \ll |H^2|. \qquad (29)$$

- From the first and second slow-roll conditions for inflation there is some overlap in convergence region of the order reduction and slow-roll conditions.

Perturbative Approximations

- To lowest order, gives the solution of T. V. Ruzmaikina and A. A. Ruzmaikin[9], which describes the slow-roll regime.

$$1 + 36\beta \dot{H}_1 = 0 \Rightarrow H_1 = \frac{1}{36\beta}(t_{e_1} - t). \tag{30}$$

- To second order,

$$H_2 = \frac{1}{36\beta}(t_{e_2} - t) + \frac{\epsilon}{6(t_{e_2} - t)}. \tag{31}$$

- To third order,

$$H_3 = \frac{1}{36\beta}(t_{e_3} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_3} - t)} - \frac{8}{3}\frac{\beta\epsilon^2}{(t_{e_3} - t)^3}. \tag{32}$$

[9] T. V. Ruzmaikina and A. A. Ruzmaikin. Quadratic Corrections to the Lagrangian Density of the Gravitational Field and the Singularity. Soviet Journal of Experimental and Theoretical Physics 30 372 (1969).

Perturbative Approximations

Until the five order,

$$H_4 = \frac{1}{36\beta}(t_{e_4} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_4} - t)} - \frac{8}{3}\frac{\beta\epsilon^2}{(t_{e_4} - t)^3} + \frac{584}{5}\frac{\beta^2\epsilon^3}{(t_{e_4} - t)^5}, \tag{33}$$

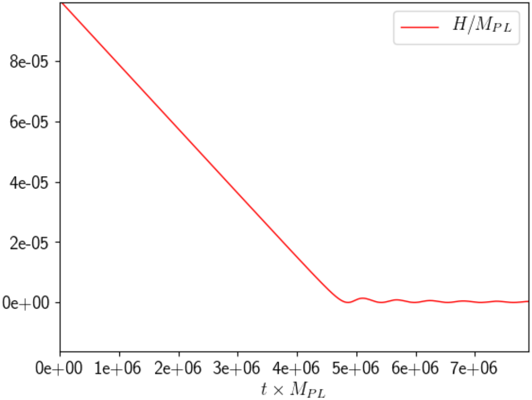
$$H_5 = \frac{1}{36\beta}(t_{e_5} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_5} - t)} - \frac{8}{3}\frac{\beta\epsilon^2}{(t_{e_5} - t)^3} + \frac{584}{5}\frac{\beta^2\epsilon^3}{(t_{e_5} - t)^5} - \frac{282048}{35}\frac{\beta^3\epsilon^4}{(t_{e_5} - t)^7}, \tag{34}$$

t_{e_i} is a constant of integration for $i = 1, 2, 3, ...$ and means the end of inflation. Both conditions for the order reduction and slow-roll are satisfied for

$$1 \ll \frac{1}{\sqrt{\beta}}(t_{e_i} - t), \tag{35}$$

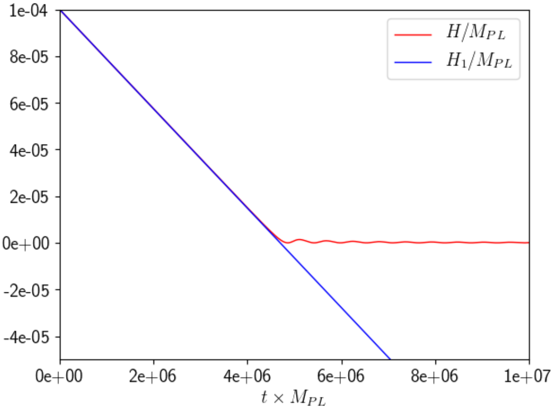
where inflation occurs for $t < t_{e_i}$. While for $t > t_{e_i}$ the conditions are inapplicable, and it presents the graceful exit from inflation.

