Linear Cosmological Perturbation Theory II

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Relativistic Cosmology

1+3 Covariant Spacetime "Threading"

The time direction

Introduce "fundamental' observers with 4-velocity

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}\,, \qquad \text{with} \qquad u^a u_a = -1\,. \tag{1}$$

Define the (proper) time derivative

$$\dot{T} = u^a \nabla_a T\,. \tag{2}$$

The 3-D space

Introduce the "projector"

$$h_{ab} = g_{ab} + u_a u_b\,, \qquad \text{with} \qquad h_{ab}u^b = 0\,, \quad h_a{}^a = 3 \quad \text{and} \quad h_{ac}h^c{}_b = h_{ab}\,. \tag{3}$$

Define the spatial derivative

$$\mathrm{D}_a T = h_a{}^b \nabla_b T\,, \qquad \mathrm{D}_b T_a = h_b{}^d h_a{}^c \nabla_d T_c\,, \qquad \text{etc}\,. \tag{4}$$

1+3 spacetime splitting

Use u_a and h_{ab} to decompose every variable, every operator and every equation into their timelike and spacelike parts.

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Use u_a and h_{ab} to decompose every variable, every operator and every equation into their timelike and spacelike parts.

Covariant Kinematics

The motion of the u_a-congruence

Decompose the 4-velocity gradient $\nabla_b u_a$ as

$$\nabla_b u_a = \mathrm{D}_b u_a - A_a u_b \,, \tag{5}$$

where $\mathrm{D}_b u_a = h_d{}^d h_a{}^c \nabla_d u_c$ and $A_a = \dot{u}_a = u^b \nabla_b u_a$. Also,

$$\mathrm{D}_b u_a = \frac{1}{3}\,\Theta h_{ab} + \sigma_{ab} + \omega_{ab} \,. \tag{6}$$

The kinematic variables

- $\Theta = \nabla_a u^a = \mathrm{D}^a u_a$ is the volume scalar. Describes expansion or contraction.
- $\sigma_{ab} = \mathrm{D}_{\langle b} u_{a\rangle}$ is the shear tensor. Describes shape changes.
- $\omega_{ab} = \mathrm{D}_{[b} u_{a]}$ is the vorticity tensor. Describes rotation. Note that

$$\omega_a = \frac{1}{2}\,\varepsilon_{abc}\omega^{bc} \,, \qquad \text{with} \qquad \varepsilon_{abc} = \eta_{abcd} u^d \tag{7}$$

  is the vorticity vector and defines the rotational axis.
- $A_a = \dot{u}_a$ is the 4-acceleration vector. Describes (non gravitational) forces.

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Gravitational Field

The Riemann tensor (total gravitational field)

$$R_{abcd} = C_{abcd} + \frac{1}{2}\left(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\right) - \frac{1}{6}Rg_{abcd}\,, \tag{8}$$

where

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}\,, \qquad R_{abcd} = R_{cdab} \qquad \text{and} \qquad R_{abcd} = R_{[ab][cd]}\,.$$

The Ricci tensor (local gravity)

$$R_{ab} = R^{c}{}_{acb}\,, \qquad \text{with} \qquad R_{ab} = R_{ba} \qquad \text{and} \qquad R = R^{a}{}_{a}\,. \tag{9}$$

The Weyl tensor (long-range gravity)

$$C_{abcd} = \left(g_{abqp}g_{cdsr} - \eta_{abqp}\eta_{cdsr}\right)u^{q}u^{s}E^{pr} - \left(\eta_{abqp}g_{cdsr} + g_{abqp}\eta_{cdsr}\right)u^{q}u^{s}H^{pr}\,, \tag{10}$$

with

$$E_{ab} = C_{acbd}u^{c}u^{d}\,, \qquad H_{ab} = \varepsilon_{a}{}^{cd} \qquad \text{and} \qquad C_{cdbe}u^{e}/2C^{c}{}_{acb} = 0\,.$$

The Einstein Field Equations

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab} - \Lambda g_{ab}\,. \tag{11}$$

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Matter Fields (1)

Imperfect fluid

$$T_{ab} = \rho u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \,. \tag{12}$$

where

$$\rho = T_{ab}u^a u^b \,, \qquad p = T_{ab}h^{ab}/3 \,, \qquad q_a = -h_a{}^b T_{bc}u^c \qquad and \qquad \pi_{ab} = h_{\langle a}{}^c h_{b\rangle}{}^d T_{cd} \,.$$

Perfect fluid

$$T_{ab} = \rho u_a u_b + p h_{ab} \,, \tag{13}$$

with $p = 0$ for dust, $p = \rho/3$ for radiation and $p = -\rho$ for a (slow-rolling) scalar field.

