Dynamics of Gravitational Systems

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1. Introduction[1]

In the universe there are many examples of dynamical systems whose components interact only (or essentially) through gravitational forces. Among them we can mention star clusters (open and globular clusters), galaxies, groups and clusters of galaxies.

The simplest gravitational system we can imagine is constituted by two bodies and its dynamics is already studied in the college. The properties of the orbit are defined basically by the two constants of the motion: the total energy **E** (kinetic + potential) and the orbital angular momentum **J**. If the total energy of the system is negative and the orbital angular momentum is different from zero, then the orbits are closed. This is not true if the motion of the considered bodies is perturbed by "external" forces. The orbit of a planet around the Sun is perturbed by the gravitational attraction of the other planets and one of the resulting effects is to produce an advance of the longitude of the perihelion. In the case of the planet Mercury this amounts to 575"/century. Urbain Le Verrier in the 19^{th} century was one of the first astronomers to estimate the contribution of the different planets to such systematic deviation of the perihelion of Mercury and he concluded that Venus should contribute to about 278"/century, the Earth to about 90"/century, Jupiter to about 154"/century and the other planets to about 10"/century, remaining an unexplained residual of 43"/century. As we know today, this residual can be explained naturally by the General Relativity Theory (GRT). Moreover, within the context of the GRT, the total energy and the angular momentum are no more strictly conserved quantities since a two-body system emits gravitational waves that carry out both energy and angular momentum, producing an inspiral motion that will lead to a "fusion" of the two bodies. This process is well observed in the binary pulsar PSR 1913+16, discovered in 1974 by Russell Hulse & Joseph Taylor, who received the Nobel Prize (in 1993) for such a discovery. Gravitational waves modify locally the space-time metric producing a back reaction effect that perturbs the orbit. This is relevant for the prediction of waveforms and the detection of gravitational waves by laser interferometers, constituting presently an important field of research.

If a third body is added to our system, then no simple solution for the orbits exists even in the Newtonian context. Different solutions have been explored in the literature (in particular when one of the bodies has a small mass in comparison with those of the other two) and a comprehensive review on this problem can be found in the two-volumes book by Michel Hènon, "The Three-Body Problem", who gave a major contribution to this field. Numerical simulations suggest that, in general, at the late evolutionary stages, one of the bodies is ejected (acquiring a positive total energy) from the system, while the other two form a strongly bonded pair.

The study of the evolution of more complex systems requires numerical methods. N-body codes compute the force acting in a given body due to the remaining ones. This approach has a high computational cost limiting the number of particles to about 10^6 with the present computers. Approaches using hierarchical schemes like the "tree-codes" permit to follow the evolution of systems constituted presently of few billions of particles and are used to study the dynamics of galaxies or to simulate a representative volume of the universe.

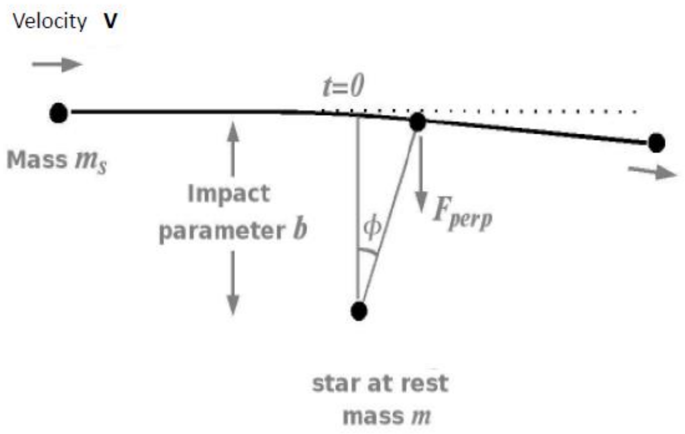
[1] This text is partially based on notes used in my past graduate lectures at the University of São Paulo.

As it will be shown in the next section, binary interactions lead, in general, to a quite long relaxation timescale. In this case, the particles constituting the system behave like a collisionless fluid and, in this case, a fundamental question is how the system relaxes to an equilibrium state. This problem will be discussed in detail along these lectures.

2. Relaxation due to binary collisions

We consider a galaxy in which the stars are assumed to be “mass points”. Consider a test star of mass m at rest and a perturbing star of mass m_s with a relative velocity **V** with respect to the test star (see figure 1)

Figure 1



Since the interaction is important essentially near the distance of the closest approach **b** (the impact parameter), a linear trajectory is considered. These approximations are reasonable if the collision time $\mathbf{t_c = b/V}$ is much smaller than the orbital period of the test star. For symmetry reasons, the momentum transferred to the test star in the direction ***x*** of the motion is zero and only the transversal component is relevant. Thus

$$\frac{dp_\perp}{dt} = -\frac{Gmm_s}{r^2}\cos\phi \tag{2.1}$$

where $\mathbf{r^2=x^2+b^2}$, $\mathbf{b=r.\cos\phi}$ and $\mathbf{x=b.tg\phi}$. Using these relations, the total variation of the transverse momentum is

$$\Delta p_\perp = -\frac{Gmm_s}{V}\int_{-\infty}^{\infty}\frac{\cos\phi}{r^2}dx = -\frac{2Gmm_s}{V}\int_0^{\pi/2}\frac{\cos\phi d\phi}{b} = -\frac{2Gmm_s}{bV} \tag{2.2}$$

From eq.(2.2) the transversal velocity of the test star in a given collision can be estimated. If the orientation of the collision direction is randomly orientated, the average variation of the velocity is zero, i.e., $\mathbf{<\Delta V_\perp> = 0}$. However, the average of the quadratic variation is not and is given by

$$dV_\perp^2 = 4\left(\frac{Gm_s}{bV}\right)^2 dN \tag{2.3}$$

where **dN** is the number of collisions in the time interval **T** having impact parameters in the range **b, b+db**. In other words

$$dN = 2\pi b(VT)n_* db \tag{2.4}$$

where n_* is the stellar density. The expression above gives essentially the number of stars inside a cylindrical shell of radius **b**, thickness **db** and height **VT**. Substituting eq.(2.4) into eq.(2.3) and performing the integration between the minimum $\boldsymbol{b_{min}}$ and the maximum $\boldsymbol{b_{max}}$ impact parameters one obtains

$$V_\perp^2 = 8\pi \frac{G^2 m_s^2}{V} T n_* \lg\left(\frac{b_{\max}}{b_{\min}}\right) \tag{2.5}$$

In general $\boldsymbol{b_{max}}$ is estimated as being comparable to the mean distance between stars, hence $b_{\max} \approx n_*^{-1/3}$. On the other hand, the minimum distance is estimated as being that in which the potential energy of the interaction is comparable to the kinetic energy, namely $\mathbf{b_{min} \sim Gm_s/V^2}$. The relaxation time $\mathbf{T_R}$ is defined as the timescale in which the quadratic variation of the velocity is equal to the mean quadratic velocity of the stars in the galaxy. Therefore, using eq.(2.5) one obtains

$$T_R = \frac{\langle V^2 \rangle^{3/2}}{8\pi G^2 m_s^2 n_* \lg \Lambda} \tag{2.6}$$

where we have defined

$$\Lambda = \frac{b_{\max}}{b_{\min}} = \frac{V^2}{G m_s n_*^{1/3}} \tag{2.7}$$

Numerically, eqs.(2.6) and (2.7) can be written as

$$T_R = 2.1\times 10^9 \frac{(V/km.s^{-1})^3}{(m_s/m_\odot)^2 (n_*/pc^{-3}) \lg \Lambda} \ years \tag{2.8}$$

and

$$\Lambda \approx 232 \frac{(V/km.s^{-1})^2}{(m_s/m_\odot)(n_*/pc^{-3})^{1/3}} \tag{2.9}$$

Typical values for the relaxation timescale can be found in table 1 below:

Table 1

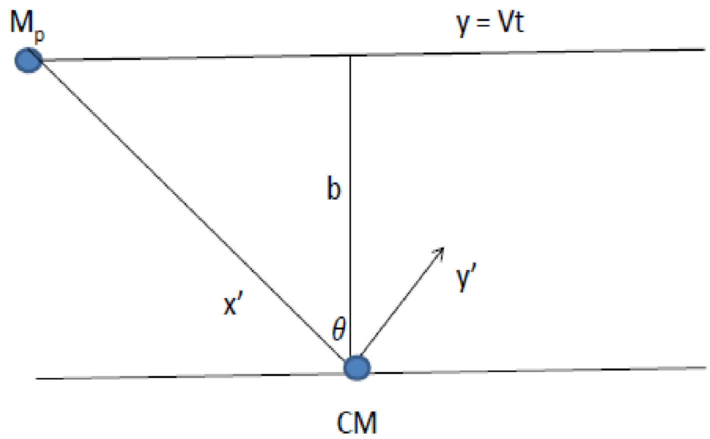
	Stellar density (pc^{-3})	V (km/s)	T_R (years)
Solar neighborhood	0.1	20	1.4×10^{13}
Globular cluster (core)	10^3	10	2.7×10^8
E-galaxy (core)	10	200	1.1×10^{14}

As we have anticipated, excepting for globular clusters, the relaxation timescale due to binary interactions is quite long raising questions concerning mechanisms by which collisionless gravitational systems relax.

3. Energy transfer by tidal forces

It is interesting to estimate also the energy transfer for a star of a given galaxy (or a body bond to any other gravitational system) from a perturbing external point mass galaxy (or another massive body). In this case, the effect of a collision between two galaxies is not limited to the transfer of energy to the center of mass (as in the previous case) since tidal forces transfer energy also to the stars, increasing their velocity dispersion or, in other words, producing a "heating" of the system.

Figure 2



In order to describe the collision process in a very simplified way, the perturbing galaxy, a point mass $\mathbf{M_p}$, moves with velocity **V** in a straight line as indicated in Fig.2. The center of mass (CM) of the perturbed galaxy is assumed to be at rest and the impact parameter of the collision is ***b***. We consider a reference frame S(x,y,z) centered at CM in which the x-axis is along the direction of the closest approach and the y-axis is parallel to the movement of the perturbing galaxy. The reference frame S'(x',y',z') follows the movement of the perturbing galaxy and the x'-axis is defined by the straight line connecting the center of mass of both galaxies (see Fig.2). In the frame S' the components of the tidal acceleration acting on a star of coordinates (x', y', z') are

$$\gamma_{x'} = \frac{2GM_p}{R^3} x' \tag{3.1}$$

$$\gamma_{y'} = -\frac{GM_p}{R^3} y' \tag{3.2}$$

$$\gamma_{z'} = -\frac{GM_p}{R^3} z' \tag{3.3}$$

where $\mathbf{R^2 = b^2 + V^2t^2}$, $\mathbf{tg}\boldsymbol{\theta} \mathbf{= Vt/b}$ and $\mathbf{b = R\cos}\boldsymbol{\theta}$ (see fig. 2). The vector components in the frame **S'** are related to those in the rest frame **S** through the equation

$$\vec{X}' = A\vec{X} \tag{3.4}$$

where **A** is a rotation matrix defined by

$$A = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{3.5}$$

From these relations, the system of equations 3.1-3.3 can be recast as

$$\frac{dV_{x'}}{dt} = \cos\theta\frac{dV_x}{dt} + \sin\theta\frac{dV_y}{dt} = \frac{2GM_p}{R^3}x' \tag{3.6}$$

$$\frac{dV_{y'}}{dt} = -\sin\theta\frac{dV_x}{dt} + \cos\theta\frac{dV_y}{dt} = -\frac{GM_p}{R^3}y' \tag{3.7}$$

$$\frac{dV_{z'}}{dt} = \frac{dV_z}{dt} = -\frac{GM_p}{R^3}z' \tag{3.8}$$

Now, multiply eq.(3.6) by $\mathbf{sin}\boldsymbol{\theta}$, eq.(3.7) by $\mathbf{cos}\boldsymbol{\theta}$, add both to obtain

$$\frac{dV_y}{dt} = \frac{2GM_p}{R^3}x'\sin\theta - \frac{GM_p}{R^3}y'\cos\theta \tag{3.9}$$