Exact Numerical Solution



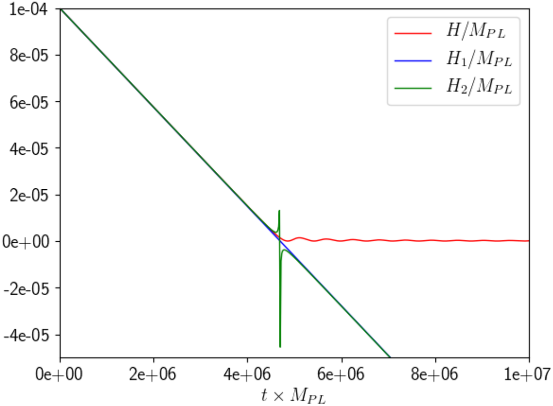
The exact numerical solution of Eq. (26) for $\beta = 1.305 \times 10^9 M_{PL}^{-2}$.

Convergence Regime



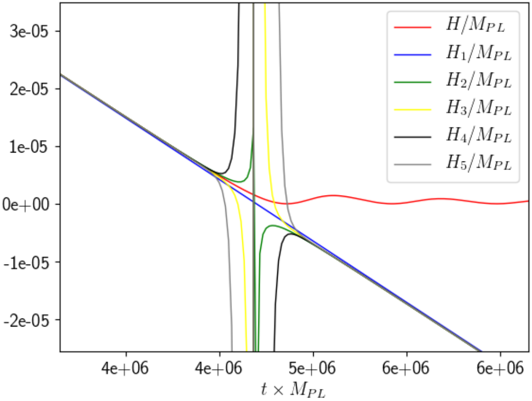
It is shown in red the exact numerical solution of Eq. (26). Plotted in blue is the perturbative solution (30), $H_1(t)$.

Convergence Regime



The perturbative solutions do not agree with the field equation in the oscillating regime of the weak coupling limit. On the other hand, both solutions show very good agreement in the slow-roll regime.

Convergence Regime



Convergence Regime

As shown in the perturbative approximation (34), in this case, the order reduction results in a Laurent series with non-zero principal part with infinite terms. As it's well known this series will not converge in the limit $t \to t_{e_i}$, shown by the asymptotes in the Figures.

The location of the asymptote in the weak-field limit of small oscillations is a consequence of the value of the constant $t = t_{e_i}$ done exclusively to best fit the initial condition chosen for H. For higher-orders, the asymptotes appear alternated in pairs due to successive powers of β, which must be positive to avoid the tachyon.

Convergence to the exact numeric solution

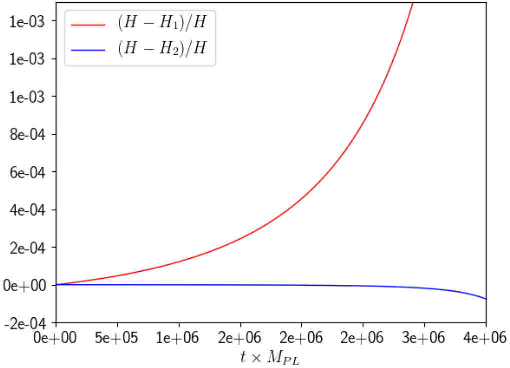
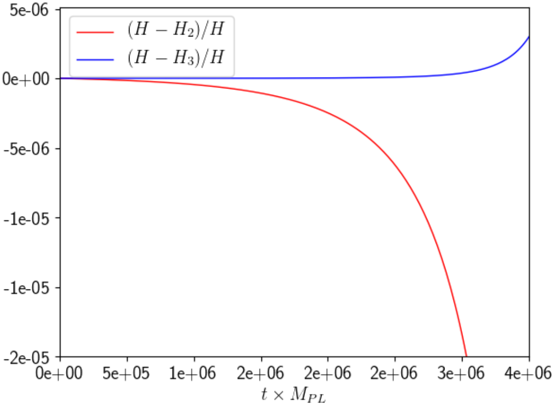


