Noncommutative supersymmetric gauge theories in 2D and 3D

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Summary

- Introduction.
- UV/IR mixing and the problem of nonintegrable singularities.
- Supersymmetric extension as a tool to improve the situation.
- Two- and three-dimensional NC supergauge theories and finiteness achieved.
- Conclusion.

This talk is based on hep-th/0309193, 0407064, 0410257, 0511059, arXiv: 0708.1002, 1112.2105.

Introduction

A fundamental relation for a noncommutative (NC) theory (Snyder, 1947):

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \quad (1)$$

Motivations: string theory (Seiberg, Witten, hep-th/9908142), space-time foam (Hawking, Ellis).

Moyal product in a noncommutative space:

$$\begin{aligned} f(\hat{x}) &= \int \frac{d^n k}{(2\pi)^n} e^{ik^\mu \hat{x}_\mu} \tilde{f}(k); \quad (2) \\ f_1(\hat{x}) f_2(\hat{x}) &= \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \tilde{f}_1(k_1) \tilde{f}_2(k_2) e^{ik_1^\mu \hat{x}_\mu} e^{ik_2^\mu \hat{x}_\mu} = \\ &= \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \tilde{f}_1(k_1) \tilde{f}_2(k_2) e^{i(k_1^\mu + k_2^\mu)\hat{x}_\mu - \frac{i}{2} k_1^\mu k_2^\nu \theta_{\mu\nu}}. \end{aligned}$$

Rule for replacement of usual product by Moyal one:

$$\int d^n x f_1(\hat{x}) f_2(\hat{x}) \dots f_N(\hat{x}) \to \tag{3}$$

$$\to \int d^n x \int \frac{d^n k_1 \dots d^n k_N}{(2\pi)^{nN}} \tilde{f}_1(k_1) \tilde{f}_2(k_2) \dots \tilde{f}_N(k_n) \times$$

$$\times \quad e^{i\left((k_1^\mu + k_2^\mu + \dots + k_N^\mu) x_\mu - \frac{1}{2} \sum_{i<j} \theta_{\mu\nu} k_i^\mu k_j^\nu\right)}$$

Other forms for noncommutativity:

(1) Fermionic non(anti)commutativity (in SUSY case) – $\{\theta^\alpha, \theta^\beta\} = \Sigma^{\alpha\beta}$ (hep-th/0305248, 0607087, arXiv: 1406.5418).

(2) Dynamical NC: $\theta^{\mu\nu}$ are treated as extra coordinates (hep-th/0312080).

(3) Kontsevich product (q-alg/9709040).

The coherent states approach describes nonlocality rather than NC.

UV/IR mixing

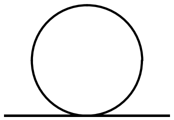
Let us consider NC $\lambda\phi^4$ theory:

$$S = \int d^4x \Big(-\frac{1}{2}\phi(\Box - m^2)\phi - \frac{\lambda}{4!}\phi * \phi * \phi * \phi \Big). \tag{4}$$

The vertex:

$$\begin{aligned} V &= \frac{\lambda}{3 \cdot 4!} \int \frac{d^4p_1 \ldots d^4p_4}{(2\pi)^{16}} (2\pi)^4 \delta(p_1 + \ldots + p_4)\phi(p_1)\phi(p_2) \times \\ &\times \phi(p_3)\phi(p_4)[\cos(p_1 \wedge p_2)\cos(p_3 \wedge p_4) \\ &+ \cos(p_1 \wedge p_3)\cos(p_2 \wedge p_4) + \cos(p_1 \wedge p_4)\cos(p_3 \wedge p_2)]. \end{aligned} \tag{5}$$

The lower contribution:



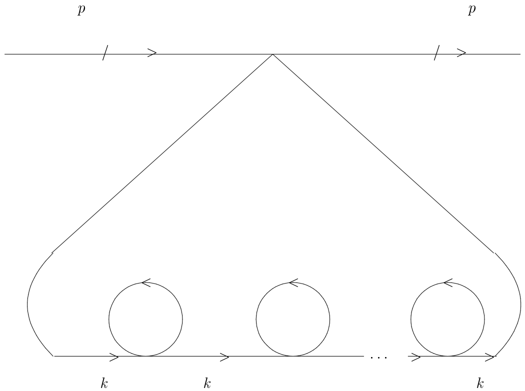
i.e. $\Gamma_1^{(2)} = -\frac{\lambda}{6} \int \frac{d^4k}{(2\pi)^4} \frac{2+\cos(2k\wedge p)}{k^2+m^2}$.