Use again eq.(3.4) to express the coordinates given in ***S'*** frame into the rest frame ***S*** to get finally

$$\frac{dV_y}{dt} = \frac{GM_p}{R^3}\left[3x\cos\theta\sin\theta + y\left(2 - 3\cos^2\theta\right)\right] \tag{3.10}$$

Similarly, multiply eq.(3.6) by cosθ, eq.(3.7) by sinθ, subtract one from another and use as before eq.(3.4) to obtain, after some algebra

$$\frac{dV_x}{dt} = \frac{GM_p}{R^3}\left[3y\cos\theta\sin\theta + x\left(2 - 3\sin^2\theta\right)\right] \tag{3.11}$$

The component along the z-axis is obtained trivially and is given by

$$\frac{dV_z}{dt} = -\frac{GM_p}{R^3}z \tag{3.12}$$

The total variation in the x-component of the velocity is

$$\Delta V_x = \int_{-\infty}^{\infty} \left(\frac{dV_x}{dt} \right) dt = \int_{-\pi/2}^{\pi/2} \left(\frac{dV_x}{dt} \right) \frac{b \sec^2 \theta d\theta}{V} \qquad (3.13)$$

Replacing eq.(3.11) into eq.(3.13) and performing the integration one obtains

$$\Delta V_x = \frac{2GM_p}{b^2 V} x \qquad (3.14)$$

The variations in the other components can be derived in a similar way and we get $\mathbf{\Delta V_y = 0}$ and

$$\Delta V_z = \frac{2GM_p}{b^2 V} z \qquad (3.15)$$

The average variation of the quadratic velocity is

$$\left\langle \Delta V^2 \right\rangle = \left\langle \Delta V_x^2 \right\rangle + \left\langle \Delta V_y^2 \right\rangle + \left\langle \Delta V_z^2 \right\rangle = \frac{4}{b^4} \left(\frac{GM_p}{V} \right)^2 \left(\overline{x}^2 + \overline{z}^2 \right) \qquad (3.16)$$

Defining

$$\left(\overline{x}^2 + \overline{z}^2 \right) = \frac{2}{3} \overline{r}_s^2 \qquad (3.17)$$

where $\overline{r}_s^2$ is the average quadratic radial distance of the test star to the center of mass, eq.(3.16) can be recast as

$$\left\langle \Delta V^2 \right\rangle = \frac{8}{3} \left(\frac{GM_p}{b^2 V} \right)^2 \overline{r}_s^2 \qquad (3.18)$$

Note that now the energy transfer depends on the position of the star. Objects near the border of the perturbed galaxy are more affected than those near the center.

Let us verify the importance of tidal interactions between galaxies in a massive cluster. The total energy of a test galaxy of mass **M** is $|E| = GM^2 / 2r_T$, where r_T is the tidal radius, supposed to fix the boundaries of the object. The energy fraction transferred after a collision is

$$\frac{\Delta E}{|E|} = \frac{M \left\langle \Delta V^2 \right\rangle}{2|E|} = \frac{8}{3} \frac{GM}{b^4 V^2} r_s^2 r_T \qquad (3.19)$$

where we have used eq.(3.18) and assumed that all galaxies in the cluster have the same (average) mass. Since the more affected stars are in the border of the galaxy, assume also that $r_s = r_T$ .Thus

$$\frac{\Delta E}{|E|} = \frac{8}{3} \frac{GM}{b^4 V^2} r_T^3 \qquad (3.20)$$

The effective cross section σ_E for the energy transfer is estimated over all possible impact parameters, i.e.,

$$\sigma_E = \int_{b_{\min}}^{b_{\max}} 2\pi b\left(\frac{\Delta E}{|E|}\right)db = \frac{8\pi}{3}\frac{GM}{V^2}r_T^3\left(\frac{1}{b_{\min}^2}-\frac{1}{b_{\max}^2}\right) \tag{3.21}$$

If $b_{\min} \simeq r_T << b_{\max} \approx n_G^{-1/3}$, where n_G is the density of galaxies in the cluster, the equation above can be recast as

$$\sigma_E = \frac{8\pi}{3}\frac{GM}{V^2}r_T \tag{3.22}$$

On the other side, the fractional rate of energy transfer is

$$\frac{1}{|E|}\frac{d\Delta E}{dt} = \sigma_E n_G V = \frac{8\pi}{3}\frac{GM}{V}r_T n_G \tag{3.23}$$

The tidal effects will be important if the timescale $|E|/(d\Delta E/dt)$ is comparable to the Hubble time, implying in the condition

$$\frac{8\pi}{3}\frac{GM}{V}r_T n_G T_H \geq 1 \quad (3.24) \tag{3.24}$$

Taking $r_T \approx 50\,kpc,\ T_H = 4.5\times10^{17}\ s$ and $M \simeq 10^{12}\,M_\odot$ as typical values, we find respectively for Virgo and Coma clusters

	n_G (Mpc^{-3})	V (km/s)	T_H/t_{tidal}
Virgo	500	1300	10
Coma	1000	1550	17

The numbers in the last column indicate that tidal effects are to be expected in both clusters.

4. Relaxation of gravitational systems

Observations indicate that E-galaxies have nearly "universal" brightness profile. As we have seen before, these galaxies have very long relaxation timescales when only binary interactions are taken into account. These "universal" profiles suggest that these objects are relaxed and, in this case, one may wonder about the nature of the relaxation mechanisms that were able to produce a state of quasi-equilibrium in these galaxies.

Numerical investigations by Hénon (Annales d'Astrophysique 27, 83, 1964) have shown that a variable potential may play a fundamental role in the relaxation of a gravitational system. Few years later, Lynden-Bell (MNRAS 135, 413, 1967) described the "**violent relaxation**" process by which a collisionless dynamical system relaxes from a chaotic initial state to quasi equilibrium, when rapid changes in the gravitational potential occur during the gravitational collapse or during a merger episode. As a consequence of such a process, Lynden-Bell concluded that the final energy distribution of particles would be Fermi-like or a Maxwellian under certain conditions. The final velocity dispersion would be the same for all particles, independent of their masses. Thus, violent relaxation leads to dynamical equilibrium but not to a state similar to the "thermal equilibrium" or, in other words, in a state in which all the particles have the same kinetic energy.

However, numerical experiments suggest that the reality is more complex. Simulations of the gravitational collapse indicate that violent relaxation enhances segregation in the energy space (Funato et al. PASJ 44, 291, 1992 and PASJ 44, 613, 1992). Moreover, simulations with N-body tree code by Merrall & Henriksen (ApJ 595, 43, 2003) indicate that relaxed systems have a Gaussian distribution and numerical studies of the relaxation process by Diemand et al. (MNRAS 352, 535, 2004) confirm that the resulting velocity distribution is Gaussian but with a negative kurtosis or, in other words, with a flat topped profile.

4.1 An estimate of the violent relaxation timescale

When the gravitational potential acting in a given test particle depends on time, the energy of the particle is not conserved. Let ε be the energy per unit of mass of the test particle. Then

$$\frac{d\varepsilon}{dt} = \frac{d}{dt}\left(\frac{V^2}{2} + \phi\right) = \vec{V}\cdot\frac{d\vec{V}}{dt} + \frac{\partial\phi}{\partial t} + \vec{V}\cdot\vec{\nabla}\phi \tag{4.1}$$

Using the equation of motion

$$\frac{d\vec{V}}{dt} + \vec{\nabla}\phi = 0 \tag{4.2}$$

eq.(4.1) becomes

$$\frac{d\varepsilon}{dt} = -\frac{\partial\phi}{\partial t} \tag{4.3}$$

This demonstrates the fact that the energy is not conserved if the potential depends explicitly on time. The effect of a time-dependent potential is to modify the statistics of the orbits leading the system to an equilibrium state. The relaxation timescale of this process is defined as $T_{VR} = \sqrt{\left[\phi/(d\phi/dt)\right]^2}$. This timescale is of the order of the dynamical timescale. To show this, assume a system originally in equilibrium. If U is the internal kinetic energy and $W = m\phi$ is the potential gravitational energy, the virial is satisfied, i.e., $2U_0 + W_0 = 0$ and the total energy as well, namely, $E = U_0 + W_0$ (the subscript "0" indicates equilibrium values). From these relations we have $2E = W_0$ or $E = -U_0$.

Let the gravitational energy of the system be given by

$$W = -\alpha\frac{GM^2}{R} \tag{4.4}$$

where α is a structure factor of the order of the unity, depending on the mass distribution. Similarly, the momentum of inertia of the system will be given by

$$I = \lambda M R^2 \tag{4.5}$$

where again a structure factor λ of the order of the unity was introduced. If the system is disturbed from its equilibrium state, the virial is no more satisfied but the Jacobi's identity remains valid, i.e.,

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2(K+U)+W \tag{4.6}$$

with K being the macroscopic kinetic energy. Using the energy conservation condition and eqs.(4.4) and (4.5), the equation above becomes

$$\lambda M \frac{d^2 R^2}{dt^2} = 4E + 2\alpha \frac{GM^2}{R} \tag{4.7}$$

Notice that in equilibrium the radius of the configuration is $R_0 = -\alpha GM^2/2E$. Let the perturbation of the equilibrium radius be of the form

$$R(t) = R_0 + \delta R(t) \tag{4.8}$$

with $\delta R(t) << R_0$. In this case, substituting eq.(4.8) into eq.(4.7), expanding in series and considering only the linear terms one obtains

$$\delta \ddot{R} + \frac{\alpha}{\lambda}\frac{GM}{R_0^3}\delta R = 0 \tag{4.9}$$

The equation above is that of a harmonic oscillator with angular frequency $\omega^2 = \alpha GM/\lambda R_0^3$. If the variation of the radius is comparable to the variation on the potential, then $\dot{\phi}_{max} \approx \left(\alpha GM/R_0^2\right)\delta\dot{R}_{max}$. But $\delta\dot{R}_{max} \approx \omega R_0$ and hence $\dot{\phi}_{max} \approx -\omega\phi$. This implies that $T_{VR} = 1/\omega \propto 1/\sqrt{G\rho}$, as we have anticipated.