Decomposing the EFE

Projecting the EFE along and orthogonal to u_a gives

$$R_{ab}u^a u^b = \frac{1}{2}(\rho + 3p) - \Lambda \qquad (total\ gravitational\ mass) \,, \tag{14}$$

$$h_a{}^b R_{bc}u^c = -q_a \,, \qquad h_a{}^c h_b{}^d R_{cd} = \frac{1}{2}(\rho - p)h_{ab} + \Lambda h_{ab} + \pi_{ab} \,. \tag{15}$$

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Matter Fields (2)

Electromagnetic field

$$T^{(EM)}_{ab} = -F_{ac}F^{c}{}_{b} - \frac{1}{4}F_{cd}F^{cd}g_{ab}\,, \qquad \textit{with} \qquad F_{ab} = 2u_{[a}E_{b]} + \varepsilon_{abc}B^{c}\,. \tag{16}$$

Fluid description of the Maxwell field

$$T^{(EM)}_{ab} = \frac{1}{2}(E^2 + B^2)u_a u_b + \frac{1}{6}(E^2 + B^2)h_{ab} + 2\mathcal{Q}_{(a}u_{b)} + \Pi_{ab}\,, \tag{17}$$

where

$$\mathcal{Q}_a = \varepsilon_{abc}E^b B^c \qquad \textit{and} \qquad \Pi_{ab} = \frac{1}{3}\left(E^2 + B^2\right)h_{ab} - E_a E_b - B_a B_b\,.$$

Scalar field

$$T^{(\varphi)}_{ab} = \nabla_a\varphi\nabla_b\varphi - \left[\frac{1}{2}\nabla_c\varphi\nabla^c\varphi + V\right]g_{ab}\,, \qquad \textit{with} \qquad V = V(\varphi)\,. \tag{18}$$

Fluid description for the φ-field

$$T^{(\varphi)}_{ab} = \rho^{(\varphi)}u_a u_b + p^{(\varphi)}h_{ab}\,, \tag{19}$$

where

$$u_a = -\frac{1}{\dot\varphi}\nabla_a\varphi\,, \qquad \rho^{(\varphi)} = \frac{1}{2}\dot\varphi^2 + V(\varphi) \qquad \textit{and} \qquad p^{(\varphi)} = \frac{1}{2}\dot\varphi^2 - V(\varphi)\,.$$

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Conservation Laws

Energy-momentum conservation

$$\textit{Bianchi Identities} \Rightarrow \nabla^b G_{ab} = 0 \Rightarrow \nabla^b T_{ab} = 0\,. \tag{20}$$

Energy density conservation

For an imperfect fluid

$$\dot{\rho} = -\Theta(\rho + p) - \mathrm{D}^a q_a - 2A^a q_a - \sigma^{ab}\pi_{ab}\,. \tag{21}$$

For a perfect fluid

$$\dot{\rho} = -\Theta(\rho + p)\,. \tag{22}$$

Momentum density conservation

For an imperfect fluid

$$(\rho + p)A_a = -\mathrm{D}_a p - \dot{q}_a - \frac{4}{3}\Theta q_a - (\sigma_{ab} + \omega_{ab})q^b - \mathrm{D}^b\pi_{ab} - \pi_{ab}A^b\,. \tag{23}$$

For a perfect fluid

$$(\rho + p)A_a = -\mathrm{D}_a p \qquad (\rho + p : \textit{ total inertial mass})\,. \tag{24}$$

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Kinematic Evolution

Ricci identities

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d\,. \tag{25}$$

Propagation equations (timelike part of (25))

$$\begin{aligned}\dot{\Theta} + \frac{1}{3}\Theta^2 = & - R_{ab}u^a u^b - 2(\sigma^2 - \omega^2) + \mathrm{D}^a A_a + A_a A^a \\ & - \left[\frac{1}{2}(\rho + 3p) - \Lambda\right] - 2(\sigma^2 - \omega^2) + \mathrm{D}^a A_a + A_a A^a\,,\end{aligned} \tag{26}$$

$$\dot{\sigma}_{\langle ab\rangle} = -\frac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma^c{}_{b\rangle} - \omega_{\langle a}\omega_{b\rangle} + \mathrm{D}_{\langle a}A_{b\rangle} + A_{\langle a}A_{b\rangle} - E_{ab} + \frac{1}{2}\pi_{ab}\,, \tag{27}$$

$$\dot{\omega}_{\langle a\rangle} = -\frac{2}{3}\Theta\omega_a - \frac{1}{2}\mathrm{curl}A_a + \sigma_{ab}\omega^b\,, \quad \textit{with} \quad \mathrm{curl}A_a = \varepsilon_{abc}\mathrm{D}^b A^c\,. \tag{28}$$