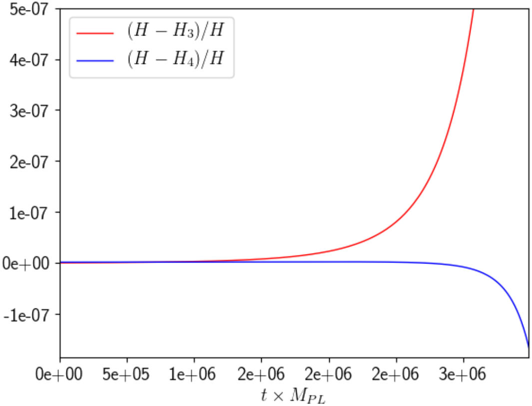
Figure: The graph in red shows the ration of the difference between the exact numeric solution (26), $H(t)$ and the analytical approximation (30), $H_1(t)$ by the exact numeric solution $H(t)$. Plotted in blue is the ratio of the difference between the exact numeric solution (26), $H(t)$ and the analytical approximation (31), $H_1(t)$ by the exact numeric solution, $H(t)$.

Convergence to the exact numeric solution



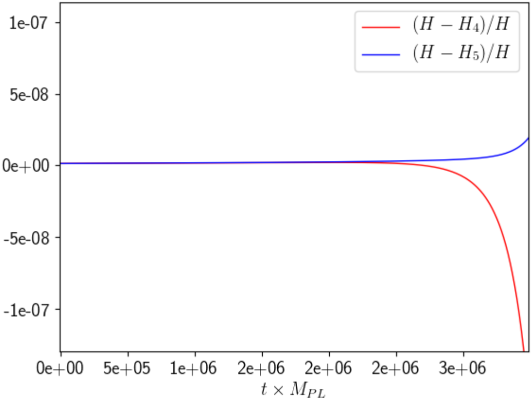
It is possible to see that higher orders of the order reduction method show some convergence to the exact numeric solution.

Convergence to the exact numeric solution



It must also be mentioned that the convergence of the order reduction is slow and becomes even slower for higher order approximations.

Convergence to the exact numeric solution



Scalar factor dependence

- The situation changes in the presence of sources or spatial curvature since then the field equations will depend explicitly on the scale factor.
- In this case, the method will necessarily present second time derivatives to lowest order instead of first time derivatives in the Hubble parameter H. If the source is also considered perturbatively, to lowest order, the first derivative equation (30) is replaced by

$$1 - 36\beta \left[\frac{\ddot{a}_1}{a_1} + \left(\frac{\dot{a}_1}{a_1} \right)^2 \right] = 0, \tag{36}$$

remind that $H = \dot{a}/a$. The explicit dependence of the field equation on the scale factor, through the source $\rho = \rho_0(a/a_0)^{-3(w+1)}$ will come in higher perturbative approximations, by assumption.

Simon and Parker work

A different choice of variables is presented compared to the work of Simon and Parker[10]. The reason for this is that this order reduction, which we are presenting here, is very sensitive to the choice of the lowest perturbative approximation.

The lowest order system in the order reduction must be chosen in accordance with which regime of the solution is going to be reproduced by the method. If higher than first time derivatives of the scale factor are neglected in the lowest perturbative approximation of the order reduction, the Ruzmaikina's regime $H = \dot{a}/a \propto -t$ is not reproduced. For example, in order to get the desired regime, the lowest order must be given by (36).

[10] L. Parker and J. Z. Simon. Einstein equation with quantum corrections reduced to second order. Phys. Rev. D47 1339 (1993).

Order reduction

- To end this presentation, the order reduction will be applied to the trace of the field equation from the action (25),

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{m_p^2}{2}\left\{R + \beta R^2 + \alpha\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right]\right\}.$$

- The trace results in,

$$R - 6\beta\Box R = 0, \tag{37}$$

for the FLRW homogeneous isotropic line element $g_{\mu\nu} = \mathrm{diag}[-1, a^2, a^2, a^2]$ with zero spatial curvature, the equation of motion becomes

$$\boxed{\frac{1}{6}\left[\dot{H} + 2H^2\right] + \beta\left[\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2\right] = 0.} \tag{38}$$