A time-dependent potential is not a guarantee that the system will find an equilibrium state through "violent relaxation". Some numerical experiments by Sridhar (MNRAS 238, 1159, 1989) and Sridhar & Nityananda (Journal of Astron. & Astrophys. 10, 279, 1989) indicate the existence of oscillating gravitational systems without relaxation and in which the relative energy distribution of the particles remains invariant. Violent relaxation demands at the same time, a time-dependent potential and phase-space mixing. The later mechanism describes the diffusion of nearby points in phase-space due to small differences in the orbital frequencies. For instance, the circular motion with a constant tangential velocity has an angular frequency that depends on the radius, i.e., $\omega(r) = V_0/r$. Thus, nearby points diverge as $\Delta\phi(t) = \Delta(V_0 t/r) = \Delta(\omega t)$ and the timescale for the mixing is of the order of $1/\Delta\omega$. This process is reversible, since the system keeps the knowledge of the initial conditions. However, this is not the case for the "chaotic mixing", an irreversible process. Here, the diffusion of nearby points in the phase-space is due to the chaotic nature of the orbits. The "chaotic mixing" operates in an timescale defined by the **Lyapunov exponents**. The exponent of Lyapunov measures how fast nearby trajectories in phase-space diverge. For each point in a phase-space of 2N dimensions (N = number of degrees of freedom) there are 2N Lyapunov exponents λ_i. Consider a sphere of dimension 2N with radius R whose center is a given point $\vec{X}$ in the phase-space. Nearby points at the surface evolve differently, producing a deformation of such a sphere, an "ellipsoid" whose principal axes are $L_i(t)$. In this case, the Lyapunov exponent for the phase-space point $\vec{X}$ is defined as

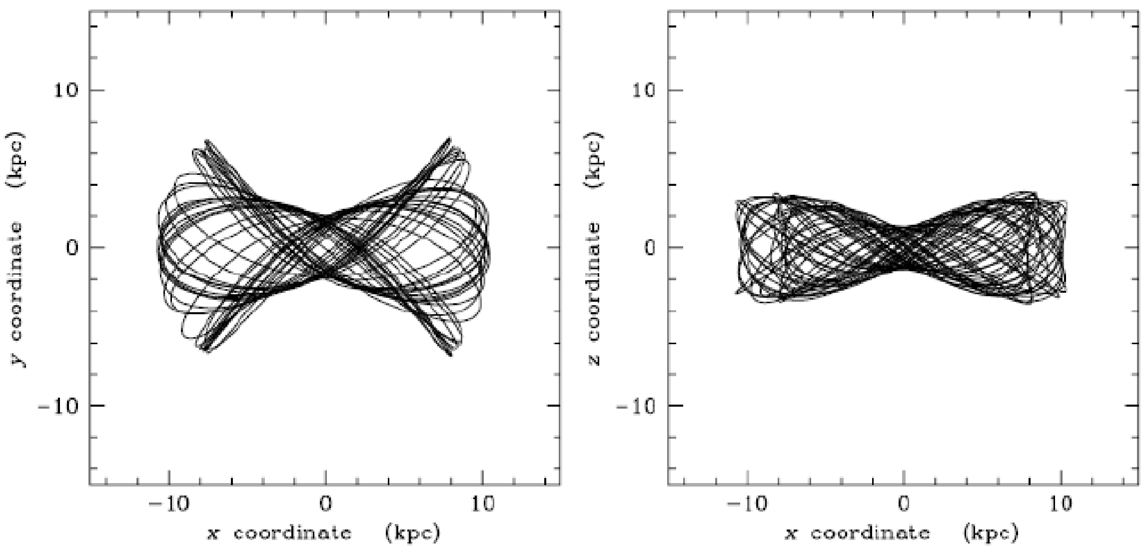
$$\lambda_i = \lim_{t\to\infty}\left(\frac{1}{t}\lg\frac{dL_i(t)}{dr}\right) \tag{4.10}$$

For a collisionless system, the following condition is satisfied

$$\sum_{i=1}^{2N}\lambda_i = 0 \tag{4.11}$$

which is an alternative way to say that the "fluid is incompressible".

Figure 3



In figure 3 it is shown an example of a chaotic mixing process, namely, the orbit of a star in a tri-axial potential. Initially the star is placed in a plane inclined with respect to the z-axis. The orbit is complex and it maps out a region of space.

4.2 Dark matter halos

The nature of dark matter (about 6-7 times more abundant than baryonic matter in the universe) is not known yet. Nevertheless, we consider that dark matter (hereafter DM) particles interact only through gravitational forces, constituting a "collisionless fluid". Similar to the question of how galaxies relax, we want to understand how DM halos find an equilibrium state.

Cosmological simulations indicate some general features concerning the properties of the equilibrium state of these halos.

The density profiles are peaked and seem to have a "universal" form. They are fitted by the so-called Navarro-Frenk-White profile (MNRAS 275, 56, 1995) and according to Klypin et al. (ApJ 554, 903, 2001), the central slopes (dlgρ/dlgr) may depend on the merger history or on the initial conditions (Ascasibar et al., MNRAS 352, 1109, 2004). Simulation indicate that dynamical young systems (clusters of galaxies) have ''cusped'' profiles while dwarf galaxies have ''cored'' ones (Henriksen, MNRAS 355, 1217, 2004).

Simulations by Ricotti (MNRAS 344, 1237, 2003) suggest that the relaxation process of DM halos is the consequence of collective effects re-excited by successive merger events and central cusps could be the consequence of the inflow of low-entropy material.

The phase-space indicator $Q = \rho / \sigma_{1D}^3$ is a useful quantity to express the dynamical evolution of DM halos. High resolution simulations indicate $Q \propto r^{-\beta}$, where the exponent is the same either for galaxy-size halos (Taylor & Navarro, ApJ 563, 483, 2001) or cluster-size halos (Rasia et al., MNRAS 351, 237, 2004), namely, $\beta \cong 1.87$. Concerning the power-law profile of the phase-space indicator observed in simulations led Williams et al. (ApJ 604, 18, 2004) to formulate two hypotheses: hierarchical assembly of DM halos preserves phase-space stratification or the power-law profile $Q \propto r^{-\beta}$ represents a generic feature of the violent relaxation mechanism.

The role of mergers (and the accretion process) in the evolution of the phase-space indicator Q of DM halos was studied in cosmological simulations by Peirani et al. (MNRAS 367, 1011, 2006) and their main results can be summarized as follows:

a) The core density of halos being assembled decreases on the average by one order of magnitude in the interval $10 < z < 0$
b) A rapid increase of the velocity dispersion associated to the relaxation process (energy transfer from bulk-to-random motions) occurs in the redshift interval $10 < z < 6.5$, followed by a late period of "slow heating"
c) Merger episodes show an energy transfer from bulk-to-random motions due to collective effects, heating all particles regardless their initial energies
d) Halos issued from simulations with particles of different masses relax as predicted by the "violent relaxation" mechanism, i.e., all particles have the very same velocity dispersion at the end of the process
e) The phase-space density indicator Q decreases on the average by a factor of 40 in the first 0.5 Gyr and then by a factor of 20 during the late evolution.
f) Every merger episode produces a sudden decrease of the phase-space density and an increase of the velocity dispersion (heating).

In conclusion of this section, it is possible to say that the relaxation of collisionless systems is not fully understood at present time. Possible issues are the relaxation via collective effects and/or relaxation in the presence of rapid changes of the gravitational potential ("violent relaxation"). The violent relaxation mechanism can be imagined as a process in which kinetic energy from bulk motions is transferred to random motions, decreasing the phase-space density of the system. At the end of the process all particles have the same velocity dispersion independently of their masses and numerical experiments confirm such a prediction. These experiments indicate also that systems relaxed via gravitational collapse and/or mergers have velocity distributions described by Gaussians with negative kurtosis ("flat-topped" profiles) – confirmed by cosmological simulations.

5. Irreversible processes

In this section will be examined the motion of a test particle under the action of the mean gravitational field of "distant" particles and the force of nearby particles supposed to be random forces.

5.1 The diffusive approximation

In this approximation, the test particle is supposed to have a random motion due to binary collisions with nearby particles of the system. Let $dP(r,t) = \rho(r,t)dV$ the probability to find the test particle at a distance **r** from the origin at the instant **t** inside the volume element **dV**. The normalization condition is given by

$$\int_V \rho(r,t)dV = 1 \tag{5.1.1}$$

and the corresponding mean values are also defined

$$\langle r^n \rangle = \int_V r^n \rho(r,t)dV \tag{5.1.2}$$

In the diffusive approximation, the probability density $\rho(r,t)$ evolves as

$$\frac{\partial \rho(r,t)}{\partial t} = \vec{\nabla} \cdot (D\vec{\nabla}\rho(r,t)) \tag{5.1.3}$$

Here, in a first approximation, the diffusion coefficient **D** will be considered as constant. Considering spherical symmetry, multiply both members of eq.(5.1.3) by $4\pi r^4$ and integrate to obtain

$$\int_0^\infty 4\pi r^4 \frac{\partial \rho}{\partial t} dr = \int_0^\infty 4\pi r^2 D \frac{\partial}{\partial r}\left(r^2 \frac{\partial \rho}{\partial r}\right) dr \tag{5.1.4}$$

The integration of the left side gives immediately $\partial \langle r^2 \rangle / \partial t$ while the integration by parts of the right side gives simply 6D (it is assumed also that $\lim_{r\to\infty} \rho, d\rho/dr = 0$ and $\lim_{r\to\infty} r^n (d\rho/dr) = 0$). In this case, in the diffusion approximation, the mean square distance of the particle evolves as

$$\langle r^2 \rangle = 6Dt \tag{5.1.5}$$

5.2 The Langevin equation

Another useful approximation for the motion of a test particle under the influence of random forces is that proposed by Langevin. The equation of motion for the test particle is given by

$$\frac{d\vec{V}}{dt} = -\beta\vec{V} + \vec{A}(t) \tag{5.2.1}$$

In the above equation $\beta\vec{V}$ is a "viscous" acceleration due to the interaction with the medium and $\vec{A}(t)$ is a random acceleration due to close binary encounters. The mean value of the random acceleration is zero, i.e., $\left\langle \vec{A}(t) \right\rangle = 0$. Now multiply both sides of eq.(5.2.1) by $\vec{r}$. One obtains

$$\vec{r}\cdot\frac{d\vec{V}}{dt} = -\beta\vec{r}\cdot\vec{V} + \vec{r}\cdot\vec{A} \tag{5.2.2}$$

On the other hand we have the following identities

$$\begin{aligned} \vec{r}\cdot\frac{d\vec{V}}{dt} &= \frac{1}{2}\frac{d^2r^2}{dt^2} - V^2 \\ \vec{r}\cdot\vec{V} &= \frac{1}{2}\frac{dr^2}{dt} \end{aligned} \tag{5.2.2a}$$

which permit to recast eq.(5.2.2) as

$$\frac{1}{2}\frac{d^2r^2}{dt^2} - V^2 = -\frac{\beta}{2}\frac{dr^2}{dt} + \vec{r}\cdot\vec{A} \tag{5.2.3}$$

Performing time averages on both sides of the equation above and recalling that $\left\langle \vec{r}\cdot\vec{A} \right\rangle = \left\langle \vec{r} \right\rangle \cdot \left\langle \vec{A} \right\rangle = 0$ because the random acceleration is uncorrelated with the particle position, we have finally

$$\frac{d^2\left\langle r^2 \right\rangle}{dt^2} + \beta\frac{d\left\langle r^2 \right\rangle}{dt} - \left\langle V^2 \right\rangle = 0 \tag{5.2.4}$$

This equation must be integrated with the initial conditions $\left\langle r^2 \right\rangle = 0$ and $d\left\langle r^2 \right\rangle / dt = 0$, which gives

$$\left\langle r^2 \right\rangle = \frac{2\left\langle V^2 \right\rangle t}{\beta} - \frac{2\left\langle V^2 \right\rangle}{\beta^2}\left(1 - e^{-\beta t}\right) \tag{5.2.5}$$

When $\beta t \ll 1$, performing series expansion of eq.(5.2.5) it results

$$\left\langle r^2 \right\rangle \approx \left\langle V^2 \right\rangle t^2 \tag{5.2.6}$$

In other words, for a short instant of time the particle does not feel the effect of the surrounding medium (viscosity) and moves in a straight line. On the contrary, when $\beta t \gg 1$, the corresponding limit of eq.(5.2.5) is

$$\langle r^2 \rangle \approx \frac{2\langle V^2 \rangle t}{\beta} \tag{5.2.7}$$

Note that for long times the particle performs a diffusive motion since its mean square distance varies linearly with time. Comparing eqs.(5.2.7) and (5.1.5) one obtains for the diffusion coefficient

$$D = \frac{\langle V^2 \rangle}{3\beta} \tag{5.2.8}$$

This equation does not depend apparently on the amplitude of the random force but depends on the mean square velocity and the viscous coefficient. What is then the role of the random force? As we shall see, the random forces are, in fact, determinant to fix the value of the velocity dispersion of the particles (or their mean square velocity). In order to see this, let us introduce the inverse Fourier transform of the velocity, i.e.,

$$\vec{V}(t) = \int_{-\infty}^{\infty} \vec{V}_\omega e^{2\pi i\omega t} d\omega \tag{5.2.9}$$

and similarly for the random acceleration

$$\vec{A}(t) = \int_{-\infty}^{\infty} \vec{A}_\omega e^{2\pi i\omega t} d\omega \tag{5.2.10}$$

Substituting eq.(5.2.9) and (5.2.10) into the equation of motion (5.2.1) one obtains after some algebra

$$\vec{V}_\omega = \frac{\vec{A}_\omega}{(\beta + 2\pi i\omega)} \tag{5.2.11}$$

Squaring this expression one obtains

$$|V_\omega^2| = \frac{|A_\omega^2|}{(\beta^2 + 4\pi^2\omega^2)} \tag{5.2.12}$$

On the other hand, by the Parseval's theorem

$$\langle V^2(t) \rangle = \frac{1}{T}\int_{-\infty}^{\infty} |V_\omega^2| d\omega \tag{5.2.13}$$

In the equation above, *T* is the time interval along which the time average is performed.

