# Tidal deformations of neutron stars with elastic phases and implications

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# Strategy

- Motivations
- Bird's-eye view on tidal deformations of compact stars
- Hybrid stars
- Elasticity of compact hybrid stars
- Some interesting results
- Conclusions

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# Why going beyond one-phase perfect fluid stars?

- Matter should be elastic at certain densities (from, e.g., the cooling of neutron stars and nuclear physics).
- GWs from NSs offer observables (e.g., tidal deformations) to probe stars' aspects.
- NSs could actually be hybrid stars (GW170817 moderately favours it [Essick et al. (2019)]) and GWs could be used to probe aspects of quark matter (e.g., phase transition order, color-superconducting phases, etc).
- Many degrees of freedom are triggered in a hybrid star with elastic phases and one should understand their impact on observables.
- When many GW observations are available, it will be possible to better constrain NS models, and one should be aware of systematic effects due to modeling.

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# Gravitational waves from NSs

- Far away from the emitting system, the scalar GW waveform (GW strain) can be written as $h(t) = \mathcal{A}(t)\exp[i\phi(t)]$.
- In the quasi-static (adiabatic) scenario, the Fourier transform of $h(t)$ is

$$\tilde{h}(f) = \tilde{\mathcal{A}}(f)\exp[i\Psi(f)]; \frac{d^2\Psi}{d\omega^2} = \frac{2}{\dot{E}}\frac{dE}{d\omega}; f = \omega/\pi. \quad (1)$$

- To find $E$ and $\dot{E}$, one works with Post-Newtonian (PN) approximations, or the Effective One Body formalism.
- Tidal interactions enter at the 5th PN order [Flanagan and Hinderer (2008)],

$$\delta\Psi = -\frac{9}{16}\frac{(\pi f M)^{5/3}}{\mu M^4}\left[\left(11\frac{m_2}{m_1} + \frac{M}{m_1}\right)\lambda + 1 \leftrightarrow 2\right], \quad (2)$$

$M$: total mass ($m_1 + m_2$), $\mu$: reduced mass, $\lambda$: tidal deformation.

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# Classical picture

- For a binary system, tidal forces could induce a quadrupolar deformation on a star. Qualitatively:

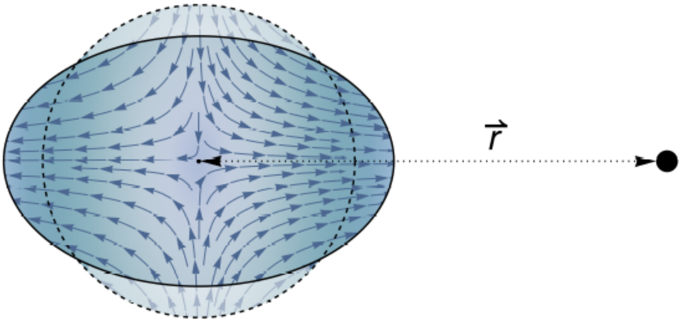


Figure: From Abdelsalhin (2019, PhD Thesis).

# Formalism

- In the weak field regime [Thorne (1998)],

$$-\frac{1+g_{tt}}{2} = -\frac{m}{r} - \frac{3}{2r^3}\mathcal{Q}_{ij}\left(\frac{x^i x^j}{r^2} - \frac{1}{3}\delta^{ij}\right) + \frac{1}{2}\mathcal{E}_{ij}x^i x^j + ..., \qquad (3)$$

$\mathcal{Q}_{ij}$: quadrupole moment. Plus,

$$\mathcal{E}_{ij} = \frac{\partial^2 \Phi_{ext}}{\partial x^i \partial x^j}, \qquad (4)$$

$\Phi_{ext}$: external grav. potential on the binary companion. Classically,

$$\mathcal{Q}_{ij} = \int d^3x \rho(\vec{x}, t)\left[x_i x_j - \frac{1}{3}r^2\delta_{ij}\right]. \qquad (5)$$

- In addition, $\mathcal{Q}_{ij} = -\lambda\mathcal{E}_{ij}$.