Order reduction

- In following, the order reduction applied to the equation of motion (38) results in the recurrence relation,

$$\boxed{\frac{\epsilon}{6}\frac{\dot{H}_n}{H_n^2} + \epsilon\beta\left[\frac{\dddot{H}_n}{H_n^2} + 7\frac{\ddot{H}_n}{H_n} + 4\frac{\dot{H}_n^2}{H_n^2}\right] + 12\left[\frac{1}{36} + \beta\dot{H}_{n+1}\right] = 0.} \tag{39}$$

- The conditions used are

$$\begin{aligned} \left|\dot{H}\right| &\ll \left|H^2\right|, & \left|\beta\ddddot{H}\right| &\ll \left|H^2\right|, \\ \left|\beta\ddot{H}\right| &\ll \left|H\right|, & \left|\beta\dot{H}^2\right| &\ll \left|H^2\right|. \end{aligned} \tag{40}$$

Order reduction

To lowest order,

$$\frac{1}{36} + \beta \dot{H}_1 = 0 \Rightarrow H_1 = \frac{1}{36\beta}(t_{e_1} - t). \tag{41}$$

Until the five order,

$$H_2 = \frac{1}{36\beta}(t_{e_2} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_2} - t)}, \tag{42}$$

$$H_3 = \frac{1}{36\beta}(t_{e_3} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_3} - t)} - \left(\frac{8}{3}\frac{\beta}{(t_{e_3} - t)^3} + \frac{108}{5}\frac{\beta^2}{(t_{e_3} - t)^5}\right)\epsilon^2, \tag{43}$$

$$H_4 = \frac{1}{36\beta}(t_{e_4} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_4} - t)} - \left(\frac{8}{3}\frac{\beta}{(t_{e_4} - t)^3} + \frac{108}{5}\frac{\beta^2}{(t_{e_4} - t)^5}\right)\epsilon^2 + \left(\frac{692}{5}\frac{\beta^2}{(t_{e_4} - t)^5} + \frac{158976}{35}\frac{\beta^3}{(t_{e_4} - t)^7} + 54432\frac{\beta^4}{(t_{e_4} - t)^9}\right)\epsilon^3, \tag{44}$$

Order reduction

$$
\begin{aligned}
H_5 = {} & \frac{1}{36\beta}(t_{e_5} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_5} - t)} - \left(\frac{8}{3}\frac{\beta}{(t_{e_5} - t)^3} + \frac{108}{5}\frac{\beta^2}{(t_{e_5} - t)^5} \right)\epsilon^2 \\
& + \left(\frac{692}{5}\frac{\beta^2}{(t_{e_5} - t)^5} + \frac{158976}{35}\frac{\beta^3}{(t_{e_5} - t)^7} + 54432\frac{\beta^4}{(t_{e_5} - t)^9} \right)\epsilon^3 \\
& - \left(\frac{441024}{35}\frac{\beta^3\epsilon^4}{(t_{e_5} - t)^7} + \frac{33642288}{35}\frac{\beta^4}{(t_{e_5} - t)^9} \right. \\
& \left. + \frac{1769382144}{55}\frac{\beta^5}{(t_{e_5} - t)^{11}} + \frac{5819869440}{13}\frac{\beta^6}{(t_{e_5} - t)^{13}} \right)\epsilon^4. \quad (45)
\end{aligned}
$$

Convergence to the exact numeric solution

The region of convergence is similar with the order reduction and slow-roll conditions applied to the equation (26) when is made $\epsilon = 1$ at the end.

Again, it can be seen that the successive analytical approximations obtained from the recurrence relation (39) show some slow convergence to the exact numerical solution (38).