Combining these two last equation one obtains

$$\left\langle V^2(t)\right\rangle = \frac{1}{T}\int_{-\infty}^{\infty}\frac{\left|A_\omega^2\right|}{\left(\beta^2+4\pi^2\omega^2\right)}d\omega \tag{5.2.14}$$

The autocorrelation function of the random acceleration is

$$C(\tau)=\left\langle \vec{A}(t)\cdot\vec{A}(t+\tau)\right\rangle = \frac{1}{T}\int_{-\infty}^{\infty}\left|A_\omega^2\right|e^{2\pi i\omega\tau}d\omega \tag{5.2.15}$$

where we have used eq.(5.2.10). The equation above indicates that the power spectrum of the random acceleration is the Fourier transform of the correlation function and can be obtained by simple inversion of eq.(5.2.15), namely

$$\left|A_\omega^2\right| = T\int_{-\infty}^{\infty}C(\tau)e^{-2\pi i\omega\tau}d\tau \tag{5.2.16}$$

Substituting this equation into eq.(5.2.14) one obtains finally for the mean quadratic velocity

$$\left\langle V^2(t)\right\rangle = \int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}C(\tau)\frac{e^{-2\pi i\omega\tau}}{\left(\beta^2+4\pi^2\omega^2\right)}d\tau \tag{5.2.17}$$

As mentioned before, the velocity dispersion of the particles is fixed by the random forces or more specifically by its autocorrelation function.

Consider a simple example in which the random forces act during a time T_c and are zero after that or, in other words, $C(\tau)=\gamma^2\ if\ 0\le t\le T_c$ and $C(\tau)=0\ if\ t\ge T_c$. In this case, from eq.(5.2.16) and taking the real part one obtains

$$\left|A_\omega^2\right| = \frac{\gamma^2 T}{2\pi\omega}\sin(2\pi\omega T_c) \tag{5.2.18}$$

Replacing this into eq.(5.2.14) and performing the integration it results

$$\left\langle V^2\right\rangle = \frac{\gamma^2}{\beta^2}e^{-\beta T_c/2}\sinh\left(\frac{\beta T_c}{2}\right) \tag{5.2.19}$$

When the correlation timescale T_c is much larger than the viscous timescale $(\beta T_c \ll 1)$ the velocity dispersion of the particles is simply

$$\left\langle V^2\right\rangle \simeq \frac{\gamma^2 T_c}{2\beta} \tag{5.2.20}$$

while in the other limit $(\beta T_c \gg 1)$ the velocity dispersion is

$$\langle V^2 \rangle \simeq \frac{\gamma^2}{2\beta^2} \tag{5.2.21}$$

This indicates that the velocity dispersion tends asymptotically to a limit fixed only by the amplitude of the random acceleration and the viscous timescale. Consequently, the diffusion coefficient becomes $D = \gamma^2 / 6\beta^3$.

As an example, let us consider the motion of a massive black hole in the center of a galaxy or the motion of a massive CD galaxy in the center of a cluster. In both cases the central object feels the general gravitational field of the host and its motion is perturbed by random forces due to nearby collisions. In this case, the equation of motion is

$$\frac{d\vec{V}}{dt} = -grad\phi + \vec{A}(t) \tag{5.2.22}$$

where the mean field characterized by the potential ϕ obeys the Poisson equation. Supposing that the homogeneous core of the galaxy (or of the cluster) with density ρ dominates the mean field we have

$$grad\phi = \frac{4\pi}{3} G\rho\vec{r} = \frac{\vec{r}}{\tau^2} \tag{5.2.23}$$

In the equation above we have introduced $\tau^2 = 3/4\pi G\rho$, the dynamical timescale of the system.

Substituting eq.(5.2.23) into eq.(5.2.22), then multiplying by $\vec{r}$ and making use of the identities (5.2.2a) one obtains

$$\frac{d^2 r^2}{dt^2} = 2V^2 - \frac{2r^2}{\tau^2} + 2\vec{r} \cdot \vec{A} \tag{5.2.24}$$

Now, performing a time average and recalling that the random acceleration is not correlated with the position lead to

$$\frac{d^2 \langle r^2 \rangle}{dt^2} + \frac{2\langle r^2 \rangle}{\tau^2} = 2\langle V^2 \rangle \tag{5.2.25}$$

The solution of this equation with the initial conditions $\langle r^2 \rangle = 0$ and $d\langle r^2 \rangle / dt = 0$ gives

$$\langle r^2 \rangle = \langle V^2 \rangle \tau^2 \left[1 - \cos\left(\sqrt{2}\,\frac{t}{\tau} \right) \right] \tag{5.2.26}$$

The equation above says that the central object performs an oscillatory motion around the center of the host (galaxy or cluster) whose amplitude depends on its mean square velocity and on the dynamical timescale. As we have shown previously, the mean square velocity will be determined by the strength of the random acceleration. In order to get a relation connecting the random acceleration to the mean square velocity, derivate first with respect to time eq.(5.2.22) making use of eq.(5.2.23). One obtains

$$\frac{d^2\vec{V}}{dt^2}+\frac{\vec{V}}{\tau^2}=\frac{d\vec{A}}{dt} \tag{5.2.27}$$

Then, make use of the Fourier transform for the velocity and the random acceleration as defined in eqs.(5.2.9) and (5.2.10). From the equation above, one obtains for the Fourier amplitude of the velocity

$$\vec{V}_\omega=\frac{2\pi i\omega\tau^2}{(1-4\pi^2\omega^2\tau^2)}\vec{A}_\omega \tag{5.2.28}$$

Therefore, from the Parseval's theorem, the mean square velocity is

$$\left\langle V^2\right\rangle=\frac{4\pi^2\tau^4}{T}\int_{-\infty}^{\infty}\frac{\omega^2\left|A_\omega^2\right|d\omega}{(1-4\pi^2\omega^2\tau^2)^2} \tag{5.2.29}$$

On the other hand, using the same correlation function as before, the power spectrum of the random acceleration is given by eq.(5.2.18) and substituting in the equation above gives

$$\left\langle V^2\right\rangle=2\pi\tau^4\int_{-\infty}^{\infty}\frac{\gamma^2\omega\sin(2\pi\omega T_c)}{(1-4\pi^2\omega^2\tau^2)^2}d\omega \tag{5.2.30}$$

The real part of the integral gives finally

$$\left\langle V^2\right\rangle=\frac{\gamma^2}{4}T_c\tau\sin\left(\frac{T_c}{\tau}\right) \tag{5.2.31}$$

The correlation time is approximately of the order of the collision time, i.e., $T_c\approx b/V_*\approx n_*^{-1/3}/V_*$, where n_* is the star density in the black hole case or the galaxy density in the CD galaxy case. The strength of the acceleration is of the order of $\gamma\approx GM_*/n_*^{-2/3}$, where M_* is the mass of the perturbing stars or galaxies, depending on the considered case. Using these approximations, we get finally

$$\left\langle V^2\right\rangle\approx\frac{G^2M_*^2}{4V_*}n_*\tau\sin\left(\frac{1}{n_*^{1/3}V_*\tau}\right) \tag{5.2.32}$$

Let us make some numerical estimates. For a massive black hole in the center of a giant elliptical galaxy where the stellar density is about $10\,pc^{-3}$ and stars have typical velocities of the order of 200 km/s, the dynamical timescale is about 2.2 million years and the correlation timescale is approximately 2300 years. Under these conditions, the square of the mean quadratic velocity acquired by the black hole is only 2.4 cm/s, a quite small value. Hence the binary collisions do not affect the motion of the central black hole. The situation is different in the case of a central galaxy in the core of a cluster like Coma, where the central density is about $10^3\, Mpc^{-3}$ and the relative velocity of the galaxies is about 1500 km/s. These figures correspond to a dynamical timescale of 230 million years and a correlation timescale of 65 million years. In this case, the central galaxy acquires a mean velocity of about 14 km/s and deviates from the central region with a maximum amplitude of about 4.6 kpc, a non-negligible value.

6. The Boltzmann Equation

Consider the one particle distribution function $f(\vec{r},\vec{V},t)$ that gives the probability for a given particle of the system be located at the instant **t** and at the position $\vec{r}$ inside the volume d^3r and with a velocity $\vec{V}$ inside the velocity-volume d^3V. The distribution function is normalized as

$$N = \int_\Gamma f(\vec{r},\vec{V},t)\, d^3r\, d^3V \tag{6.1}$$

where N is the total number of particles constituting the system. The distribution function or the phase-space density is conserved in a collisionless system, namely

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} + \frac{\partial f}{\partial \vec{V}} \cdot \frac{d\vec{V}}{dt} = 0 \tag{6.2}$$

Using the equations of motion, eq.(6.2) can be rewritten as

$$\frac{\partial f}{\partial t} + \vec{V} \cdot \frac{\partial f}{\partial \vec{r}} - \vec{\nabla}\phi \cdot \frac{\partial f}{\partial \vec{V}} = 0 \tag{6.3}$$

The equation above is known as the collisionless Boltzmann equation (CBE) or Vlasov equation. If the distribution function does not depend explicitly on time, i.e., we have a steady situation, the solutions of eq.(6.3) can be expressed in terms of integrals of motion $I_i = I_i(\vec{r},\vec{V})$. This is the Jeans theorem. Note that the vectors $\vec{r}$ and $\vec{V}$ depend on time. If the $I_i' s$ are constant along the particle trajectory we have

$$\frac{dI_i}{dt} = \frac{\partial I_i}{\partial \vec{r}} \cdot \vec{V} - \frac{\partial I_i}{\partial \vec{V}} \cdot \vec{\nabla}\phi = 0 \tag{6.4}$$

Let $f = f(I_i)$ and differentiating this relation one obtains

$$\frac{df}{dt} = \frac{\partial f}{\partial I_i} \frac{\partial I_i}{\partial \vec{r}} \vec{V} - \frac{\partial f}{\partial I_i} \frac{\partial I_i}{\partial \vec{V}} \vec{\nabla}\phi \tag{6.5}$$

Now using eq.(6.4) it results $df / dt = 0$, which proves the Jeans theorem.