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# Quasi-static perturbations and tidal deformations

- In the early-inspiral phase of a binary system, the stars are far away so that the tidal influence one has on another can be treated as small, quasi-static perturbations.
- In this case, metric perturbations ($h_{ab}$) are assumed to be

$$h_{ab} = \text{diag}[H_0(r)e^{\nu(r)}, H_2(r)e^{\lambda(r)}, r^2k(r), r^2\sin^2\theta k(r)]Y_l^m(\theta,\phi); \tag{6}$$

we take for the metric of the background spacetime

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7}$$

- For a perfect-fluid star, one can show that $H_2 = H_0$,

$$k'(r) = H_0' + H_0\nu' \tag{8}$$

and, most importantly [Hinderer (2008)],

- $$H_0'' + \mathcal{A}_1 H_0' + \mathcal{A}_0 H_0 = 0, \quad (9)$$

  with

  $$\mathcal{A}_0 = e^{\lambda}\left[-\frac{l(l+1)}{r^2} + 4\pi(\varepsilon + p)\frac{d\varepsilon}{dp} + 4\pi(5\varepsilon + 9p)\right] - (\nu')^2, \quad (10)$$

  $$\mathcal{A}_1 = \frac{2}{r} + \frac{1}{2}(\nu' + \lambda'). \quad (11)$$

- From the exterior solution, one could find the external $g_{tt}$ solution to a star. Comparison with the asymptotic form of this metric component and the assumption that $\mathcal{Q}_{ij} = -\lambda_2 \mathcal{E}_{ij}$ leads to $\lambda_2 = (2/3)R^5 k_2$ [$y \equiv H_0' R/H_0$, $R$: star's radius, $C$: compactness],

$$\begin{aligned} k_2 &= \frac{8}{5}C^5(1-2C)^2[2+2C(y-1)-y]\{2C[6-3y+3C(5y-8)] \\ &+ 4C^3[13-11y+C(3y-2)+2C^2(1+y)] + \\ &+ 3(1-2C)^2[2-y+2C(y-1)]\ln(1-2C)\}^{-1}. \end{aligned} \quad (12)$$

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# Neutron stars

- Neutron stars are superdense systems. They have radii of order 10 km and masses of the order of the Sun's.
- They present a very rich phenomenology: random bursts, light emission in practically all electromagnetic spectrum, glitches, quasi-periodic oscillations, beamed and quasi-thermal emission, very high magnetic fields, etc.
- They are split into several categories according to their properties: pulsars, magnetars, CCOs, etc.
- What are they made of?

# Neutron star constitution

- Not totally known.

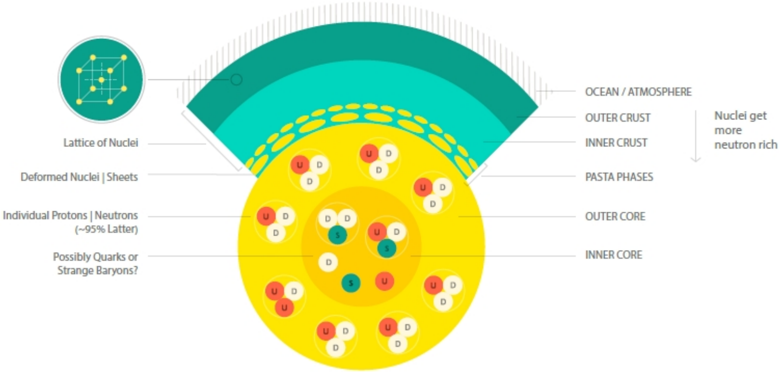


Figure: From Watts (2017). Not to scale.