Constrains (spacelike part of (25))

$$\mathrm{D}^b\sigma_{ab} = \frac{2}{3}\mathrm{D}_a\Theta + \mathrm{curl}\omega_a + 2\varepsilon_{abc}A^b\omega^c - q_a\,, \qquad \mathrm{D}^a\omega_a = A_a\omega^a \tag{29}$$

$$H_{ab} = \mathrm{curl}\sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} + 2A_{\langle a}\omega_{b\rangle}\,, \quad \textit{with} \quad \mathrm{curl}\sigma_{ab} = \varepsilon_{cd\langle a}\mathrm{D}^c\sigma^d{}_{b\rangle}\,. \tag{30}$$

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Weyl Curvature

Weyl tensor

Describes tidal forces and gravitational waves (GWs)

$$C_{ab}{}^{cd} = 4\left(u_{[a}u^{[c} + h_{[a}{}^{[c}\right)E_{b]}{}^{d]} + 2\varepsilon_{abe}u^{[c}H^{d]e} + 2u_{[a}H_{b]e}\varepsilon^{cde}\,. \tag{31}$$

where $E_{ab} = E_{\langle ab \rangle}$ and $H_{ab} = H_{\langle ab \rangle}$.

To isolate the GWs we impose the constrains

$$\mathrm{D}^b E_{ab} = 0 = \mathrm{D}^b H_{ab}\,. \tag{32}$$

"Field equations"

$$Bianchi \quad Identities \Rightarrow \nabla^d C_{abcd} = \nabla_{[b}R_{a]c} + \frac{1}{6}\,g_{c[b}\nabla_{a]}R\,. \tag{33}$$

Lead to propagation and constrain formulae that resemble Maxwell's equations.

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Spatial Curvature

The 3-Riemann tensor

By definition

$$\mathcal{R}_{abcd} = h_a{}^q h_b{}^s h_c{}^f h_d{}^p R_{qsfp} - v_{ac} v_{bd} + v_{ad} v_{bc}\,, \qquad \textit{where} \qquad v_{ab} = \mathrm{D}_b u_a\,. \tag{34}$$

Also,

$$\mathcal{R}_{abcd} = \mathcal{R}_{[ab][cd]}\,, \qquad \textit{but} \qquad R_{abcd} \neq R_{cdab} \qquad (\textit{when} \qquad \omega_a \neq 0)\,. \tag{35}$$

The 3-Ricci tensor

By definition

$$\mathcal{R}_{ab} = h^{cd} \mathcal{R}_{acbd} \qquad \textit{and} \qquad \mathcal{R} = h^{ab} \mathcal{R}_{ab}\,. \tag{36}$$

Then,

$$\mathcal{R}_{ab} = \frac{1}{3} \mathcal{R} h_{ab} + E_{ab} + \frac{1}{2} \pi_{ab} - \frac{1}{3} \Theta(\sigma_{ab} + \omega_{ab}) + \sigma_{c\langle a} \sigma^c{}_{b\rangle} - \omega_{c\langle a} \omega^c{}_{b\rangle} + 2\sigma_{c[a} \omega^c{}_{b]}\,, \tag{37}$$

which means that $\mathcal{R}_{ab} \neq \mathcal{R}_{ba}$ (when $\omega_a \neq 0$). Finally,

$$\mathcal{R} = 2\left(\rho - \frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 + \Lambda\right)\,, \qquad (\textit{generalised Friedmann equation})\,. \tag{38}$$

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Spatially Homogeneous Universes

The FRW Models (1)

The FRW symmetries

Spatial homogeneity and isotropy ensure that the only non-vanishing variables are

$$\rho = \rho(t)\,, \qquad p = p(t) \qquad and \qquad \Theta = \Theta(t) = 3H\,. \tag{39}$$

Moreover,

$$\mathcal{R}_{abcd} = \frac{K}{a^2}\left(h_{ac}h_{bd} - h_{ad}h_{bc}\right)\,, \qquad with \qquad K = 0, \pm 1 \qquad and \qquad \frac{\dot{a}}{a} = H\,. \tag{40}$$

The Friedmann equations

$$H^2 = \frac{1}{3}\rho - \frac{K}{a^2} + \frac{1}{3}\Lambda\,, \qquad since \qquad \mathcal{R} = \frac{6K}{a^2}\,. \tag{41}$$

$$\dot{H} = -H^2 - \frac{1}{6}(\rho + 3p) + \frac{1}{3}\Lambda\,. \tag{42}$$

Conservation laws

$$\dot{\rho} = -3H(\rho + p)\,. \tag{43}$$

Then, $\rho \propto a^{-3}$ when $p = 0$, $\rho \propto a^{-4}$ when $p = \rho/3$ and $\rho =$ constant when $p = -\rho$.