As an example, consider a spherical stellar system in which the distribution function depends only on the total energy as

$$f(E) = A|E|^p \tag{6.6}$$

where A and p are constants. The density of stars $n(r)$ is derived simply by integrating the distribution function in the velocity space, i.e.,

$$n(r) = \int f(E)d^3V = 4\pi A \int_0^{V_{max}} \left|\frac{V^2}{2} + \phi\right|^p V^2 dV \tag{6.7}$$

Note that there is a maximum velocity, which is a local quantity, corresponding to the escape velocity at the point r , given by $V_{max}^2 = -2\phi(r)$ or, in other words, corresponding to particles with zero total energy that are not bond to the system. After some algebra (the reader is invited to perform all the missing steps) and integration, one obtains from eq.(6.7)

$$n(r) = (2\pi)^{3/2} A \frac{\Gamma(1+p)}{\Gamma(p+5/2)} (-\phi)^{p+3/2} \tag{6.8}$$

(recall that the potential ϕ is negative). The next step is to calculate the mean quadratic velocity of the stars, which is given by

$$\langle V^2 \rangle = \int V^2 f(E) d^3V / \int f(E) d^3V \tag{6.9}$$

Using eqs.(6.6), (6.7) and (6.8), one obtains after some algebra and integration

$$\langle V^2 \rangle = \frac{3}{(p+5/2)}(-\phi) \tag{6.10}$$

Hence, the local mean quadratic velocity (or the velocity dispersion) tracks the potential. Let *m* be the average stellar mass. In this case, the Poisson equation can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 4\pi G m n(r) \tag{6.11}$$

Substituting eq.(6.8) in the equation above one obtains

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 2(2\pi)^{5/2} GmA \frac{\Gamma(1+p)}{\Gamma(p+5/2)} (-\phi)^{p+5/2} \tag{6.12}$$

Defining the dimensionless variables $\psi = \phi/\phi_0$, where ϕ_0 is the potential at the center and $x = r/r_0$, where r_0 is a suitable scale, eq.(6.12) can be written as

$$\psi'' + \frac{2}{x}\psi' - (-1)^\lambda \psi^\lambda = 0 \tag{6.13}$$

In the equation above we have introduced the notation $\psi'' \equiv d^2\psi/dx^2$ *and* $\psi' \equiv d\psi/dx$ and the parameter $\lambda = p + 3/2$. The equation becomes completely dimensionless by fixing the scale r_0 as

$$r_0^2 = \frac{\Gamma(p+5/2)}{2(2\pi)^{5/2} GmA\phi_0^{p+1/2}\Gamma(p+1)} \tag{6.14}$$

For odd values of λ, the eq.(6.13) is the well-known equation of Lane-Emden which should be solved with the boundary conditions $\psi(0)=1$ *and* $d\psi/dx=0$ at $x = 0$. Exact solutions are known for some cases as, for instance those shown in table 3 below

Table 3

λ = p + 3/2	p	Ψ(x)
1	-1/2	$\sin x/x$
5	7/2	$1/(1+x^2/3)^{1/2}$

The solution p = 7/2 is known as Schuster solution and describes quite well the structure of some globular clusters. Using this solution, the velocity dispersion in the line-of-sight is (isotropic distribution)

$$\sigma_p^2 = \frac{1}{3}\langle V^2\rangle = \left(-\frac{\phi_0}{6}\right)\frac{1}{\left(1+\frac{x^2}{3}\right)^{1/2}} \tag{6.15}$$

In fact, in order to compare with observations, we have to compute the average value of σ_p^2 projected in the line-of-sight and weighted by the light distribution. If the mass-to-light ratio is assumed to be constant, we can use the mass density distribution instead of the light distribution. In particular, for the central region, one obtains using eqs.(6.8) and (6.15)

$$\bar{\sigma}_p^2 = \frac{\int_0^\infty \sigma_p^2 n(r)dr}{\int_0^\infty n(r)dr} = \frac{3\pi}{64}(-\phi_0) \tag{6.16}$$

On the other side, the total mass of the system is given by

$$M = 4\pi\int_0^\infty mn(r)r^2dr = \frac{7\sqrt{3}\pi^3}{8\sqrt{2}} mAr_0^3(-\phi_0)^5 \tag{6.17}$$

where again we have used eq.(6.8). Now, combining eqs.(6.14), (6.16) and (6.17), one obtains respectively for the total mass and the central mass density

$$M = \frac{64}{\sqrt{3}\pi}\frac{r_0\bar{\sigma}_p^2}{G} \tag{6.18}$$

and

$$\rho(0) = \frac{16}{3\pi^2}\frac{\bar{\sigma}_p^2}{Gr_0^2} \tag{6.19}$$

Observations provide the value of the central velocity dispersion and by fitting the project light distribution and/or the velocity dispersion profile, the scale r_0 can be estimated (if the distance to the object is known). Table 4 below gives some examples

Table 4

cluster	$\bar{\sigma}_p\,(km/s)$	r_0 (parsec)	$M/M_{\odot}$	$\rho_0\,(M_{\odot}/pc^3)$	M/L_B
47 Tuc	10.4	4.2	1.2 x 10^6	778	3.1
ω Cen	16.5	3.4	2.5 x 10^6	2930	2.1

If λ is even, eq.(6.13) has singular solutions given by $\psi \propto 1/x^{\beta}$ where $\beta = 4/(1+2p)$. In particular, the case p = 9/2 or λ = 6 and $\beta = 2/5$ is interesting because it predicts a power-law density profile, i.e., $\rho \propto 1/x^{12/5}$ and a phase-space profile $Q \propto 1/x^{9/5}$, which are quite compatible with profiles derived from cosmological simulations for the structure of dark matter halos.

These results can be also applied to estimate the density profile of stars in the core of a galaxy that are under the gravitational influence of a central massive black hole. The influence radius is fixed by the equality between the kinetic energy of the star and its interaction gravitational energy with the black hole, i.e., $r_{in} = GM/\sigma^2$, where M is the black hole mass and σ is the velocity dispersion of the stars. The black hole captures stars having angular momentum less than the critical value $J_* = 4GM/c$ (for a Schwarzschild black hole) and the phase-space volume defined by $J \le J_*$ is dubbed the "loss-cone". The infall of these stars in eccentric orbits onto the black hole is set by the rate at which the relaxation processes repopulate the "loss-cone" orbits. If a steady situation is reached and a power-law for the energy distribution of bond stars is assume, then from eq.(6.8) the stellar density must have a profile of the form

$$n(r) \propto \frac{1}{r^{p+3/2}} \tag{6.20}$$

On the other side, if the "loss cone" orbits are repopulated via a violent relaxation process, the relaxation time is proportional to $t_{RV} \propto 1/n^{1/2} \propto r^{p/2+3/4}$ (see Section 4). Therefore, the flux of stars captured by the black hole is

$$\Phi \propto \frac{nr^3}{t_{RV}} \propto r^{3/4-3p/2} \tag{6.21}$$

In a steady situation, the flux should be constant along distances less the influence radius. Thus p = ½ and consequently, $n \propto r^{-2}$. This implies a logarithmic density slope equal to -2.0.

It is interesting to remark that dwarf ellipticals have near the core a slope of -1.8 ± 0.5, close to the expected for stellar profile influenced by a central massive black hole.

6.1 Spherical systems

If the potential depends only on the radial coordinate *r* and the system is in a steady state, the collisionless Boltzmann equation can be written as

$$V_r \frac{\partial f}{\partial r} + \left(\frac{V_\theta^2 + V_\varphi^2}{r} - \frac{\partial \phi}{\partial r} \right) \frac{\partial f}{\partial V_r} + \left(\frac{V_\varphi^2 \cot\theta}{r} - \frac{V_r V_\theta}{r} \right) \frac{\partial f}{\partial V_\theta} - \left(\frac{V_r V_\varphi}{r} + \frac{V_\theta V_\varphi}{r} \cot\theta \right) \frac{\partial f}{\partial V_\varphi} = 0 \qquad (6.1.1)$$

In this case, the characteristic equations of (6.1.1) are

$$V_r dV_r = \frac{1}{r}\left(V_\theta^2 + V_\varphi^2\right) dr - \frac{\partial \phi}{\partial r} dr \qquad (6.1.2)$$

$$V_r dV_\theta = \frac{1}{r}\left(V_\varphi^2 \cot\theta - V_r V_\theta\right) dr \qquad (6.1.3)$$

and

$$V_r dV_\varphi = -\frac{1}{r}\left(V_r V_\varphi + V_\varphi V_\theta \cot\theta\right) dr \qquad (6.1.4)$$

Now multiply eq.(6.1.3) by V_θ, eq.(6.1.4) by V_φ and add both relations, we get

$$V_\theta dV_\theta + V_\varphi dV_\varphi = -\frac{1}{r}\left(V_\theta^2 + V_\varphi^2\right) dr \qquad (6.1.5)$$

Substitute the equation above into eq.(6.1.2) and integrate to obtain

$$\frac{1}{2}\left(V_r^2 + V_\theta^2 + V_\varphi^2\right) + \phi = E \qquad (6.1.6)$$

where E is an integration constant equal to the total particle energy . Introducing the tangential velocity V_t by the relation

$$V_t^2 = V_\theta^2 + V_\varphi^2 \qquad (6.1.7)$$

Differentiate the equation above to get

$$V_t dV_t = V_\theta dV_\theta + V_\varphi dV_\varphi \qquad (6.1.8)$$

Using eqs.(6.17) and (6.1.8), the equation (6.1.5) can be recast as

$$V_t dV_t = -\frac{V_t^2}{r}dr \tag{6.1.9}$$

The integration of the equation above is immediate and introduces another conserved quantity, the angular momentum

$$J = rV_t \tag{6.1.10}$$

By the Jeans theorem, the distribution function depends on the integrals of motion. So, we can write

$$f(E, J^2) = \exp\left[-\left(\alpha E + \beta J^2\right)\right] \equiv \exp\left[-G\right] \tag{6.1.11}$$

where α and β are constants. Note that the spherical symmetry requires that the dependence on the angular momentum be with the square of this quantity. Using the expressions for the energy (eq. 6.1.6) and the angular momentum (eq. 6.1.10), the function G in terms of coordinates and velocities can be written as

$$G = \frac{\alpha}{2}V_r^2 + \left(\frac{\alpha}{2} + \beta r^2\right)\left(V_\theta^2 + V_\varphi^2\right) + \alpha\phi \tag{6.1.12}$$

It can be verified easily that the velocity dispersion components are given by

$$\sigma_r^2 = \frac{1}{\alpha} \;\; and \;\; \sigma_\theta^2 = \sigma_\varphi^2 = \frac{1}{\alpha + 2\beta r^2} \tag{6.1.13}$$

In this case, the radial velocity dispersion is constant but the ratio with the other components is not, i.e.,

$$\frac{\sigma_\theta^2}{\sigma_r^2} = \frac{\sigma_\varphi^2}{\sigma_r^2} = \frac{\alpha}{\alpha + 2\beta r^2} \tag{6.1.14}$$

The relations above say that if $\beta = 0$ or in the central regions when $r \to 0$, the velocity distribution is isotropic. On the other side, if $\beta \neq 0$, in the regions far from the center, the orbits are essentially radial. In fact, during the gravitational collapse of an initially spherical system, energy from the radial bulk motion is transferred to random motions (a characteristic of the violent relaxation mechanism), producing radial instabilities due to an anisotropic velocity distribution. The study of these instabilities is presented in great detail in the book by A.M. Fridman & V.I. Polyachenko, "Physics of Gravitating Systems" (Springer-Verlag).