# Hybrid stars

- Phase transitions could take place in neutron stars (NSs):

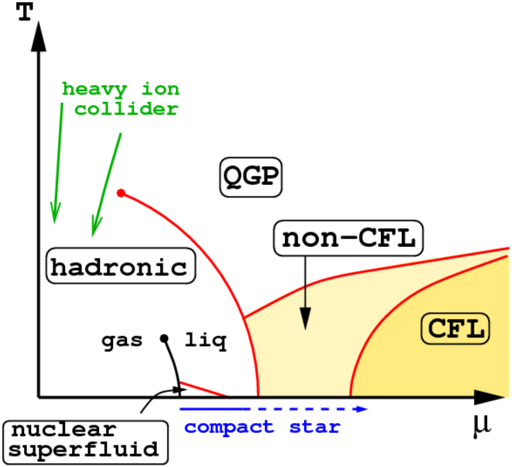


Figure: From Alford (2009).

# Hybrid stars II

- How to “construct” a hybrid star’s equation of state?

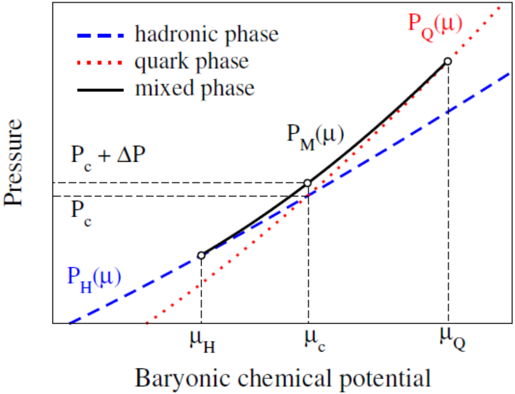


Figure: Schematic EOSs for a hybrid star. From Blaschke et al. 2018.

- Example 1: Just one phase transition from hadronic to quark matter at a given transition pressure ($p_{trans}$) and with a given energy density jump ($\Delta\varepsilon$):

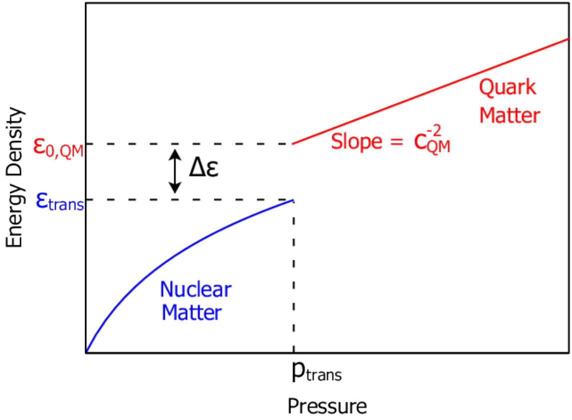


Figure: From Han and Steiner (2019).

- $M - R$ relations for hybrid stars with energy jumps.

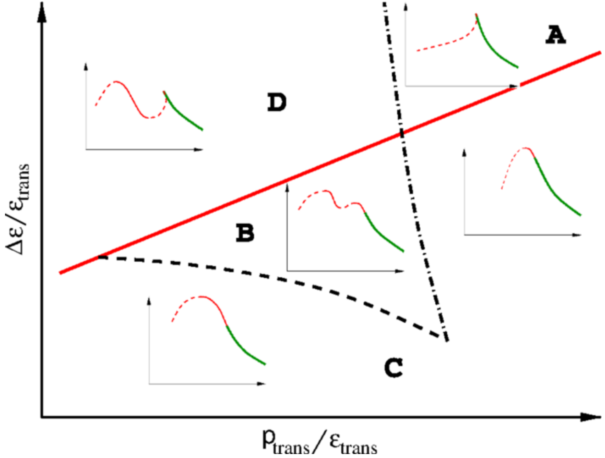


Figure: Topologies of hybrid stars. From Alford et al. (2013).

# cont.II

- Example 2: two phase transitions. One from hadrons to quarks and a sequential quark phase transition.

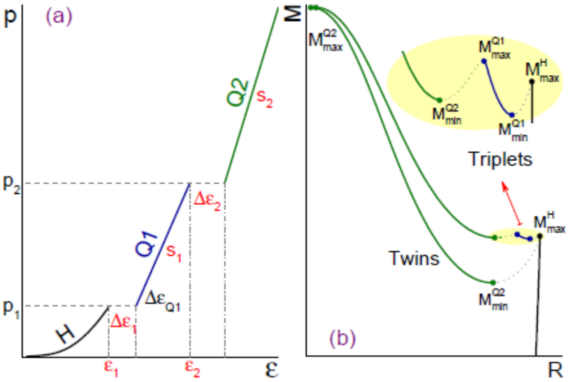


Figure: Sequential phase transitions. From Li, Sedrakian and Alford (2019).