$$\Gamma_1^{(2)pl} = \frac{\lambda}{48\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2}\right) \quad (6)$$

$$\Gamma_1^{(2)np}|_{m\to 0} = \frac{\lambda}{96\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-\frac{\tilde{p}^2}{\alpha}} = \frac{\lambda}{96\pi^2\tilde{p}^2}, \quad (7)$$

where $\tilde{p}^2 = p^\mu\theta_{\mu\nu}\theta^{\nu\lambda}p_\lambda$ – example of the UV/IR mixing mechanism (*hep-th/9912072*).

Main difficulty: presence of all orders in $\frac{1}{\tilde{p}^2}$ breaks perturbative expansion! Indeed, sum of these graphs yields $\sum_{n=0}^{\infty}\frac{1}{(p^2\tilde{p}^2)^n}$.



Supersymmetric extension as a tool to improve the situation

Let us consider a very simplified model

$$L=-\frac{1}{2}\phi(\Box+m^2)\phi+\frac{\lambda}{4!}\phi^4+\bar{\Psi}(i\partial\!\!\!/-m+h\phi)\Psi. \qquad (8)$$

Two contributions to the two-point function:

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Each fermionic loop carries a (-1) factor.

So, these contributions are

$$\begin{aligned} I_a &= \frac{\lambda}{2}\int\frac{d^4p}{(2\pi)^4}\phi(-p)\phi(p)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}; \qquad (9)\\ I_b &= -h^2\int\frac{d^4p}{(2\pi)^4}\phi(-p)\phi(p)\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}\frac{(k\!\!\!/+m)(k\!\!\!/+p\!\!\!/+m)}{k^2-m^2}, \end{aligned}$$

or

$$I_a = \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\phi(-p)\phi(p)\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}; \quad (10)$$

$$I_b = -4h^2\int \frac{d^4p}{(2\pi)^4}\phi(-p)\phi(p)\int \frac{d^4k}{(2\pi)^4}\frac{(k^2+m^2)}{(k^2-m^2)^2},$$

The quadratic divergences vanish in the sum $I_a + I_b$, for $\lambda = 8h^2$.

A natural continuation: if we have Moyal factors in vertices, the cancellations also must occur! – verified for the Wess-Zumino model, see hep-th/0005272.

Moreover, SUSY kills dangerous quadratic and linear divergences.

Moyal product does not affect Grassmannian coordinates and thus does not affect SUSY transformations in the superfield form! So, we can apply it to superfields!

Superspace in 2D and 3D

In $2D$ and $3D$, structure of superspace is the same.

Grassmannian coordinates: θ^α with $\alpha = 1, 2$, and $\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}$.

The spinors are transformed under spinor representation of $SO(1,1)$ or $SO(1,2)$.

The Dirac matrices: $(\gamma^0)^\alpha{}_\beta = -i\sigma^2, (\gamma^1)^\alpha{}_\beta = \sigma^1, (\gamma^2)^\alpha{}_\beta = \sigma^3$, $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$, and $V^m \to V^{\alpha\beta} = V^m(\gamma_m)^{\alpha\beta}$, for **any** V^m, with $V_{\alpha\beta}$ (two indices below) is symmetric.

For $D = 2$, we keep γ^0 and γ^1, and γ^2 plays the role of γ_5.

SUSY generators:

$$Q_\alpha = i\partial_\alpha + \theta^\beta \partial_{\beta\alpha}, \tag{11}$$

Anticommutation relation:

$$\{Q_\alpha, Q_\beta\} = 2i\partial_{\alpha\beta}. \tag{12}$$

Supercovariant derivatives:

$$D_\alpha = \partial_\alpha + i\theta^\beta \partial_{\beta\alpha}. \tag{13}$$

Their key relation:

$$D_\alpha D_\beta = i\partial_{\alpha\beta} - C_{\alpha\beta} D^2, \tag{14}$$

with $C_{\alpha\beta} = -i\epsilon_{\alpha\beta}$.
One has $\{D_\alpha, D_\beta\} = 2i\partial_{\alpha\beta}$.
It is clear that in $2D$ and $3D$ the convergence is better than in 4D.

Noncommutative superfield gauge theories in 2D and 3D

The $A_\alpha = i\theta^\beta V_{\beta\alpha} + \ldots$ is the basic gauge superfield, $V_{\beta\alpha}$ is a bispinor corresponding to the usual gauge vector V_m.

