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Classical and quantum cosmology in the Brans-Dicke theory

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Contents

Motivation

General Relativity theory

- The Einstein–Hilbert action

$$S = \int \left(\frac{R}{16\pi G} + L_m \right) \sqrt{-g} d^4x. \tag{1}$$

- The Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2}$$

- Confirmed by experiments in the Solar System.
- Confirmed by the emission of gravitational waves by binary systems and it is in accordance with the bounds on the velocity of gravitational waves.

Why study scalar-tensor theories

- The necessity of the dark sector to explain cosmological observational data within GR's framework.
- The initial singularity before the Big Bang.
- The theoretical motivation to unify GR and QM in a single theoretical framework.

Brans-Dicke theory

Brans-Dicke action

- Introduced by Brans and Dicke in 1961 (based on the work of Pascual Jordan 1959).
- The B-D action

$$S = \frac{1}{16\pi} \int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\nu\phi\right) + \int d^4x\sqrt{-g}S_m, \qquad (3)$$

where ϕ is a scalar field, ω is the scalar field coupling constant and S_m is the matter term.

Coupling constant ω

- It is commonly understood that in the $\omega \to \infty$ limit BD theory becomes identical to GR.
- When $\omega \gg 1$, the field equations seem to show that

$$\phi = \frac{1}{G_N} + \mathcal{O}\left(\frac{1}{\omega}\right). \tag{4}$$

- There are some examples where exact solutions cannot be continuously deformed into the corresponding a GR solutions by taking the $|\omega| \to \infty$ limit:

$$\phi = \frac{1}{G_N} + \mathcal{O}\left(\frac{1}{\sqrt{\omega}}\right). \tag{5}$$

Brans-Dicke field equations

- Variation with respect to the metric

$$G_{\mu\nu} = \frac{1}{\phi}\left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu}\Box\phi\right) + \frac{\omega}{\phi^2}\left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi\right) + \frac{8\pi}{\phi} T_{\mu\nu}. \quad (6)$$

- Variation with respect to the scalar field

$$\Box\phi = \frac{8\pi}{3+2\omega} T. \quad (7)$$

- Energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0. \quad (8)$$

Brans-Dicke field equations

- The matter content described by a perfect fluid $T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} - pg^{\mu\nu}$ and equation of state $p = \alpha\rho$ with $-1 \leq \alpha \leq 1$.
- The field equations in the FLRW universe with spatial section curvature

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi\frac{\rho}{\phi} - 3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2, \tag{9}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)}\left(\rho - 3p\right). \tag{10}$$

- The continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0. \tag{11}$$

Conformal transformation

- A conformal transformation on the metric

$$g_{\mu\nu} = \phi^{-1}\tilde{g}_{\mu\nu} \tag{12}$$

- The action in the Einstein frame

$$\mathcal{S} = \int d^4x\sqrt{-\tilde{g}}\left\{\tilde{R} - \epsilon\sigma_{;\mu}\sigma^{;\mu}\right\} + \int d^4x\sqrt{-\tilde{g}}\mathcal{S}_m, \tag{13}$$

where

$$\epsilon = \text{sign}\left(\omega + \frac{3}{2}\right) \tag{14}$$

and σ is a new scalar field defined as

$$\sigma = \sqrt{\left|\omega + \frac{3}{2}\right|}\ln\left(\frac{\phi}{\phi_0}\right). \tag{15}$$

Regular cosmological solutions in BD theory

Models with bounce

- Replace an initial singularity with a bounce - a smooth transition from contraction to expansion - in order to solve fundamental problems in cosmology (the horizon and flatness problems).
- The conditions to have a bounce are

$$\ddot{a}(t) > 0, \tag{16}$$

$$\dot{a}(t_0) = 0. \tag{17}$$

Models with bounce

- To obtain a bouncing solution in GR: violation of the null energy condition (NEC) $\rho + p \geq 0$ is required (usually).
- Consequence: the appearance of exotic kind of matter fields, for example, a scalar field with negative energy density (ghosts).
- The crucial point in bouncing models is to construct a regular model in which such ghosts are absent while still having a bouncing phase.

Gurevich’s solutions

- Gurevich, Finkelstein and Ruban obtained a class of flat space solutions for the equation of state $p = \alpha\rho$, where $0 \leq \alpha < 1$.