# Tidal deformations of perfect-fluid hybrid stars

- Phase transitions basically decrease tidal deformations.
- Universal relations are weakened. In some cases departures could be as large as $10\%$ [Sieniawska et al. (2018), Han and Steiner (2019)].
- It would be possible to probe hybrid stars just when there will be measurements of tidal deformations of stars with similar masses.
- In addition, lower limits to tidal deformations are much stricter for purely hadronic stars. So, they could be used to distinguish a hybrid star from a one-phase one [Most et al. (2018)].
- However, currently NICER seems to be the best hope for probing hybrid stars [Ozel et al. (2016)].
- In principle, NICER is not able to constrain elastic effects in any form of stars.

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# Elastic properties of compact stars

- It is known that part of the outer crust of a neutron star should arrange as a lattice. In this case, shear stresses should appear. The shear modulus of matter should be around $1\%$ of the pressure [Chamel and Haensel (2007)].
- Although this is small, it could already lead to many phenomena (e.g., models for QPOs in magnetars, elastic modes, etc).
- However, it is possible that stars have mixed and/or crystalline quark phases, where their shear moduli would be up to a thousand times larger than the outer crust!
- All the above naturally motivates studies on the impact of elasticity onto GW observables, such as tidal deformations.

# Formalism of elasticity in compact stars

- The classical theory of elasticity has been extended to general relativity by Carter and Quintana (1972).
- When the background spacetime is static (and unstrained) and perturbations are of first-order, strain (shear) corrections to the energy-momentum tensor can be found through geometric relations [Penner et al. (2011)].
- More specifically,

$$\delta T_a^b = \delta(T_a^b)_{\text{perf.}} + \delta\Pi_a^b, \tag{13}$$

($\Delta_L$: the Lagrangian perturbation) [Penner et al. (2011)]

$$\delta\Pi_a^b = -\tilde{\mu}\left(\mathcal{P}_a^c\mathcal{P}^{db} - \frac{1}{3}\mathcal{P}_a^b\mathcal{P}^{cd}\right)\Delta_L g_{cd}, \tag{14}$$

with $\tilde{\mu}$ the shear modulus and $\mathcal{P}_{ab} \equiv g_{ab} + u_a u_b$ the projector onto the orthogonal direction of the four-velocity $u^a$.

# cont.

- Fluid perturbations are assumed to be given by

$$\xi^r = \frac{W(r)}{r} Y_l^m; \;\; \xi^\theta = \frac{V(r)}{r^2}\frac{dY_l^m}{d\theta}; \;\; \xi^\phi = \frac{V(r)}{r^2 \sin^2\theta}\frac{dY_l^m}{d\phi}. \tag{15}$$

- Tidal deformations are taken as static perturbations. This simplifies tremendously the problem since $\delta u^j = 0$ ($j = 1, 2, 3$) and

$$u_t = u_t^0 + \delta u_t = e^{\frac{\nu}{2}} - \frac{1}{2} e^{\frac{\nu}{2}} H_0 Y_l^m. \tag{16}$$

- Finally, from

$$\begin{aligned} \Delta_L g_{cd} &= h_{cd} + \xi_{c;d} + \xi_{d;c} \\ &= h_{cd} + \partial_c \xi_d + \partial_d \xi_c - 2\Gamma^a_{cd}\xi_a, \end{aligned} \tag{17}$$

one can easily obtain all $\delta\Pi^a_b$.