We have two following terms (in $2D$, we only change $d^5 z$ by $d^4 z$):

$$S_{YM} = \frac{1}{2g^2} \mathrm{tr} \int d^5 z W^\alpha * W_\alpha. \quad (15)$$

and

$$\begin{aligned} S_{CS} &= \frac{m}{2g^2} \mathrm{tr} \int d^5 z (A^\alpha * W_\alpha + \frac{i}{6}\{A^\alpha, A^\beta\} * D_\beta A_\alpha + \\ &+ \frac{1}{12}\{A^\alpha, A^\beta\} * \{A_\alpha, A_\beta\}), \quad (16) \end{aligned}$$

where $W_\alpha = \frac{1}{2} D^\beta D_\alpha A_\beta - \frac{i}{2}[A^\beta, D_\beta A_\alpha] - \frac{1}{6}[A^\beta, \{A_\beta, A_\alpha\}]$. We can have also $S_{MCS} = S_{YM} + S_{CS}$ (in $2D$, a CS term is trivial).

The nontrivial gauge group is implemented through $A^\alpha \to A^{\alpha a} T^a$.

Matter action (adjoint representation):

$$S_m = \mathrm{tr} \int d^5 z (D^\alpha \Phi + i[\Phi, A^\alpha]) * (D_\alpha \bar{\Phi} - i[A_\alpha, \bar{\Phi}]). \qquad (17)$$

In the fundamental representation case we have

$$S_m = \mathrm{tr} \int d^5 z (D^\alpha \Phi + i\Phi * A^\alpha) * (D_\alpha \bar{\Phi} - iA_\alpha * \bar{\Phi}). \qquad (18)$$

One can show that in the last case, in one loop all phase factors are cancel out, and the results replay the commutative case (see f.e. hep-th/0612223).

Ghosts action:

$$S_{gh} = \frac{1}{2g^2}\mathrm{tr}\int d^5 z c' * D^\alpha (D_\alpha c + i\{A_\alpha, c\}). \tag{19}$$

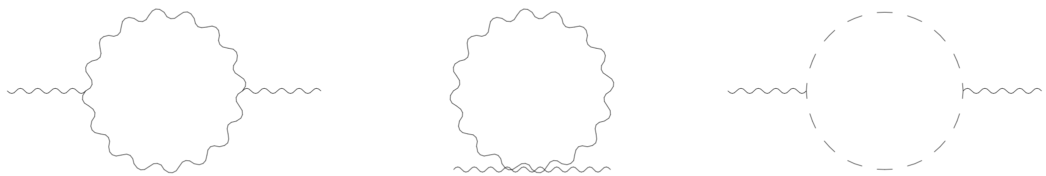
The superficial degree of divergence in $3D$ (V_i carries i derivatives):

$$\begin{aligned} \omega_{YM} &= 2 - \frac{1}{2}V_c - 2V_0 - \frac{3}{2}V_1 - V_2 - \frac{1}{2}(V_3 + V_c) - \\ &- \frac{1}{2}E_\phi - \frac{1}{2}V_\phi^D - V_\phi^0; \\ \omega_{CS} &= 2 - \frac{E_A + E_\phi}{2} \end{aligned} \tag{20}$$

In $2D$, ω_{YM} is decreased by 1.

So, the NC CS theory is renormalizable, and NC Maxwell and MCS theories are super-renormalizable, with no divergences beyond two loops (in 3D) or beyond one loop (in 2D).

The only one-loop (linearly) divergent contributions in the purely gauge sector are:



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The divergent contributions coincide in all three theories. In $U(1)$ case we have (with $d = 2, 3$)

$$
\begin{aligned}
S_a(p) &= \frac{\xi}{2}\int d^2\theta A^\alpha(-p)A_\alpha(p)\int\frac{d^dk}{(2\pi)^d}\frac{\sin^2(k\wedge p)}{k^2};\\
S_b(p) &= \frac{1}{2}(1-\xi)\int d^2\theta A^\alpha(-p)A_\alpha(p)\int\frac{d^dk}{(2\pi)^d}\frac{\sin^2(k\wedge p)}{k^2};\\
S_c(p) &= -\frac{1}{2}\int d^2\theta A^\alpha(-p)A_\alpha(p)\int\frac{d^dk}{(2\pi)^d}\frac{\sin^2(k\wedge p)}{k^2}. \qquad (21)
\end{aligned}
$$

Their sum is zero.