ON THE PROBLEM OF THE INITIAL STATE IN THE ISOTROPIC SCALAR-TENSOR COSMOLOGY OF BRANS-DICKE

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First family of solutions

- The first family of solutions $\omega < -\frac{3}{2}$

$$a = a_0 \left[(\theta + \theta_-)^2 + \theta_+^2\right]^{\sigma/2A} e^{\pm\sqrt{\frac{2}{3}|\omega|-1}\,\frac{1}{A}\,\arctan\left(\frac{\theta+\theta_-}{\theta_+}\right)}, \tag{18}$$

$$\phi = \phi_0 \left[(\theta + \theta_-)^2 + \theta_+^2\right]^{(1-3\alpha)/2A} e^{\mp 3(1-\alpha)\sqrt{\frac{2}{3}|\omega|-1}\,f(\theta)}, \tag{19}$$

  where θ is the parameterized time : $dt = a^{3\alpha} d\theta$.
- This model admits a cosmological bounce: when $\theta \to \infty$, a does not vanish. The infinite contraction occurs till a regular minimum a_{min} and after it is followed by the expansion.

Second family of solutions

- The second family of solutions $\omega > -\frac{3}{2}$

$$a = a_0\,(\theta-\theta_+)^{\omega/3(\sigma\mp\sqrt{1+\frac{2}{3}\omega})}\,(\theta-\theta_-)^{\omega/3(\sigma\pm\sqrt{1+\frac{2}{3}\omega})}\,, \tag{20}$$

$$\phi = \phi_0\,(\theta-\theta_+)^{\left(1\mp\sqrt{1+\frac{2}{3}\omega}\right)/(\sigma\mp\sqrt{1+\frac{2}{3}\omega})}\,(\theta-\theta_-)^{\left(1\pm\sqrt{1+\frac{2}{3}\omega}\right)/(\sigma\pm\sqrt{1+\frac{2}{3}\omega})}\,. \tag{21}$$

- Regular bounce: $\frac{1}{4} < \alpha < 1$ and $-\frac{3}{2} < \omega \leq -\frac{4}{3}$.
- The energy conditions for the scale factor are satisfied.

Universe filled with radiative fluid

- Equation of state: $p = \frac{1}{3}\rho$.
- $\omega > -\frac{3}{2}$:

$$a(\eta) = a_0(\eta-\eta_+)^{\frac{1}{2}\pm\frac{1}{2\sqrt{1+\frac{2}{3}\omega}}}(\eta-\eta_-)^{\frac{1}{2}\mp\frac{1}{2\sqrt{1+\frac{2}{3}\omega}}}, \quad (22)$$

$$\phi(\eta) = \phi_0(\eta-\eta_+)^{\mp\frac{1}{\sqrt{1+\frac{2}{3}\omega}}}(\eta-\eta_-)^{\pm\frac{1}{\sqrt{1+\frac{2}{3}\omega}}}. \quad (23)$$

- $\omega < -\frac{3}{2}$:

$$a(\eta) = a_0\left[(\eta+\eta_-)^2+\eta_+^2\right]^{\frac{1}{2}} e^{\pm\frac{1}{\sqrt{\frac{2}{3}|\omega|-1}}\arctan\frac{\eta+\eta_-}{\eta_+}}, \quad (24)$$

$$\phi(\eta) = \phi_0 e^{\mp\frac{2}{\sqrt{\frac{2}{3}|\omega|-1}}\arctan\frac{\eta+\eta_-}{\eta_+}}. \quad (25)$$

Energy conditions

- The strong and null energy conditions in General Relativity are given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0, \tag{26}$$

$$-2\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right) = 8\pi G(\rho + p) > 0. \tag{27}$$

- In the Einstein frame: both energy conditions are satisfied as far as $\omega < -\frac{3}{2}$.
- This is consistent with the fact that in the Einstein frame the cosmological scenarios are singular unless $\omega < -\frac{3}{2}$.
- In the Jordan frame: there are non singular models if $-\frac{3}{2} < \omega < -\frac{4}{3}$. In this range the scalar field obeys the energy condition.

Canonical quantization of the BD theory

Quantum formulation

- The construction of a quantum cosmological model encounters many problems.
- Absence of an explicit time coordinate due to the invariance by time reparametrizations in the classical theory.
- Way to solve: to allow the matter fields to play the role of time, which can be achieved through Schutz's description of a fluid.
- Choice of the suitable formalism to interpret the quantum theory and thus obtain specific predictions.

Hamiltonian formulation

- Quantize the Hamiltonian constraint $\mathcal{H}_{\text{tot}} \approx 0$ to obtain the Wheeler-DeWitt Equation $\hat{H}_{\text{tot}}\Psi = 0$.
- Gravitational Lagrangian

$$\mathcal{L}_G = \frac{1}{N}\left[6\left(\phi a\dot{a}^2 + a^2\dot{a}\dot{\phi}\right) - \omega a^3\frac{\dot{\phi}^2}{\phi}\right]. \tag{28}$$

- $\mathcal{L}_G$ as a function of the conjugated momenta $\pi_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ and a Legendre transformation:

$$\mathcal{H}_G = \frac{N}{(3+2\omega)}\left(\frac{\omega}{12\phi a}\pi_a^2 + \frac{1}{2a}\pi_a\pi_\phi - \frac{\phi}{2a^3}\pi_\phi^2\right). \tag{29}$$

Total Hamiltonian

- Total Hamiltonian $\mathcal{H}_{tot} = \mathcal{H}_G + \mathcal{H}_M$.
- Radiative fluid $p = \frac{1}{3}\rho$, Schutz's formalism:

$$\mathcal{H}_M = \frac{N}{a}\pi_T, \tag{30}$$

where T is directly related to the entropy of the fluid.