# cont. II

- From $\varepsilon = \varepsilon(n)$ [$n$: baryon number density] and the first law of thermodynamics, one has

$$\Delta_L \varepsilon = (p+\varepsilon)\frac{\Delta_L n}{n}. \quad (18)$$

- Complemented with the law of baryon number conservation, which implies that $\Delta_L n = -\frac{n}{2}\mathcal{P}^{ab}\Delta_L g_{ab}$, one has all ingredients for deriving the field equations.
- For instance, a generic result from Einstein equations in the present case is that

$$H_2(r) = H_0(r) + 32\pi\tilde{\mu}V(r). \quad (19)$$

- The case of stars with sharp phase transitions is basically a eigenvalue problem, and hence appropriate jump conditions should be given.

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# Boundary conditions for sharp phase transitions

- Assuming the absence of surface degrees of freedom, it follows that at the interface of two phases [Finn (1990)]:
  - $[\delta\Pi^r_r + \Delta_L p]^+_- = 0$ (continuity of the radial traction);
  - $[\delta\Pi^r_\theta]^+_- = 0$ (continuity of the transverse traction);
  - $\left[H_0' + \frac{1}{2}H_2\nu' - \xi^r\nu'' + \frac{1}{2}\xi^r\nu'\lambda'\right]^+_- = 0$ (continuity of the 2nd fundamental form);
  - $\left[rH_2 - r^2k' + r\xi^r\lambda'\right]^+_- = 0$ (continuity of the 2nd fundamental form).

- One has coupled ODEs to solve fulfilling the above conditions at each interface. They are highly restrictive.

- For instance, one learns that $H_0'$ has a non-zero jump whenever $\Delta\varepsilon \neq 0$ or elastic phases are in touch with perfect fluids.

- Hence, in general, the jump of $H_0'$ could influence $\lambda_2$ through $y(R) = H_0'(R)R/H_0(R)$.

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# Stars with elastic crusts

- Penner et al. (2011) were the first to find the consequences of an elastic crust on tidal deformations.
- By using a phenomenological model ($\tilde{\mu} = \kappa p + \mu_0$), they have shown that tidal deformations of elastic stars change negligibly with respect to perfect fluid ones.
- Caveats:
  - Their analysis has been limited to stars without phase transitions.
  - Some boundary conditions there have not been calculated precisely.
- There is more to the story...

# Compact stars with elastic cores

- In a hybrid star, it is also possible for the quark core to be crystalline. This is known as the LOFF phase. It leads to $\tilde{\mu}_{core}/\tilde{\mu}_{crust} \sim 1000$ [Mannarelli et al. (2007)].
- If such a phase exists, in principle it could be probed by LIGO because it could lead to large stellar deformations [Haskell et al. (2007)].
- Lin et al (2017) have been the first ones to calculate the effects of a solid quark core on tidal deformations. For strange quark stars, they have shown that $(\lambda_{fluid} - \lambda_{elast})/\lambda_{fluid} \lesssim 0.6$.
- When stars are wrapped up by a hadronic phase, the above change would decrease but could still be large for several cases.

- Lin has found that:

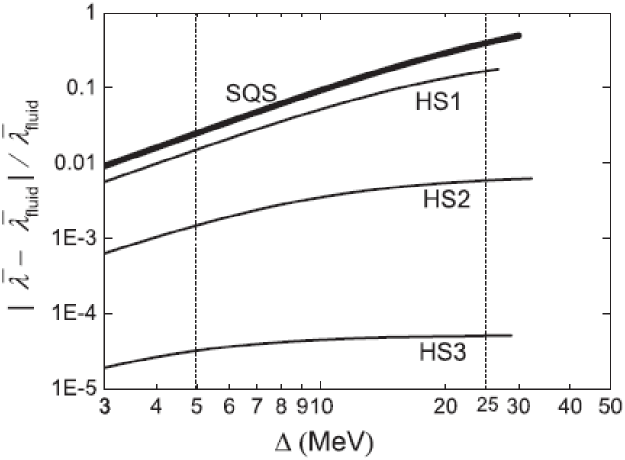



Figure: From Lin et al. (2019).