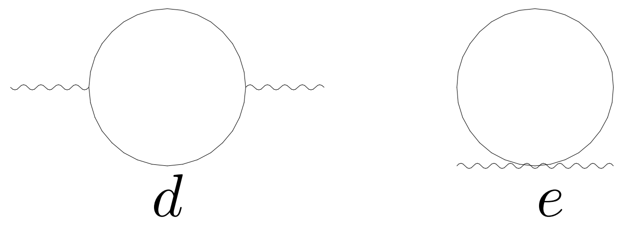
The 1-loop formally logarithmically divergent terms in $3D$ either vanish by symmetric integration or are proportional to

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^\mu \sin(2k \wedge p)}{k^4} = -\frac{i}{4\pi} \frac{\tilde{p}^\mu}{\sqrt{\tilde{p}^2}}. \tag{22}$$

The only more divergences in $3D$ (for Maxwell and MCS) are two-loop ones. They are proportional to $A^\alpha A_\alpha$, but this term is forbidden by the gauge symmetry therefore it is natural to expect that this divergence vanishes at least at one special gauge **which however can be highly unusual – for example, in the commutative QED with scalar matter the two-loop finiteness is achieved for $\xi = -8$, see 0709.3501**.

In $2D$ there is no divergences beyond one loop, so, the theory is clearly all-loop finite.

In the matter sector we have only divergent graphs:



The sum of these graphs is finite:

$$iS_3(p) = \int d^2\theta f(p)(W_0^\alpha(-p)W_{0\alpha}(p) + 2mW_0^\alpha(p)A_\alpha(p));$$

$$f(p) = \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2(k\wedge p)}{(k^2+m^2)[(k+p)^2+m^2]}, \qquad (23)$$

where $W_{0\alpha}$ is a linear in A_α part of W_α, and

$$f(p) = \frac{1}{16\pi}\int_0^1 dx \frac{1-e^{-M\sqrt{\tilde{p}^2}}}{M}, \qquad M = \sqrt{m^2+p^2x(1-x)}. \quad (24)$$

Now let us go from $U(1)$ to more generic group. Repeating all calculations we find that the leading UV (and as well UV/IR infrared) divergences vanish in all theories we consider if

$$\mathrm{tr}(T^bT^aT^bT^c) = 2\mathrm{tr}(T^bT^dT^a)\mathrm{tr}(T^bT^dT^c), \quad (25)$$

which is satisfied in the fundamental representation of $U(N)$ where the generators are normalized to satisfy the relation
$\frac{1}{2}(T^a)_{ij}(T^a)_{kl} = \delta_{il}\delta_{jk}$.
So, not only the gauge group but also the representation is restricted!
We note that this quantum restriction on the representation is consistent with the classical restrictions.

Indeed, just noncommutative $u(N)$ algebra is consistent at the classical level since it closes in the noncommutative case (hep-th/0107037).

While $su(N)$ by definition is composed by zero-trace matrices, its noncommutative extension is not closed.

Let us consider $\mathrm{tr}(A * B - B * A)$:

$$\begin{aligned} \mathrm{tr}[A, B]_* &= \mathrm{tr}(AB - BA) + \mathrm{tr}\,\theta^{mn}(\partial_m A \partial_n B - \partial_m B \partial_n A) + \ldots = \\ &= \theta^{mn}\mathrm{tr}\,\{\partial_m A, \partial_n B\} + \ldots . \end{aligned} \tag{26}$$

In general it is not zero.

The same situation occurs for all algebras purely composed by traceless matrices like $SO(N)$, while the groups $O(N)$ and $Usp(2N)$ (and other non-traceless ones) are acceptable.

One can proceed as well with the finite part of two-point contributions. For the CS theory, with $\mathrm{tr}(T^a T^b) = \kappa \delta^{ab}$, one has from the gauge sector a Maxwell-like contribution

$$\Gamma^{(1)}_{gauge} = \frac{1}{4\kappa^2}(\xi^2 - 1) \int d^2\theta \frac{d^3p}{(2\pi)^3} \frac{1}{2} W_0^{\alpha a}(p) W^b_{0\alpha}(-p) I^{ab}(p), \quad (27)$$

where $I^{ab}(p)$ is defined from expressions:

$$\int \frac{d^3k}{(2\pi)^3} \frac{F^{ab}(k,p)}{k^2(k-p)^2} = I^{ab}(p); \quad (28)$$

$$F^{ab} = A^{cad}A^{dbc} - A^{cad}A^{cbd}\cos(2k \wedge p); \qquad A^{abc} = \mathrm{tr}(T^a T^b T^c).$$

For the matter sector, one has the non-Abelian analogue of (23).
We see that Maxwell and CS terms arise as quantum corrections.