Figure 4

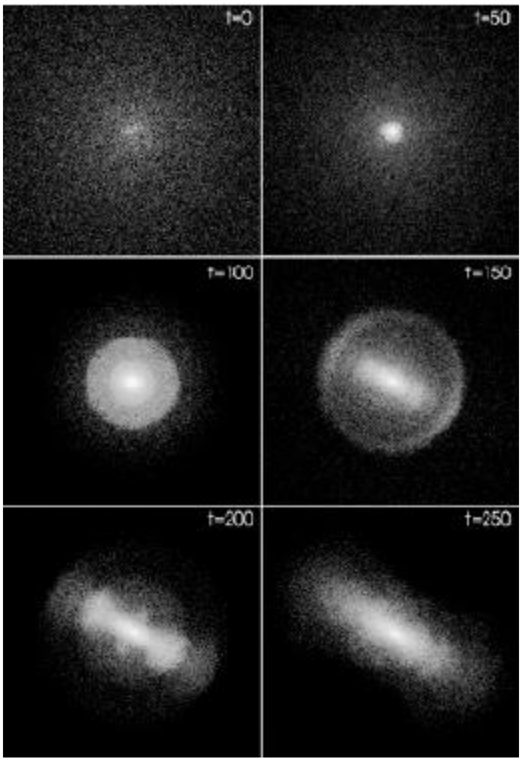


Figure 4 above shows snapshots of different phases of the gravitational collapse of an initially spherical and cold (zero velocity dispersion) system derived from numerical simulations. As first demonstrated by Hénon (A&A 24, 229, 1973) and van Albada (MNRAS 201, 939, 1982), such an infall is subjected to violent instabilities. This is because self-gravity in the tangential directions bunch the radial orbits together, disturbing sphericity and producing a flattened prolate structure. Numerical simulations also indicate that if, in the one hand, the initial kinetic energy is high the resulting density profile can be fitted either by a de Vaucouleurs law or a Sersic family of profiles, which represent quite well elliptical galaxies. In the other hand, if the initial kinetic energy is low, the resulting density profile can be fitted by a Navarro-Frenk-White profile, which represents the mass distribution of dark halos observed in cosmological simulations.

The Jeans equation in spherical symmetry are derived easily in the following way: first put together the terms in $\cot\theta$ appearing in eq.(6.1.1). Then, since the distribution function does not depends on θ, put the resulting term equal to zero or, in other words

$$\left(V_\varphi \frac{\partial f}{\partial V_\theta} - V_\theta \frac{\partial f}{\partial V_\varphi}\right)\cot\theta = 0 \;\rightarrow\; V_\varphi \frac{\partial f}{\partial V_\theta} = V_\theta \frac{\partial f}{\partial V_\varphi} \tag{6.1.15}$$

Making use of this result and of the relation

$$V_\theta \frac{\partial f}{\partial V_\theta} + V_\varphi \frac{\partial f}{\partial V_\varphi} = V_t \frac{\partial f}{\partial V_t} \tag{6.1.16}$$

eq.(6.1.1) can be rewritten as

$$V_r \frac{\partial f}{\partial r} + \left(\frac{V_t^2}{r} - \frac{\partial \phi}{\partial r} \right) \frac{\partial f}{\partial V_r} - \frac{V_r V_t}{r} \frac{\partial f}{\partial V_t} = 0 \qquad (6.1.17)$$

Multiply this equation by V_r and integrate in the velocity space. The first term gives

$$\int V_r^2 \frac{\partial f}{\partial r} d^3V = \frac{\partial}{\partial r} \int V_r^2 f d^3V = \frac{\partial}{\partial r} \left(n_* \sigma_r^2 \right) \qquad (6.1.18)$$

The first integral of the second term gives

$$\int \frac{V_r V_t^2}{r} \frac{\partial f}{\partial V_r} d^3V = \frac{1}{r} \left[\int V_t^2 V_r d^2V \, f \Big|_{-\infty}^{\infty} - \int V_t^2 f d^3V \right] = -\frac{1}{r} n_* \sigma_t^2 \qquad (6.1.19a)$$

while the second integral of the second term gives

$$\int V_r \frac{\partial \phi}{\partial r} \frac{\partial f}{\partial V_r} d^3V = \frac{\partial \phi}{\partial r} \left[\int V_r \, d^2V f \Big|_{-\infty}^{\infty} - \int f d^3V \right] = -n_* \frac{\partial \phi}{\partial r} \qquad (6.1.19b)$$

Making use of (6.1.16), the integral of the third and last term is

$$\int \frac{V_r^2}{r} \left(V_\theta \frac{\partial f}{\partial V_\theta} + V_\varphi \frac{\partial f}{\partial V_\varphi} \right) d^3V = -2 \frac{n_* \sigma_r^2}{r} \qquad (6.1.20)$$

Since both terms in the equation above give the same result after integration by parts. Finally, grouping eqs.(6.1.18), (6.1.19) and (6.1.20) gives

$$\frac{\partial}{\partial r} \left(n_* \sigma_r^2 \right) + \frac{n_*}{r} \left(2\sigma_r^2 - \sigma_t^2 \right) + n_* \frac{\partial \phi}{\partial r} = 0 \qquad (6.1.21)$$

It is worth mentioning that in the equation above, n_* and σ_i are respectively the density and the velocity dispersion components of a given stellar population whereas the potential corresponds to the contribution of all stellar populations (including dark matter) present in the galaxy.

Here is given an application of the Jeans equation (6.1.21), which permits to estimate the central mass density of elliptical galaxies at high redshift (Biressa & de Freitas Pacheco, Gen.Relativ.&Gravit. 43, 2649, 2011). The dark matter halo of these galaxies act as a gravitational lens producing Einstein ring images of objects located far beyond. The halo of these galaxies are modeled by a non-singular dark matter profile given by

$$\rho(r) = \frac{K}{(R^2 + r^2)} \qquad (6.1.22)$$

where K is a constant and R is the so-called core radius. Supposing a isotropic velocity distribution, the mass distribution above and the Jeans equation (6.1.21) gives for the velocity dispersion profile

$$\sigma^2(r) = \frac{4\pi GK}{R^2}\left(R^2 + r^2\right)\left[\frac{\pi^2}{8} - \frac{arctg(r/R)}{(r/R)} - \frac{1}{2}arctg^2(r/R)\right] \qquad (6.1.23)$$

As mentioned before, observers measure the velocity dispersion projected in the line-of-sight and weighted by the luminosity (mass distribution), i.e., $\bar{\sigma}^2 = \int \sigma^2(r)\rho(r)dr / \int \rho(r)dr$. Using eqs.(6.1.22) and (6.1.23) one obtains for the central projected velocity dispersion

$$\bar{\sigma}^2 \cong 1.222\pi GK \qquad (6.1.24)$$

This equation permits to express the constant K in terms of the projected central velocity dispersion, an observed quantity. Under these conditions, the Einstein ring angular radius is given by (see Biressa & de Freitas Pacheco, cited)

$$\frac{\theta_E^2}{\left(\sqrt{1+u^2\theta_E^2}-1\right)} \cong 6.547\pi R\left(\frac{\bar{\sigma}^2}{c^2}\right)\frac{D_{LS}}{uD_S} \qquad (6.1.25)$$

In this equation $u = D_L / R$, D_L, D_{LS} *and* D_S are respectively the angular distance to the lens, the lens to source and to the source. In the equation above, if the Einstein angular radius is measured and the velocity dispersion as well as the redshift of the lens and of the source are known, the core radius R can be computed and then the central density from eqs.(6.1.22) and (6.1.24).

Table 5

Object	θ_E	$\bar{\sigma}(km/s)$	z_{lens}	z_{source}	R(kpc)	$\rho(M_\odot pc^{-3})$	$Q(M_\odot pc^{-3}km^3s^{-3})$
Q0047	1.4”	229	0.48	3.60	2.49	0.43	3.58 x 10^{-8}
CFR803	1.1”	256	0.94	2.94	3.40	0.30	1.78 x 10^{-8}
MG1549	0.9”	227	0.11	1.17	1.78	0.94	8.04 x 10^{-8}

In table 5, the first column gives the lensed object according to the catalog COSMOS (Faure et al., ApJS 176, 19, 2008), the second column gives the measured radius of the Einstein ring in arcsec (opus cited), the fourth and the fifth columns give respectively the redshift of the lens and of the source. The third column gives the measured central stellar velocity dispersion, supposed to be equal to the velocity dispersion of dark matter particles, since this is expected from the violent relaxation mechanism. The remaining columns give the derived quantities: the core radius of the halo, the central density and the central phase-space density.

6.2 Systems with cylindrical symmetry

Systems with cylindrical symmetry have a potential of the form $\phi = \phi(R, Z)$, where R is the radial coordinate in the plane xy and Z the coordinate along an axis perpendicular to the plane xy. In this case, for a steady state system, the collisionless Boltzmann equation can be written as

$$V_R \frac{\partial f}{\partial R} + V_Z \frac{\partial f}{\partial Z} + \left(\frac{V_\theta^2}{R} - \frac{\partial \phi}{\partial R} \right) \frac{\partial f}{\partial V_R} - \frac{V_R V_\theta}{R} \frac{\partial f}{\partial V_\theta} - \frac{\partial \phi}{\partial Z} \frac{\partial f}{\partial V_Z} = 0 \qquad (6.2.1)$$

The characteristic equations of this equation are

$$V_R dV_R = \left(\frac{V_\theta^2}{R} - \frac{\partial \phi}{\partial R} \right) dR \qquad (6.2.2)$$

$$-V_R dV_\theta = \frac{V_R V_\theta}{R} dR \qquad (6.2.3)$$

and

$$V_Z dV_Z = -\frac{\partial \phi}{\partial Z} dZ \qquad (6.2.4)$$

Equation (6.2.4) can be integrated immediately giving the energy conservation along the Z-axis

$$\frac{1}{2}\left(V_R^2 + V_\theta^2 \right) + \phi = C_2(R) \qquad (6.2.5)$$

Note that the integration "constant depends on the radial coordinate. Thus, the energy constant along the Z-axis depends on R. Similarly, eq.(6.2.3) can be integrated giving the angular momentum conservation with respect to the Z-axis, i.e.,

$$J_Z = V_\theta R \qquad (6.2.6)$$

Now multiply eq.(6.2.2) by V_R and eq.(6.2.3) by V_θ. Then subtract one from another and integrate to obtain

$$\frac{1}{2}\left(V_R^2 + V_\theta^2 \right) + \phi = C_2(Z) \qquad (6.2.7)$$

which expresses the energy conservation in the xy plane. Again the integration "constant" here is not a true constant but depends on the coordinate Z. In order to obtain the true energy conservation proceeds as before, namely, multiply eq.(6.2.2) by V_R and eq.(6.2.3) by V_θ. Subtract one from another, add side by side eq.(6.2.4) and then integrate to obtain

$$\frac{1}{2}\left(V_R^2 + V_\theta^2 + V_Z^2 \right) + \phi = E \qquad (6.2.8)$$

In this case, the distribution function will be a linear combination of the integrals of the movement, namely,

$$G = \alpha E + \beta J_Z + \gamma J_Z^2 \qquad (6.2.9)$$

Using eqs.(6.26) and (6.2.8), the equation above can be rewritten as

$$G=\frac{\alpha}{2}\left(V_R^2+V_Z^2\right)+\frac{\alpha}{2}\left(1+\frac{2\gamma}{\alpha}R^2\right)\left(V_\theta-V_c\right)^2-\frac{\beta^2R^2}{2(\alpha+2\gamma R^2)}+\alpha\phi \tag{6.2.10}$$

where we have defined

$$V_c=\frac{\beta R}{(\alpha+2\gamma R^2)} \tag{6.2.11}$$

as the mean circular velocity of the bulk motion at R. Note that this general solution implies that the velocity dispersion along the radial direction and along the Z-axis are constant and equal. If $\gamma\neq 0$, the velocity distribution is quasi-isotropic near the central regions and anisotropic far beyond.