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$$\dot{H} = -H^2 - \frac{1}{6}\,(\rho + 3p) + \frac{1}{3}\,\Lambda\,. \tag{42}$$

Conservation laws

$$\dot{\rho} = -3H(\rho + p)\,. \tag{43}$$

Then, $\rho \propto a^{-3}$ when $p = 0$, $\rho \propto a^{-4}$ when $p = \rho/3$ and $\rho = \text{constant}$ when $p = -\rho$.

The FRW Models (2)

Time Evolution ($K = 0, \pm 1$)

$a \propto t^{2/3}$ for dust, $a \propto t^{1/2}$ for radiation, $a \propto e^t$ for a slow-rolling scalar field.
$a \propto \sin^2(\eta/2)$ for dust, $a \propto \sin\eta$ for radiation.
$a \propto \sinh^2(\eta/2)$ for dust, $a \propto \sinh\eta$ for radiation

The density parameter(s)

$$\Omega_\rho = \rho/3H^2 \qquad \Omega_K = -K/(aH)^2 \qquad \Omega_\Lambda = \Lambda/3H^2$$

Then,

$$\Omega_\rho + \Omega_K + \Omega_\Lambda = 1\,. \tag{44}$$

The deceleration parameter

$$q = -\ddot{a}a/\dot{a}^2 = -[1 - (\dot{H}/H^2)]$$

Then,

$$qH^2 = \frac{1}{6}(\rho + 3p) - \frac{1}{3}\Lambda\,. \tag{45}$$

Characteristic scales

Hubble horizon $\lambda_H = 1/H$ Curvature scale $\lambda_K = a$

Then,

$$\left(\frac{\lambda_K}{\lambda_H}\right)^2 = -\frac{K}{1-\Omega_\rho}\,, \qquad \textit{when} \quad \Lambda = 0\,. \tag{46}$$

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The Bianchi Models

Bianchi symmetries

Spatial homogeneity, but no isotropy, ensure that the additional nonzero values are

$$\pi_{ab} = \pi_{ab}(t)\,, \quad \sigma_{ab} = \sigma_{ab}(t)\,, \quad E_{ab} = E_{ab}(t) \quad \text{and} \quad H_{ab} = H_{ab}(t)\,. \tag{47}$$

The Bianchi Classification

Group class	Group type	FLRW as special case
A	I	$K = 0$
	II	–
	VI_0	–
	VII_0	$K = 0$
	$VIII$	–
	IX	$K = +1$
B	V	$K = -1$
	IV	–
	VI_h	–
	VII_h	$K = -1$

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The Bianchi I Universe

Basic features ($H_{ab}=0$, $K=0$)

$$H^2=\frac{1}{3}\left(\rho+\sigma^2\right)\,, \qquad \dot{H}=-H^2-\frac{1}{6}(\rho+3p)-\frac{2}{3}\sigma^2\,, \quad \textit{with} \quad \sigma^2=\sigma_{ab}\sigma^{ab}/2\,, \tag{48}$$

$$\dot{\rho}=-3H(\rho+p)-\sigma_{ab}\pi^{ab}\,, \tag{49}$$

$$\dot{\sigma}_{ab}=-3H\sigma_{ab}+\pi_{ab}\,, \qquad E_{ab}=H\sigma_{ab}-\sigma_{c\langle a}\sigma^{c}{}_{b\rangle}-\frac{1}{2}\pi_{ab}\,. \tag{50}$$

The Kasner limit (vacuum)

Shear dominates at early times

$$H^2=\frac{1}{3}\sigma^2 \quad \textit{and} \quad \dot{\sigma}_{ab}=-3H\sigma_{ab}\,. \tag{51}$$

Average evolution $a\propto t^{1/3}$, but the expansion is confined along one or two axes only.

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Inghomogeneous & Ansisotropic Cosmologies

The Gauge Problem

Gauge freedom & and the fitting problem

To study cosmological perturbations we need to establish an one-to-one mapping (a gauge)

$$\phi : \overline{\mathcal{W}} \rightarrow \mathcal{W}\,, \tag{52}$$

between the fictitious background spacetime ($\overline{\mathcal{W}}$) and a more realistic perturbed one ($\mathcal{W}$).