-

$$\mathcal{H} = N\left\{\frac{1}{(3+2\omega)}\left[\frac{\omega}{12\phi a}\pi_a^2 + \frac{1}{2a}\pi_a\pi_\phi - \frac{\phi}{2a^3}\pi_\phi^2\right] - \frac{1}{a}\pi_T\right\}. \tag{31}$$

Canonical quantization

- $\pi_k \to -i\partial_k$ (Jordan's frame)

$$\partial_a^2\Psi + \frac{p}{a}\partial_a\Psi + \frac{6}{\omega}\frac{\phi^2}{a^2}\left\{\frac{a}{\phi}\partial_a\partial_\phi - \left(\partial_\phi^2\Psi + \frac{q}{\phi}\partial_\phi\Psi\right)\right\} = -12i\frac{(3+2\omega)}{\omega}\phi\partial_T\Psi, \tag{32}$$

where p, q are ordering factors.

- Einstein's frame

$$\partial_b^2\Psi + \frac{\bar{p}}{b}\partial_b\Psi - \tilde{\omega}\frac{\phi^2}{b^2}\left\{\partial_\phi^2\Psi + \frac{\bar{q}}{\phi}\partial_\phi\Psi\right\} = -i\partial_T\Psi\,. \tag{33}$$

- Equations (32) and (33) are Schrödinger-like:

$$\hat{H}\Psi = i\frac{\partial}{\partial t}\Psi\,, \tag{34}$$

if we consider the matter field playing the role of time.

Solution

- Einstein's frame:

$$a = e^{-\frac{\sigma}{\sqrt{|1+\frac{2}{3}\omega|}}} b. \tag{35}$$

- The Schrödinger equation in terms of the σ

$$\partial_b^2 \Psi + \frac{1}{b}\partial_b \Psi - \epsilon \frac{1}{b^2}\partial_\sigma^2 \Psi = -i\partial_T \Psi. \tag{36}$$

- The regular solution is

$$\Psi(b, \sigma) = A(k, E) J_\nu(\sqrt{E} b) e^{i(k\sigma - ET)}, \quad \nu = \sqrt{-\epsilon}|k|, \tag{37}$$

where k is a separation constant.

Analysis via the de Broglie-Bohm approach

Many worlds & dBB interpretations

- The many-world interpretation: every state of the wave function is real, existing in parallel with each other (our universe is one of many). Does not require the collapse of the wave function, it is possible to investigate different states separately from the wave packet.
- The de Broglie-Bohm (dBB) interpretation: the wave function is a guide to the possible evolution of the universe $\Rightarrow$ it is observer-independent.
- Bohmian mechanics:

$$\Psi(x_i, t) = R(x_i, t)e^{iS(x_i,t)}. \tag{38}$$

- Probability density of the trajectory: $\rho(x_i, t) = |\Psi(x_i, t)| = R^2(x_i, t)$.
- Bohmian trajectories:

$$p_j = \partial_j S = \left(\frac{i}{2}\right) \frac{\Psi\Psi^*{}_{,j} - \Psi^*\Psi_{,j}}{|\Psi|^2}. \tag{39}$$

Wave packet

- General wave packet

$$\Psi(b,\sigma,T) = \int_0^{\infty}\int_{-\infty}^{+\infty} A(k)x^{\nu+1}e^{-(\gamma+iT)x^2}J_\nu(xb)e^{ik\sigma}dk\,dx, \quad (40)$$

with the definition $x=\sqrt{E}$ and $\nu=|k|$.

- Integration in x

$$\Psi(b,\sigma,T) = \int_{-\infty}^{+\infty} A(k)\frac{b^\nu}{(\alpha)^{\nu+1}}e^{-\frac{b^2}{4\alpha}}e^{ik\sigma}dk. \quad (41)$$

The scalar field is absent - Many worlds

- $A(k) = \delta(k)$ (the world where $\nu = 0$), the contribution of the scalar field vanishes:

$$\Psi(b, \sigma, T) = \frac{e^{-\frac{b^2}{4\alpha}}}{\alpha}. \tag{42}$$

- Expected value of the scale factor:

$$< b >= \frac{1}{\gamma \mathbf{N}^2}\sqrt{\gamma^2 + T^2}, \tag{43}$$

where **N** is the normalization factor of the wave function.