The Jeans equation in cylindrical symmetry is obtained in a similar way since the collisionless Boltzmann equation along the Z-axis can be decoupled as

$$V_Z\frac{\partial f}{\partial Z}-\frac{\partial\phi}{\partial Z}\frac{\partial f}{\partial V_z}=0 \tag{6.2.12}$$

Multiplying the above equation by V_Z and integrating over velocities one obtains easily

$$\frac{\partial}{\partial Z}\left(n_*\sigma_Z^2\right)+n_*\frac{\partial\phi}{\partial Z}=0 \tag{6.2.13}$$

The remaining equation is

$$V_R\frac{\partial f}{\partial R}+\left(\frac{V_\theta^2}{R}-\frac{\partial\phi}{\partial R}\right)\frac{\partial f}{\partial V_R}-\frac{V_RV_\theta}{R}\frac{\partial f}{\partial V_\theta}=0 \tag{6.2.14}$$

Again, multiplying eq.(6.214) by V_R and integrating in velocity space (steps are similar as in the previous case and the reader is invited to perform in detail the integration of all terms) one obtains

$$\frac{\partial}{\partial R}\left(n_*\sigma_R^2\right)+\frac{n_*}{R}\left(\sigma_R^2-\sigma_\theta^2-V_c^2\right)+n_*\frac{\partial\phi}{\partial R}=0 \tag{6.2.15}$$

Note that the circular velocity appearing in eq.(6.215) is that of the mean circular motion. The expected circular velocity is defined as $V_{cir}^2=R(\partial\phi/\partial R)$. Using the equation above one obtains

$$V_c^2-V_{cir}^2=\left(\sigma_R^2-\sigma_\theta^2\right)+\frac{R}{n_*}\frac{\partial}{\partial R}\left(n_*\sigma_R^2\right) \tag{6.2.16}$$

which expresses the so-called velocity ”lag” of the circular motion.

As an application of these equations, we will estimate the dark matter density in the solar neighborhood. The motion of stars of a given population along the Z-axis is decoupled and is governed by eq.(6.2.13). On the other side, the gravitational potential obeys the Poisson equation, which in cylindrical coordinates can be written as

$$\frac{\partial^2\phi}{\partial R^2}+\frac{1}{R}\frac{\partial\phi}{\partial R}+\frac{\partial^2\phi}{\partial Z^2}=4\pi G\rho_t \qquad (6.2.17)$$

Recall that the density in the right side of eq.(6.2.17) includes all matter components, namely, baryons (stars+gas) and dark matter. Introducing now the Oort constants *A* and *B* defined by

$$A=\frac{1}{4\omega}\left(\frac{\partial^2\phi}{\partial R^2}-\frac{1}{R}\frac{\partial\phi}{\partial R}\right) \qquad (6.2.18)$$

and

$$B=\frac{1}{4\omega}\left(\frac{\partial^2\phi}{\partial R^2}+\frac{3}{R}\frac{\partial\phi}{\partial R}\right) \qquad (6.2.19)$$

with $\omega=A-B$. After some algebra (the reader is invited once more to perform all the missing steps) one obtains

$$2\left(A^2-B^2\right)=\frac{\partial^2\phi}{\partial R^2}+\frac{1}{R}\frac{\partial\phi}{\partial R} \qquad (6.2.20)$$

Substituting this relation in the Poisson equation one gets

$$2\left(A^2-B^2\right)+\frac{\partial^2\phi}{\partial Z^2}=4\pi G\rho_t \qquad (6.2.21)$$

The equation above can be recast in an alternative form since the acceleration along the Z-axis is $K_Z=-\partial\phi/\partial Z$. In other words

$$2\left(A^2-B^2\right)-\frac{\partial K_Z}{\partial Z}=4\pi G\rho_t \qquad (6.2.22)$$

Studies of F giants give for the acceleration gradient in the galactic plane $(\partial K_Z/\partial Z)=-6.18\times10^{-30}\,s^{-2}$ while Hipparco's data, derived from F and A stars indicate $A=14.8\,km/s/kpc$ *and* $B=-12.4\,km/s/kpc$. Substituting these values in eq.(6.2.22) one obtains for the **total** matter density near the Sun $\rho_t=0.110\pm0.03M_\odot pc^{-3}$. On the other side, the estimated contribution of baryonic matter corresponds to $\rho_b=0.09\pm0.02\,M_\odot pc^{-3}$. Thus, the expected dark matter density in the solar vicinity is around $\rho_{dm}=0.020\pm0.036\,M_\odot pc^{-3}$. Note that when the uncertainties are taken into account the dark matter density is also consistent with a ***zero*** density! For more details on the Jeans equations in cylindrical symmetry applied to the dynamics of the Galaxy, we suggest to the reader the series of papers by Kuijken & Gilmore (MNRAS 239, 571, 1989; idem 239, 605, 1989; idem 239, 651, 1989)

7. The Fokker-Planck equation

When collisions are important, the Boltzmann equation has a right side term that takes into account the interaction between particles. Under this condition, the distribution function evolves with time, reaching a statistical equilibrium state only after a time interval greater than the relaxation timescale. In this case, the solution of the Boltzmann equation is quite complex and, in general, is obtained under different approximations as, for instance, the Fokker-Planck equation.

Consider a stochastic process in which a random force acting in a test particle produces a random acceleration $\vec{A}(t,\vec{v})$. This term includes not only the random force due to collisions with nearby particles but also the interaction with the medium ("viscous" forces). In this case, the equation of motion of the test particle is

$$\frac{d\vec{v}}{dt} = \vec{A}(t,\vec{v}) \tag{7.1}$$

Let $\vec{R}(\vec{v}) = \left\langle \vec{A}(t,\vec{v}) \right\rangle$ the average value of the random acceleration, computed in a time interval smaller than the correlation timescale. From eq.(7.1), the expected variation of the particle velocity in a time interval Δt is

$$\Delta\vec{v} = \int_t^{t+\Delta t} \vec{A}(\tau,\vec{v})d\tau \tag{7.2}$$

Performing a time average (within a time interval less than Δt) and considering the limit

$$\lim_{\Delta t\to 0}\frac{\left\langle \Delta\vec{v}\right\rangle}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left\langle \vec{A}(\tau,\vec{v})\right\rangle d\tau = \vec{R}(\vec{v}) \tag{7.3}$$

In a similar way

$$\lim_{\Delta t\to 0}\frac{\left\langle \Delta\vec{v}\cdot\Delta\vec{v}\right\rangle}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left\langle \vec{A}(\tau_1,\vec{v})\cdot\vec{A}(\tau_2,\vec{v})\right\rangle d\tau_1 d\tau_2 = 2D(\vec{v}) \tag{7.4}$$

The eq.(7.4) defines the diffusion coefficient in the velocity space, as we shall see later. Let us introduce now the probability density $P(\vec{v},\vec{v}',\Delta t)$. This measures the probability for a given particle, having an initial velocity $\vec{v}$, has after a time interval Δt a velocity $\vec{v}'$. This probability density satisfies the normalization condition

$$\int P(\vec{v},\vec{v}',\Delta t)d^3v = \int P(\vec{v},\vec{v}',\Delta t)d^3v' = 1 \tag{7.5}$$

With these definitions the distribution function evolves as

$$f(\vec{v},t+\Delta t) = \int f(\vec{v}',t)P(\vec{v},\vec{v}',\Delta t)d^3v' \tag{7.6}$$

The equation above is a fundamental equation describing the evolution of a stochastic process. Let now $\varphi(\vec{v})$ be an arbitrary function that is continuous and differentiable. Consider the following identity

$$\int \varphi(\vec{v}) \frac{\partial f(\vec{v},t)}{\partial t} d^3v = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\int \varphi(\vec{v}) f(\vec{v},t+\Delta t) d^3v - \int \varphi(\vec{v}) f(\vec{v},t) d^3v\right] \tag{7.7}$$

Substitute now eq.(7.6) into the first integral in the right side of the equation above gives

$$\int \varphi(\vec{v}) d^3v \int f(\vec{v}',t) P(\vec{v},\vec{v}',\Delta t) d^3v' - \int \varphi(\vec{v}) f(\vec{v},t) d^3v \tag{7.8}$$

Expand the arbitrary function $\varphi(\vec{v})$ in Taylor series ($\vec{v} = \vec{v}' + \Delta\vec{v}$) as

$$\varphi(\vec{v}) = \varphi(\vec{v}') + \Delta t\left[\left(\frac{\partial \varphi}{\partial \vec{v}'}\right)\left(\frac{\Delta \vec{v}}{\Delta t}\right) + \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial \vec{v}'\partial \vec{v}'}\right)\left(\frac{\Delta\vec{v}\cdot\Delta\vec{v}}{\Delta t}\right) + \dots\right] \tag{7.9}$$

Performing a time average on the velocity variation and using the definitions (7.3) and (7.4) one obtains

$$\varphi(\vec{v}) = \varphi(\vec{v}') + \Delta t\left[\left(\frac{\partial \varphi}{\partial \vec{v}'}\right) R(\vec{v}') + \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial \vec{v}'\partial \vec{v}'}\right) D(v') + \dots\right] \tag{7.10}$$

Now replace this equation into the first integral of eq.(7.8) to get

$$\int \varphi(\vec{v}') f(\vec{v}',t) d^3v' \int P(\vec{v},\vec{v}',\Delta t) d^3v + \int P(\vec{v},\vec{v}',\Delta t) d^3v \int f(\vec{v}',t)\left[R(\vec{v}')\frac{\partial \varphi}{\partial \vec{v}'} + D(\vec{v}')\frac{\partial^2 \varphi}{\partial \vec{v}'\partial \vec{v}'}\right]\Delta t d^3v'$$
$$-\int \varphi(\vec{v}) f(v',t) d^3v \tag{7.11}$$

Integrate the first and the second integrals in d^3v under the normalization condition (7.5) to obtain finally after simplification

$$\Delta t\int d^3v' f(\vec{v}',t)\left[R(\vec{v}')\frac{\partial \varphi}{\partial \vec{v}'} + D(\vec{v}')\frac{\partial^2 \varphi}{\partial \vec{v}'\partial \vec{v}'}\right]$$

(7.12)

Integrating by parts the first term of the equation above and twice the second term one obtains

$$\Delta t\int d^3v'\left[-\frac{\partial (Rf)}{\partial \vec{v}'} + \frac{\partial^2 (Df)}{\partial \vec{v}'\partial \vec{v}'}\right]\varphi(\vec{v}') \tag{7.13}$$

Substituting this equation into (7.7) we get

$$\int \varphi(\vec{v}) \frac{\partial f}{\partial t} d^3v = -\int\left[\frac{\partial (Rf)}{\partial \vec{v}} - \frac{\partial^2 (Df)}{\partial \vec{v}\partial \vec{v}}\right]\varphi(\vec{v}) d^3v \tag{7.14}$$

Due to the arbitrariness of the function $\varphi(\vec{v})$ one obtains finally

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \vec{v}}\left[R(\vec{v})\, f(\vec{v},t) - \frac{\partial}{\partial \vec{v}}\left(D(\vec{v})\, f(\vec{v},t)\right)\right] \tag{7.15}$$

Equation (7.15) is the Fokker-Planck (F-P) equation in velocity space. It gives an approximate solution for the evolution of the distribution function in the velocity space, including a viscous term $R(\vec{v}) = -\beta\vec{v}$, which represents the interaction of the test particle with the ambient medium (interaction with the "mean field" due to the remaining particles) and a diffusion term $D(\vec{v})$, which describes binary collisions with nearby particles. The coefficient β, the dynamical viscosity coefficient (Chandrasekhar, ApJ 97, 255, 1943; idem 98, 54, 1943) is in general a function of the velocity of the particle with respect to the ambient medium. An approximate expression for the dynamical viscosity coefficient is

$$\beta(\vec{v}) = 4\pi\rho_* \frac{G^2 m}{V^3} \lg \Lambda \tag{7.16}$$

where

$$\Lambda = \frac{\sigma_*^2}{Gmn_*^{1/3}} \tag{7.17}$$

In these equations, ρ_*, n_*, σ_* are respectively the mass density, the particle density and the velocity dispersion of the particles constituting the ambient medium. This expression for the dynamical viscosity coefficient requires that the test particle mass *m* be much larger than the mass of those constituting the medium. *V* is the relative velocity between the test particle and the medium.