- Problem No 1: Results are generally gauge-dependent (even for scalars)
- Problem No 2: Selecting the best gauge is not always straight forward (fitting problem)

Gauge independence

Variables that remain invariant under gauge-transformations are (Stewart & Walker lema):

- Scalars that are constant in the background spacetime
- Tensors that vanish in the background spacetime
- Linear combinations of products of the Kronecker deltas with constant coefficients

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Description of Inhomogeneities

Density inhomogeneities

Define the comoving fractional density gradient

$$\Delta_a = \frac{a}{\rho}\mathrm{D}_a\rho = \frac{a}{\rho}h_a{}^b\nabla_b\rho\,. \tag{53}$$

- Δ_a measures spatial density variations between neighbouring observers
- Δ_a vanishes when the background is spatially homogeneous ($\mathrm{D}_a\rho \equiv 0$)

Isolating scalar, vector and (trace-free) tensor perturbations

- Δ_a contains collective information for all three types of density perturbations.
- To decode this information define the gradient $\Delta_{ab} = a\mathrm{D}_b\Delta_a$ and introduce the splitting

$$\Delta_{ab} = \frac{1}{3}\Delta h_{ab} + \Delta_{[ab]} + \Delta_{\langle ab\rangle}\,. \tag{54}$$

- Δ describes overdensities/underdensities ($\Delta \to \delta = \delta\rho/\rho$)
- $\Delta_{[ab]}$ describes density vortices (rotational distortions)
- $\Delta_{\langle ab\rangle}$ describes shape-distortions (under constant volume)

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Nonlinear Evolution of Inhomogeneities (Perfect Fluid)

Evolution of density inhomogeneities

$$\dot{\Delta}_{\langle a\rangle} = \frac{p}{\rho}\,\Theta\Delta_a - \left(1+\frac{p}{\rho}\right)\mathcal{Z}_a - \left(\sigma^b{}_a + \omega^b{}_a\right)\Delta_b\,, \tag{55}$$

where $\mathcal{Z}_a = a\mathrm{D}_a\Theta$ describes inhomogeneities in the expansion.

Evolution of expansion inhomogeneities

$$\begin{aligned}\dot{\mathcal{Z}}_{\langle a\rangle} \quad = \quad & -\frac{2}{3}\,\Theta\mathcal{Z}_a - \frac{1}{2}\,\rho\Delta_a - \frac{3}{2}\,a\mathrm{D}_a p - a\left[\frac{1}{3}\,\Theta^2 + \frac{1}{2}\,(\rho+3p) - \Lambda\right]A_a + a\mathrm{D}_a\mathrm{D}^b A_b \\ & - \left(\sigma^b{}_a + \omega^b{}_a\right)\mathcal{Z}_b - 2a\mathrm{D}_a\left(\sigma^2 - \omega^2\right) + 2aA^b\mathrm{D}_a A_b \\ & - a\left[2\left(\sigma^2 - \omega^2\right) - \mathrm{D}^b A_b - A^b A_b\right]A_a\,,\end{aligned} \tag{56}$$

with

$$(\rho+p)A_a = -\mathrm{D}_a p\,. \tag{57}$$

Barotropic fluid ($p = p(\rho)$)

$$\mathrm{D}_a p = c_s^2\mathrm{D}_a\rho = a^{-1}c_s^2\rho\Delta_a\,, \qquad \textit{with} \qquad c_s^2 = \dot{p}/\dot{\rho} \qquad (\textit{adiabatic sound speed}) \tag{58}$$

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Almost FRW Cosmologies

The Linear Regime

The linearisation scheme

- Quantities with nonzero background value are of zero perturbative order (ρ, p, H)
- Quantities that vanish in the background (and their gradients) are first order variables
- Terms of higher than the first order are dropped

Linear evolution of inhomogeneities ($K = 0$)

$$\dot{\Delta}_a = 3wH\Delta_a - (1+w)\mathcal{Z}_a \tag{59}$$

and

$$\dot{\mathcal{Z}}_a = -2H\mathcal{Z}_a - \frac{1}{2}\rho\Delta_a - \frac{c_s^2}{1+w}\mathrm{D}^2\Delta_a - 6ac_s^2 H \mathrm{curl}\,\omega_a\,, \tag{60}$$

with $w = p/\rho$ and

$$\dot{w} = -3H(1+w)(c_s^2 - w)\,. \tag{61}$$

Then,

$$\dot{w} = 0 \Leftrightarrow w = c_s^2\,, \qquad when \qquad w \neq -1\,. \tag{62}$$