- In this world a bounce occurs when $T \gg \gamma$, $\langle b \rangle \to T$.

The scalar field is absent - dBB

- Phase of the wave function:

$$S = T\frac{b^2}{4|\alpha|^2} - \arctan\left(\frac{T}{\gamma}\right). \tag{44}$$

- Bohmian trajectories (conjugate momenta $p_b = \dot{b}/2$):

$$\dot{b} = \frac{bT}{|\alpha|^2}. \tag{45}$$

- Solution:

$$b = b_0\sqrt{\gamma^2 + T^2}. \tag{46}$$

- This universe also has a bounce.

A single scalar mode

- Superposition function: $A(k) = \delta(k - k_0)$.
- Wave function:

$$\Psi(b, \sigma, T) = \frac{b^{\nu_0}}{(\alpha)^{\nu_0+1}} e^{-\frac{b^2}{4\alpha}} e^{ik_0\sigma}, \tag{47}$$

  with $\nu_0 = \sqrt{-\epsilon} k_0$.
- We will analyze the Bohmian scenario for $\epsilon = -1$ and $\epsilon = 1$.

A single scalar mode, $\epsilon = -1$

- The phase of the wave function:

$$S = \frac{b^2}{4|\alpha|^2}T - (k+1)\arctan\left(\frac{T}{\gamma}\right) + k_0\sigma. \tag{48}$$

- Solutions:

$$b = b_0\sqrt{\gamma^2 + T^2}, \tag{49}$$

$$\sigma = \sigma_0 \arctan\left(\frac{T}{\gamma}\right). \tag{50}$$

- Jordan frame:

$$a \propto \exp\left[-\frac{\arctan\left(\frac{T}{\gamma}\right)}{\sqrt{|1+\frac{2}{3}\omega|}}\right]\sqrt{\gamma^2 + T^2}. \tag{51}$$

- The classical solution is recovered asymptotically, but the bounce in this case is asymmetric.

A single scalar mode, $\epsilon = 1$

- We cannot compute the evolution of the scale factor by evaluating the expectation values because the wave function is not finite (energy spectrum is not bounded from below).
- New feature: $\nu = ik$.
- Solutions:

$$b = b_0\sqrt{\gamma^2 + T^2}\sqrt{\sigma_0 - \bar{k}_0 \arctan\left(\frac{T}{\gamma}\right)}, \quad (52)$$

$$\sigma = 3\ln\left\{\sigma_0 - \bar{k}_0 \arctan\left(\frac{T}{\gamma}\right)\right\}, \quad (53)$$

where b_0 and σ_0 are constants and $\bar{k}_0 = 36k_0/(b_0^2\gamma)$.

- Solutions are non-singular only if $\sigma_0 > \frac{\pi}{2}\bar{k}_0$ and $\sigma_0 > 0$.

A single scalar mode, $\epsilon = 1$

- Jordan frame:

$$a = a_0\sqrt{\gamma^2 + T^2}\left\{\sigma_0 + \bar{k}_0 \arctan\left(\frac{T}{\gamma}\right)\right\}^r, \tag{54}$$

with

$$r = \frac{1}{2}\frac{\sqrt{1+\frac{2}{3}\omega}+6}{\sqrt{1+\frac{2}{3}\omega}}. \tag{55}$$

- The classical solution is recovered asymptotically.

Multiple scalar modes

- Combines different scalar modes:

$$A(k) = \delta(k - k_0) + \eta\delta(k + k_0), \tag{56}$$

  with $\eta = \pm 1$.
- The scalar field is not present in the phase of the wave function, and we recover the same solutions already given in the case the scalar field is absent.

Conclusions

Classical scenario

- It is possible to obtain a singularity-free cosmological solution if the Brans-Dicke parameter ω varies as $-3/2 < \omega < -4/3$.
- In this range, the energy conditions are satisfied in the Einstein frame: the avoidance of the singularity is driven by the non-minimal coupling.

Conclusions

Quantum scenario

- The energy is bounded only for $\omega < -3/2$ (when the energy condition is violated in the Einstein frame).
- For $\omega > -3/2$ the energy conditions are satisfied, but the energy is not bounded from below, it becomes problematic to employ the usual interpretation scheme based on the Copenhagen formulation of quantum mechanics: the wave function is not finite anymore.
- de Broglie-Bohm: for $\omega > -3/2$ we have either a singular or non-singular solution. This implies that in the interval $-\frac{3}{2} < \omega < -\frac{4}{3}$ the classical model displays singularity-free scenarios, while the quantum models may display either singular or non-singular solution.

Obrigada