Convergence to the exact numeric solution

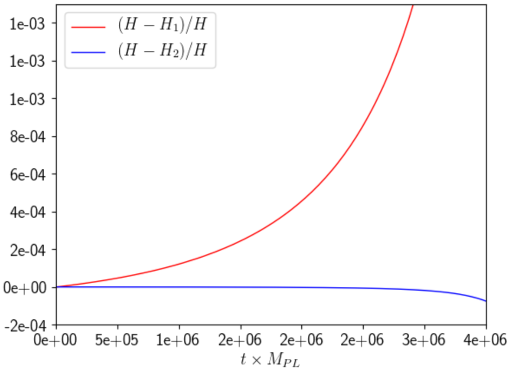
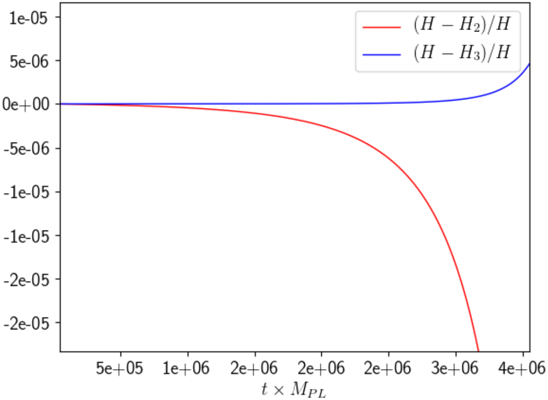
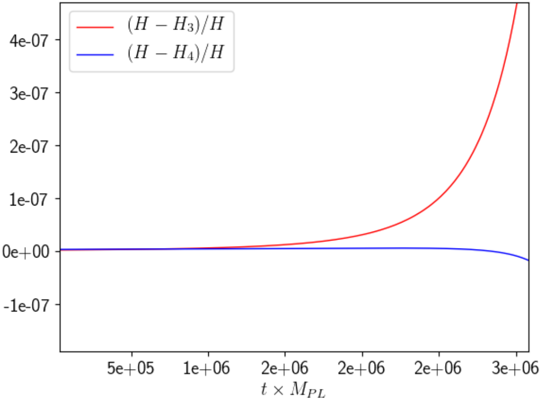


Figure: The graph in red shows the ratio of the difference between the exact numeric solution (38), $H(t)$ and the analytical approximation (41), $H_1(t)$ by the exact numeric solution, $H(t)$. Plotted in blue is the ratio of the difference between the exact numeric solution (38), $H(t)$ and the analytical approximation (42), $H_2(t)$ by the exact numeric solution, $H(t)$.

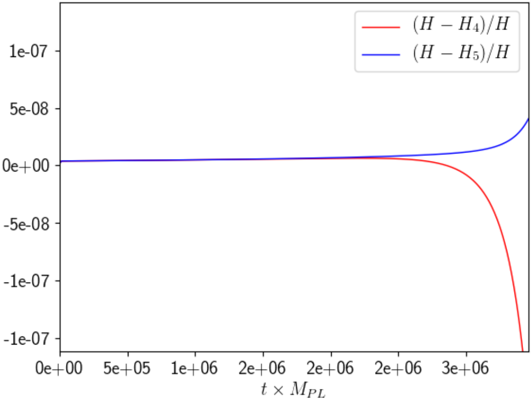
Convergence to the exact numeric solution



Convergence to the exact numeric solution



Convergence to the exact numeric solution



Conclusions

- In this work, a simple extension of the order reduction technique as an iterative method of solution of higher order differential equations was presented for some simple examples.
- Both situations, with or without a source, are analyzed. The order reduction presents a good agreement in the strong coupling regimes, non-oscillating which slowly approaches equilibrium, at least when applied to these examples. While also considering these specific examples, in the in the oscillating regime of a weak coupling limit, the ORT is inapplicable.
- The cases with external sources fall into the class of problems mentioned in the introduction. It is possible to control the external frequency to be much smaller than the natural frequency of the system and order reduction converges to the expected solution.

Conclusions

- As is well known, the order reduced equations present fewer solutions. This was one intention of the ORT to make it easier to select the ones that are physically relevant[11].
- This present work agrees with this reasoning. For all solutions analyzed, the perturbative order reduction in its convergence region approaches the physical solutions.
- Also, there could be physical solutions that will not be detected by order reduction.

The main results discussed are based on the work[12].

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[12] W. P. F. De Medeiros and D. Müller. On the order reduction. Eur. Phys. J. C 81 231 (2021). arXiv:2009.00629.

Thanks for your attention!