# New results [JPP et al. (2020)]

- For hybrid stars with sharp phase transitions and elastic crusts, tidal deformations are weakly dependent on the crust-core energy jumps. Relative changes (w.r.t. perfect fluid stars) are up to some percent ($\sim 5\%$).
- Maximum changes are expected for a star with an elastic region around $60\%$ of its radius (mixed phase is a natural candidate) .
- Amazingly enough, general relativity does not allow induced interface degrees of freedom for quasi-adiabatic perturbations.
- Future GW analysis based on missions such as the Cosmic Explorer and the Einstein telescope should not ignore hybrid stars with elastic regions.

# Models investigated

- We assume a hybrid star with a quark core and hadronic external layers; they are separated by a sharp interface.
- The models on which we focus are

| Hybrid Model | Quark Model $\left(\frac{B^{\frac{1}{4}}}{\text{MeV}}, a_4, \frac{a_2^{\frac{1}{2}}}{\text{MeV}}\right)$ | $\eta$ $\left(\frac{\epsilon_q}{\epsilon_h} - 1\right)$ | Hadronic Model |
|---|---|---|---|
| HS1 | $(137, 0.40, 100)$ | free | Polyt. (n=1) |
| HS2 | $(140, 0.55, 100)$ | $\approx 0.45$ | BPS+NL3 |
| HS3 | Bag with $c_s^2 = 1$ | free | SLy4+Polyt. ($\gamma = 4.5$) |

Table: Main aspects of the hybrid star models used in our analysis.

- We basically work with soft and stiff EOSs:

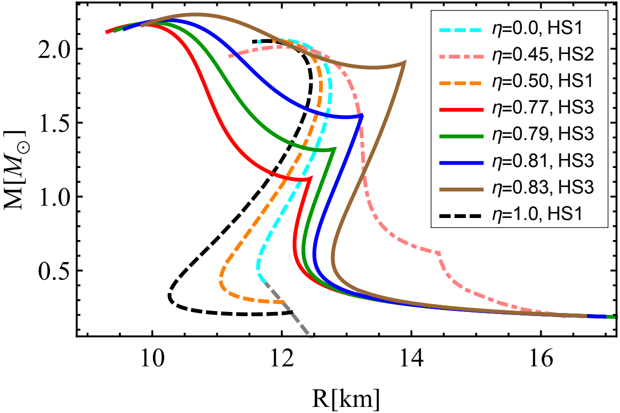



Figure: M-R relations.

# Density jumps and tidal deformations

- Density jumps affect more pronouncedly tidal deformations only when the elastic crust directly touches the quark phase.

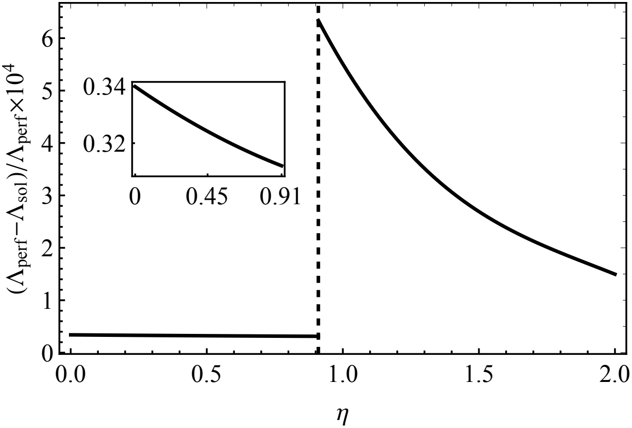
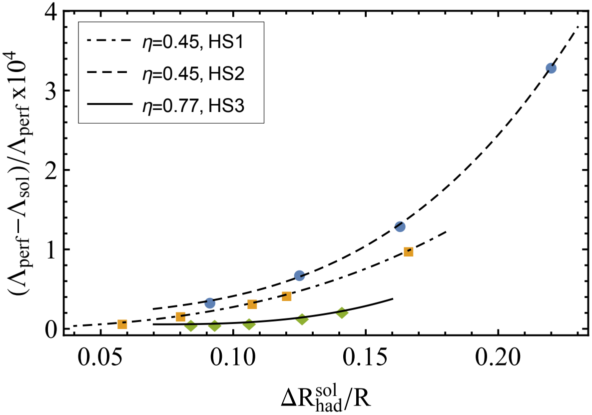



Figure: Tidal deformations of 1.4 $M_{\odot}$ solid hybrid stars as a function of the density jump ($\eta$) for the HS1 EOS.