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Linear Density Perturbations

The $w =$ constant case

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$$\dot{\mathcal{Z}} = -2H\mathcal{Z} - \frac{1}{2}\,\rho\Delta - \frac{c_s^2}{1+w}\mathrm{D}^2\Delta\,. \tag{64}$$

Then,

$$\ddot{\Delta} = -2\left(1 - \frac{3}{2}\,w\right)H\dot{\Delta} + \left[\frac{1}{2}\,\rho(1-w)(1+3w)\right]\Delta + c_s^2\mathrm{D}^2\Delta\,. \tag{65}$$

Harmonic splitting

Set $\Delta = \Delta_{(n)}\mathcal{Q}^{(n)}$, with $\mathrm{D}_a\Delta_{(n)} = 0 = \dot{\mathcal{Q}}^{(n)}$, $\mathrm{D}^2\mathcal{Q}^{(n)} = -(n/a)^2\mathcal{Q}^{(n)}$ and $n^2 \geq 0$.

Then,

$$\ddot{\Delta}^{(k)} = -2\left(1 - \frac{3}{2}\,w\right)H\dot{\Delta}^{(k)} + \left\{\frac{1}{2}\,[\rho(1-w)(1+3w)] - \frac{n^2c_s^2}{a^2}\right\}\Delta^{(k)}\,. \tag{66}$$

- The first term on the RHS describes an effective “friction” stress due to the expansion
- The second term on the RHS reflects the counteracting effects of gravity and pressure

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Growth of Linear Density Perturbations (1)

The Einstein-de Sitter universe ($w = 0 = c_s^2$)

$$\dot{\Delta} = -\mathcal{Z}\,, \qquad \dot{\mathcal{Z}} = -2H\mathcal{Z} - \frac{1}{2}\rho\Delta \tag{67}$$

and

$$\ddot{\Delta} = -2H\dot{\Delta} + \frac{1}{2}\rho\Delta\,. \tag{68}$$

Linear growth of Δ

When $H = 2/3t$ and $\rho = 4/3t^2$, Eq. (68) leads to

$$\Delta = \Delta_+ t^{2/3} + \Delta_- t^{-1}\,. \tag{69}$$

Therefore,

$$\Delta \propto t^{2/3} \propto a\,, \tag{70}$$

on all scales.

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Growth of Linear Density Perturbations (2)

The case of radiation ($w = 1/3 = c_s^2$)

When $H = 1/2t$ and $\rho = 3/4t^2$, Eq. (66) takes the form

$$\frac{d^2\Delta_{(n)}}{dt^2} + \frac{1}{2t}\frac{d\Delta_{(n)}}{dt} - \frac{1}{2t^2}\left[1 - \left(\frac{\lambda_J}{\lambda_n}\right)^2\right]\Delta_{(n)} = 0\,, \qquad (71)$$

where $\lambda_J = \lambda_H/\sqrt{6}$ defines the Jeans length, $\lambda_H = 1/H$ and $\lambda_n = a/n$.

Large-scale growth

When $\lambda_n \gg \lambda_J$ Eq. (71) gives

$$\Delta = \Delta_+ t + \Delta_- t^{-1/2}\,. \qquad (72)$$

Therefore,

$$\Delta \propto t \propto a^2\,. \qquad (73)$$

Small-scale oscillation

When $\lambda_n \ll \lambda_J$ Eq. (71) gives

$$\Delta_{(n)} = \Delta_1 \sin\left[\sqrt{3}\left(\frac{\lambda_H}{\lambda_n}\right)\right] + \Delta_2 \cos\left[\sqrt{3}\left(\frac{\lambda_H}{\lambda_n}\right)\right]\,, \qquad \textit{where} \qquad \lambda_H/\lambda_n \propto t^{1/2}\,. \qquad (74)$$

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Linear Density Vortices

Density & expansion vortices

Following the Frobenius theorem $\mathrm{D}_{[b}\mathrm{D}_{a]}f = \dot{f}\omega_{ab}$. Then, to linear order,

$$\Delta_{[ab]} = -3a^2(1+w)H\omega_{ab} \qquad \text{and} \qquad \mathcal{Z}_{[ab]} = 3a^2\dot{H}\omega_{ab}\,. \tag{75}$$