SUSY NC CP^{N-1} model

This example is interesting in the context of the emergent dynamics.

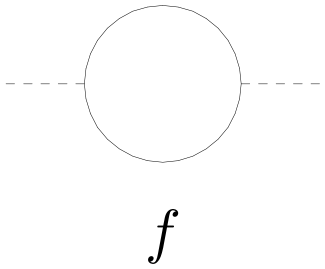
So, we will give some its review.

We start with the action

$$S_m = \sum_{i=1}^{N} \mathrm{tr} \int d^5z \Big((D^\alpha \Phi_i + i[\Phi_i, A^\alpha]) * (D_\alpha \bar{\Phi}_i - i[A_\alpha, \bar{\Phi}_i]) + m\bar{\Phi}_i \Phi_i + \eta * [\Phi_i, \bar{\Phi}_i] \Big). \qquad (29)$$

The dynamics of the gauge superfield A_α will emerge. The η is a Lagrange multiplier.

We have the two-point function of the η given by



Hence the propagator of η is

$$< \eta(-p,\theta_1)\eta(p,\theta_2) >= -\frac{1}{N}\frac{D^2-2m}{4f(p)(p^2+4m^2)}\delta_{12}, \qquad (30)$$

where $f(p)$ is given by (23).

The quadratic action of the gauge field A_α is given by the (23) whose r.h.s. is multiplied by N since we have N fields. And the propagator of A_α (under an appropriate gauge fixing) is

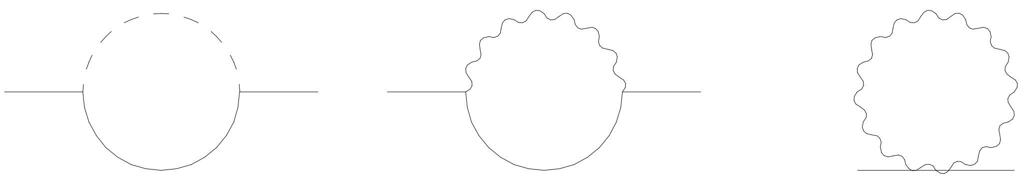
$$< A^\alpha(-p,\theta_1)A^\beta(p,\theta_2) >= \frac{1}{4Nf(p)}\times$$
$$\times \quad \left[\frac{(D^2-2m)D^\beta D^\alpha}{p^2(p^2+4m^2)}+\xi\frac{D^2D^\alpha D^\beta}{p^4}\right]\delta_{12}. \qquad (31)$$

Since $f(p)|_{p\to\infty}\sim\frac{1}{\sqrt{p^2}}$, we see that the asymptotics of this propagator is worse than that one in QED ($\sim p^{-1}$ instead of p^{-2}).

As a result, the theory is only renormalizable rather than super-renormalizable. One has

$$\omega_{CS} \quad = \quad 2 - \frac{E_A + E_\phi}{2} - E_\eta. \tag{32}$$

The graphs with external matter legs are



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In the UV limit, their sum is

$$S_2 = -\frac{4(1+\xi)}{N\pi^2\epsilon} \int d^5 z \bar{\phi}_a (D^2 - m) \phi_a + \text{fin}, \tag{33}$$

so, the divergence vanishes in a certain gauge.

Conclusions

1. The SUSY gauge theories in $2D, 3D$ are free of nonintegrable infrared divergences.

2. $2D$ NC QED (and SYM) is proved to be finite in all loop orders and $3D$ NC QED (and SYM and MCS) is almost proved to be finite at all loop orders since the only possible (two-loop) divergence is forbidden by gauge invariance.

3. Classical restrictions on the gauge group are maintained at the quantum level. The conditions on the gauge group are the same for all theories (SYM, CS and MCS).

Remaining problems:

a. Explicit calculation of the two-loop contribution to the two-point function of A^α in $3D$.

b. Study of the NC gauge theories with extended supersymmetry (especially $N = 6, 8$ super-Chern-Simons theory) in superfield approach and explicit proof of all-loop finiteness.