# Tidal deformations and crust thickness

- Fractional tidal deformation changes of elastic hybrid stars (w.r.t. perfect fluid stars) are negligible if their elastic parts are small:




- Maximum changes ($\sim 5\%$) occur when the elastic region of a hybrid star is larger than approximately $60\%$ of the star's radius.

# No induced surface degrees of freedom

- In general, a glue of two spacetimes induces the following energy-momentum tensor $S^a_b$ at the hypersurface splitting them ($K^a_b$: extrinsic curvature and $K = K^a_a = K^{ab}$) [Poisson (2004, book)]:

$$S^a_b = -\frac{1}{8\pi}[K^a_b - \delta^a_b K]^+_-, \tag{20}$$

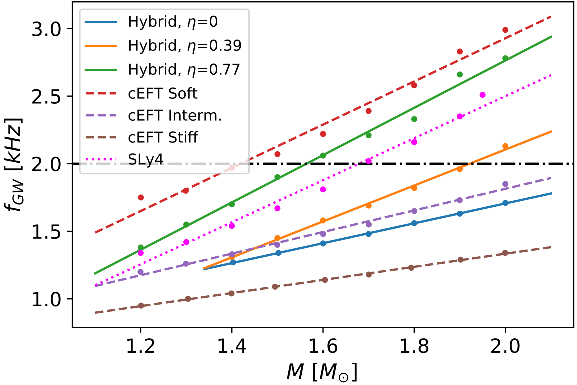
- For spherically symmetric backgrounds,

$$S_{ab} = (\sigma + \mathcal{P})u_a u_b + \mathcal{P}\bar{h}_{ab}. \tag{21}$$

- From perturbation equations in elastic phases, one can show that $\mathcal{P} = 0$; since one should have in general $\mathcal{P} = \mathcal{P}(\sigma) \to \sigma = 0$.
- In the (i) non-adiabatic case, (ii) when degrees of freedom are present in the background or (iii) outside general relativity, this issue is open.

# Crust yielding

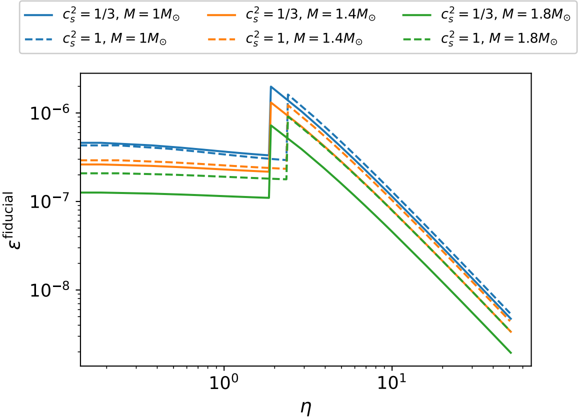
- There's a critical GW frequency above which tidal deformations can strain the crusts of hybrid stars beyond repair:




- So, for not too soft EOSs, precursors could exist.

# Ellipticities

- Ellipticity of hybrid stars:




- Thus, energy density jumps may be constrained with ellipticity observations.

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# Some conclusions and outlook

- Elasticity might change tidal deformations up to the level of detections just when the elastic phase is considerable.
- It can happen when the quark phase is small and the density jump is large enough, or even when a hybrid star has a mixed phase or the quark phase is not a perfect fluid.
- Tidal deformations might also probe these aspects of hybrid stars.
- In the future, when precision for tidal deformations is higher or there are more observations, the elasticity of stars and their possible phase transition aspects might not be ignored due to their relevance on at least systematic uncertainties.
- Tidal deformations in the late inspiral phase are also important to constrain dense matter, such as their phase transition properties.
- Tests of general relativity could also be conceived through tidal deformations.
- Precursors/ellipticity may reveal hybrid stars' aspects.