Consequently,

$$\mathcal{Z}_{[ab]} = -\frac{\dot{H}}{(1+w)H}\Delta_{[ab]}\,. \tag{76}$$

Evolution of density vortices

Define the spacelike vector $\mathcal{W}_a = \varepsilon_{abc}\Delta^{bc}$. Then, Eq. (63) leads to

$$\dot{\mathcal{W}}_a = -\frac{1}{2H}(1-w)\rho\mathcal{W}_a\,. \tag{77}$$

As a result,

$$\mathcal{W} \propto t^{-1} \propto a^{-3/2} \quad \text{for dust} \quad \text{and} \quad \mathcal{W} \propto t^{-1/2} \propto a^{-1} \quad \text{for radiation}\,. \tag{78}$$

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Dissipative Effects

Pressure support & the Jeans mass

Define $M_J \sim \rho_b \lambda_J^3$ (baryonic Jeans mass). No growth for Δ below M_J.

- Before decoupling: $M_J \sim 10^{16} (\Omega_b/\Omega)(\Omega h^2)^{-1/2}\,\mathrm{M}_\odot$ (*supercluster*)
- After decoupling: $M_J \sim 10^{4} (\Omega_b/\Omega)(\Omega h^2)^{-1/2}\,\mathrm{M}_\odot$ (*globular cluster*)

Photon diffusion & Silk damping

- Photons diffuse from overdense to underdense regions carrying the baryons along
- No structure below

$$M_S \propto \rho_b \ell_S^3 \simeq 6.2 \times 10^{12} \left(\frac{\Omega}{\Omega_b}\right)^{3/2} \left(\Omega h^2\right)^{-5/4} \mathrm{M}_\odot \,. \qquad (\textit{Silk mass}) \qquad (79)$$

Collisionless free-streaming & and Landau damping

- Collisionless (dark matter) particles move freely erasing structures in their distribution
- No structure below

$$\ell_{FS} \simeq 0.5\, m_{DM}^{-4/3} \left(\Omega_{DM} h^2\right)^{1/3} \mathrm{Mpc}\,, \qquad \textit{with} \quad m_{DM} \quad \textit{in} \quad \mathrm{keV}\,. \qquad (80)$$

- When $m_{DM} \simeq 30$ eV, $\ell_{FS} \simeq 30$ Mpc. When $m_{DM} \simeq 1$ keV, $\ell_{FS} \simeq 0.5$ Mpc.

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Cold Dark Matter Effects

The "baryons only" problem

At recombination: $\Delta^{(b)} \simeq 10^{-5}$. After recombination: $\Delta^{(b)} \propto a$. Today: $\Delta^{(b)} \simeq 10^{-2}$.

A CDM-baryon universe

Linear baryonic perturbations evolve according to

$$\dot{\Delta}^{(b)} = -\mathcal{Z} - a\mathrm{D}^2 v^{(b)}\,, \qquad \dot{\mathcal{Z}} = -2H\mathcal{Z} - \frac{1}{2}\rho\Delta \quad \text{and} \quad \dot{v}^{(b)} = -Hv^{(b)}\,, \tag{81}$$

where $\rho = \rho^{(CDM)} + \rho^{(b)}$ and $\rho\Delta = \rho^{(CDM)}\Delta^{(CDM)} + \rho^{(b)}\Delta^{(b)}$. Then,

$$\ddot{\Delta}^{(b)} + 2H\dot{\Delta}^{(b)} = \frac{1}{2}\rho^{(CDM)}\Delta^{(CDM)}\,. \tag{82}$$

A CDM boosting of baryonic perturbations

At recombination: $\rho^{(b)} \ll \rho^{(CDM)}$, $\Delta^{(b)} \ll \Delta^{(CDM)}$, $\rho^{(CDM)} \propto a^{-3}$.
After recombination: $\Delta^{(CDM)} \propto a$. Then,

$$\Delta^{(b)} = \Delta^{(CDM)}\left(1 - \frac{a_{\rm rec}}{a}\right)\,, \qquad \text{with} \qquad \Delta^{(b)} \to \Delta^{(CDM)}\,. \tag{83}$$

Cold Dark Matter Effects

The "baryons only" problem

At recombination: $\Delta^{(b)} \simeq 10^{-5}$. After recombination: $\Delta^{(b)} \propto a$. Today: $\Delta^{(b)} \simeq 10^{-2}$.

A CDM-baryon universe

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The Recent Universal Acceleration

Interpreting the SN Observations

The luminosity distance

The luminosity distance at redshift z is

$$D_L = a_0(1+z)r_0\,, \qquad \textit{with} \qquad \mathrm{d}t = a\mathrm{d}r \qquad (\textit{for a null radial geodesic})\,. \tag{84}$$

Then,

$$r_0 = a_0^{-1}\int_0^z H^{-1}\mathrm{d}x\,, \qquad \textit{where} \qquad \int_H^{H_0} H^{-1}\mathrm{d}H = -\int_0^z (1+q)\mathrm{d}[\ln(1+x)]\,, \tag{85}$$

since $\mathrm{d}z = -(1+z)H\mathrm{d}t$. Consequently,

$$a_0 r_0 = H_0^{-1}\int_0^z \mathrm{e}^{-\int_0^x (1+q)\mathrm{d}[\ln(1+y)]}\mathrm{d}x\,, \qquad \textit{with} \qquad q = -[1+(\dot{H}/H^2)]\,. \tag{86}$$

Finally,

$$D_L = (1+z)H_0^{-1}\int_0^z \mathrm{e}^{-\int_0^x (1+q)\mathrm{d}[\ln(1+y)]}\mathrm{d}x\,. \tag{87}$$

Universal acceleration (!?)

- Substituting the SN data into Eq. (87) has repeatedly given $q < 0$ (out to $z \simeq 1$).
- It appears that the universe started to accelerate about two billion years ago.
- How can we explain this?

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The Dark Energy Paradigm

Raychaudhuri's equation

In an FRW universe the Raychaudhuri equation takes the form

$$qH^2 = \frac{1}{6}\rho(1+3w) - \frac{1}{3}\Lambda \quad \Rightarrow \quad q = \frac{1}{2}(1+3w)\Omega_\rho - \Omega_\Lambda\,. \tag{88}$$

Therefore accelerated expansion (i.e. $q < 0$) requires

- $\Omega_\Lambda \gtrsim \frac{1}{2}(1+3w)\Omega_\rho$ (positive cosmological constant)

or

- $w = p/\rho < -\frac{1}{3}$ (dynamic dark energy)

The concordance (ΛCDM) model

Advocates a flat FRW universe with: $\Omega_\Lambda \simeq 0.7$, $\Omega_{CDM} \simeq 0.25$ and $\Omega_b \simeq 0.05$.

- Advantages: Good agreement with observations.
- Caveats: Considerable fine tuning.

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Alternatives to Dark Energy

Raychaudhuri's equation

In general, the Raychaudhuri equation reads

$$\frac{1}{3}\Theta^2 q = \frac{1}{2}(\rho + 3p) + 2(\sigma^2 - \omega^2) - \mathrm{D}^a A_a - A_a A^a - \Lambda\,, \tag{89}$$

Therefore accelerated expansion without dark energy is theoretically possible.

Alternative scenarios

The most popular alternatives to dark energy advocate:

- Introducing new physics (e.g. modify GR, add extra dimensions, etc).
- Accounting for the (recent) effects of the large-scale structure.

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The Backreaction Paradigm (1)

Spatial averaging

Non-commutativity between spatial averaging and time evolution of scalars

$$\langle\phi\rangle^{\cdot}_D - \langle\dot{\phi}\rangle_D = \langle\Theta\phi\rangle_D - \langle\Theta\rangle_D\langle\phi\rangle_D\,, \tag{90}$$

where D is the averaged domain.

The averaged equations

Assuming irrotational dust, the Raychaudhuri equation averages to

$$\langle\Theta\rangle^{\cdot} + \frac{1}{3}\langle\Theta\rangle^2 + \frac{1}{2}\langle\rho\rangle - \Lambda = \mathcal{Q}_D\,, \tag{91}$$

with

$$\mathcal{Q}_D = -\frac{2}{3}\left(\langle\Theta\rangle^2 - \langle\Theta^2\rangle\right) - 2\langle\sigma^2\rangle\,. \tag{92}$$

is the nonlinear backreaction term. Also note that

$$\mathcal{Q}_D = \frac{1}{2}\langle\mathcal{R}\rangle - \langle\rho\rangle + \frac{1}{3}\langle\Theta\rangle^2 - \Lambda\,, \qquad where \quad \mathcal{R} \quad is\ the\ 3D\ Rici\ scalar\,. \tag{93}$$

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The Backreaction Paradigm (2)

Acceleration from backreaction

Following (91), accelerated expansion is possible when $\mathcal{Q} > 0$ and

$$\mathcal{Q}_D > \frac{1}{2}\langle\rho\rangle\,. \tag{94}$$

However,

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Attractive aspects and caveats

- Simple and with no “coincidence problem”.
- Considerable information loss, ambiguous results.

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