As a simple example of solution of the F-P equation, consider the case in which β, D are constants and that initially all particles of the system have the same velocity V_0. Under this condition, the initial distribution function is

$$f(v,0) = K\delta(v - v_0) \tag{7.18}$$

where *K* is a normalization constant. In this case, the solution of the F-P is simply

$$f(v,t) = K\left[\frac{\beta}{2\pi D\left(1-e^{-2\beta t}\right)}\right]^{1/2} \exp\left[-\frac{\beta(v - v_0 e^{-\beta t})^2}{2D(1-e^{-2\beta t})}\right] \tag{7.19}$$

This solution is a Gaussian whose median is $v_{med} = v_0 e^{-\beta t}$ and the dispersion is $\sigma^2 = \frac{D}{\beta}\left(1-e^{-2\beta t}\right)$. When $t \to \infty$, the median goes to zero and the velocity dispersion becomes constant, equal to the ratio between the diffusion and the viscosity coefficients. The distribution function is a pure Gaussian, i.e.,

$$f(v,\infty) = K\sqrt{\frac{\beta}{2\pi D}}\exp\left[-\frac{\beta v^2}{2D}\right] \qquad (7.20)$$

When $\beta = 0$ the distribution function evolves only due to a diffusion process in the velocity space driven by binary collisions with nearby particles or, in other words,

$$f(v,t) = \frac{K}{\sqrt{4\pi Dt}}\exp\left[-\frac{(v-v_0)^2}{4Dt}\right] \qquad (7.21)$$

In this situation, the distribution function is still a Gaussian in which the median is equal to the initial velocity of the particles but the velocity dispersion increases with time as $\sigma^2 = 2Dt$. In the case in which *D = 0*, there is no diffusion in the velocity space but the velocity of the particles decrease due to the dynamical viscosity since the distribution function evolves as

$$f(v,t) = K\delta(v - v_0 e^{-\beta t}) \qquad (7.22)$$

Note that the apparent violation of the energy conservation is not real. In a gravitational system, a small lost in the kinetic energy produces a contraction which "reheats" the particles, maintaining a situation of quasi-equilibrium, which will end with the total collapse of the system. Note also if eq.(7.18) is put into (7.6) one obtains for the probability density *P*

$$P(v,v_0,t) = \sqrt{\frac{\beta}{2\pi D(1-e^{-2\beta t})}}\exp\left[-\frac{\beta(v-v_0e^{-\beta t})^2}{2D(1-e^{-2\beta t})}\right] \qquad (7.23)$$

This expression gives the probability for a particle having initially a velocity v_0 be found with a velocity v at the instant *t*.

As an application of the dynamical friction, consider the merger of a galaxy with one of its satellites. During the process of fusion, the central black hole will be probably ejected from the center of the main galaxy. Then, it will orbit in a stellar halo, supposed here to have an isothermal structure, whose density profile is given by

$$\rho(r) = \rho_0\left(\frac{r_0}{r}\right)^2 \qquad (7.24)$$

The black hole, supposed to be in a circular orbit, has a constant velocity V_c given by

$$V_c^2 = 4\pi G\rho_0 r_0^2 \qquad (7.25)$$

The viscous force per unit of mass acting on the black hole is

$$F_v = -\beta V_c = -4\pi G^2 M\rho(r)\frac{\lg\Lambda}{V_c^2} \qquad (7.26)$$

where *M* is the black hole mass and we have make use of (7.16). Combining eqs.(7.24), (7.25) and (7.26) gives for the viscous force per unit of mass

$$F_v = -\frac{GM}{r^2}\lg\Lambda \quad (7.27)$$

Although the orbital velocity of the black hole remains constant in an "isothermal" halo, the dynamical viscosity produces a torque that decreases the angular momentum of the black hole, leading to an inspiral motion (since angular momentum is a conserved quantity, what is lost by the black hole is transferred to the medium). Thus, the variation of the angular momentum of the black hole is

$$\frac{d\vec{J}}{dt} = \vec{F}_v \cdot \vec{r} \quad or \quad V_c \frac{dr}{dt} = -\frac{GM}{r}\lg\Lambda \quad (7.28)$$

The equation above can be integrated easily if one assumes an average value for Λ, neglecting its variation with the radial coordinate, since we are interested only in an order of magnitude estimate of the inspiral timescale. If r_{in} is the initial distance of the black hole from the center, the time required for reaching the center is

$$t_v = \frac{r_{in}^2 V_c}{2GM\lg\Lambda} \quad (7.29)$$

For typical values: V_c = 200 km/s, M = $10^8 M_\odot$, $r_{in} = 10\,kpc$ and $\lg\Lambda = 6.0$ one obtains an inspiral timescale of 4 billion years. This indicates that the dynamical viscosity plays a fundamental role to keep very massive black holes at the center of galaxies.

7.1 The viscosity coefficient in the velocity space

In order to compute the diffusion coefficient in the velocity space, we consider a system where particles interact only through gravitational forces. In a region of dimension $L \propto n^{-1/3}$ the variations of the mean field are supposed to be very small or negligible while random forces due to binary collisions are responsible for variations in the velocity of the particles. In these circumstances, the Boltzmann equation has a right side term different from zero, which represents the aforementioned binary interactions. In other words

$$\frac{\partial f}{\partial t} + \vec{V}\cdot\frac{\partial f}{\partial \vec{r}} + \vec{g}\cdot\frac{\partial f}{\partial \vec{V}} = -\left(\frac{\partial f}{\partial t}\right)_{coll} = -\nu\left(f - f_{eq}\right) \quad (7.1.1)$$

Note that the "collision" term in the right side was approximated by the collision frequency ν times the difference between the real value of the distribution function with that expected in a state of equilibrium. We will be interested in the limit in which $\nu \to 0$.

Consider now that the distribution function $f(t,\vec{r},\vec{V})$ could be expressed as a sum of two terms: the first one $f_0(t,\vec{V})$, homogeneous over the considered scale *L* and slowly varying in time and a second one $f_1(t,\vec{r},\vec{V})$, which varies very rapidly inside the scale *L*. Thus

$$f(t,\vec{r},\vec{V}) = f_0(t,\vec{V}) + f_1(t,\vec{r},\vec{V}) \tag{7.1.2}$$

Substitute eq.(7.1.2) into eq.(7.1.1) and separate in two equivalent equations as follows

$$\frac{\partial f_1}{\partial t} + \vec{V}\cdot\frac{\partial f_1}{\partial \vec{r}} + \vec{g}\cdot\frac{\partial f_0}{\partial \vec{V}} = -\nu f_1 \tag{7.1.3}$$

and

$$\frac{\partial f_0}{\partial t} + \vec{g}\cdot\frac{\partial f_1}{\partial \vec{V}} = -\nu(f_0 - f_{eq}) \tag{7.1.4}$$

Assume that the fast varying component could be expressed as

$$f_1(t,\vec{r},\vec{V}) = e^{-2\pi i\omega t}\int_{-\infty}^{\infty}\tilde{f}_1(\vec{V},\vec{k})e^{2\pi i\vec{k}\cdot\vec{r}}d^3k \tag{7.1.5}$$

In a similar way, we express the acceleration due to binary collisions as a sum of Fourier components

$$\vec{g}(\vec{r}) = \int_{-\infty}^{\infty}\tilde{g}(\vec{k})e^{2\pi i\vec{k}\cdot\vec{r}}d^3k \tag{7.1.6}$$

Now replace eqs.(7.15) and (7.1.6) into eq.(7.1.3) and after some algebra, one obtains for the fast component

$$\tilde{f}_1(\vec{V},\vec{k}) = -\tilde{g}(\vec{k})\cdot\frac{\partial f_0}{\partial \vec{V}}\frac{e^{2\pi i\omega t}}{\left[\nu - 2\pi i(\omega - \vec{k}\cdot\vec{V})\right]} \tag{7.1.7}$$

In the next step we perform a space average of eq.(7.1.4) over a volume L^3, i.e.,

$$\left\langle\frac{\partial f_0}{\partial t}\right\rangle + \left\langle\vec{g}\cdot\frac{\partial f_1}{\partial \vec{V}}\right\rangle = -\left\langle\nu(f_0 - f_{eq})\right\rangle = 0 \tag{7.1.8}$$

The right side average is equal to zero because on the considered volume the average value of f_0 approaches the equilibrium value of the distribution function.

Now replace into (7.1.8) the acceleration given by (7.1.6) and the fast component of the distribution function given by (7.1.5) and (7.1.7). After rearranging the terms one obtains

$$\frac{\partial\langle f_0\rangle}{\partial t} = \frac{1}{L^3}\frac{\partial}{\partial \vec{V}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{L^3} d^3r\, d^3k\, d^3k'\,\tilde{g}(\vec{k})\tilde{g}(\vec{k}')\frac{\partial f_0}{\partial \vec{V}}\frac{e^{2\pi i(\vec{k}+\vec{k}')\cdot\vec{r}}}{\left[\nu - 2\pi i(\omega - \vec{k}'\cdot\vec{V})\right]} \tag{7.1.9}$$

Integrating first over space

$$\frac{\partial\langle f_0\rangle}{\partial t} = \frac{1}{L^3}\frac{\partial}{\partial \vec{V}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3k\, d^3k'\,\tilde{g}(\vec{k})\tilde{g}(\vec{k}')\frac{\partial f_0}{\partial \vec{V}}\frac{\delta(\vec{k}+\vec{k}')}{\left[\nu - 2\pi i(\omega - \vec{k}'\cdot\vec{V})\right]} \tag{7.1.10}$$

Integrating next over *k'* gives

$$\frac{\partial\langle f_0\rangle}{\partial t}=\frac{1}{L^3}\frac{\partial}{\partial\vec{V}}\int_{-\infty}^{\infty}\tilde{g}(\vec{k})\tilde{g}(-\vec{k})\frac{\partial f_0}{\partial\vec{V}}\frac{d^3k}{\left[\nu-2\pi i(\omega+\vec{k}\cdot\vec{V})\right]} \qquad (7.1.11)$$

Recalling that $\tilde{g}(-\vec{k})=\vec{g}^*(\vec{k})$, the equation above can be recast

$$\frac{\partial\langle f_0\rangle}{\partial t}=\frac{1}{L^3}\frac{\partial}{\partial\vec{V}}\int_{-\infty}^{\infty}\left|\tilde{g}(\vec{k})\right|^2\frac{\partial f_0}{\partial\vec{V}}\frac{d^3k}{\left[\nu-2\pi i(\omega+\vec{k}\cdot\vec{V})\right]} \qquad (7.1.12)$$

Using the identity

$$\frac{1}{\left[\nu-2\pi i(\omega+\vec{k}\cdot\vec{V})\right]}=\frac{\nu+2\pi i(\omega+\vec{k}\cdot\vec{V})}{\left[\nu^2+4\pi^2(\omega+\vec{k}\cdot\vec{V})^2\right]} \qquad (7.1.13)$$

the integral in the right side of eq.(7.1.12) can be split into a real and an imaginary part, namely

$$\int_{-\infty}^{\infty}\left|\tilde{g}(\vec{k})\right|^2\frac{\partial f_0}{\partial t}\frac{\nu d^3k}{\left[\nu^2+4\pi^2(\omega+\vec{k}\cdot\vec{V})^2\right]}+2\pi i\int_{-\infty}^{\infty}\left|\tilde{g}(\vec{k})\right|^2\frac{\partial f_0}{\partial t}\frac{(\omega+\vec{k}\cdot\vec{V})d^3k}{\left[\nu^2+4\pi^2(\omega+\vec{k}\cdot\vec{V})^2\right]} \qquad (7.1.14)$$

Use now the following identity for the delta function (Dirac's distribution)

$$\delta(z)=\frac{1}{\pi}\lim_{x\to 0}\frac{x}{(x^2+z^2)} \qquad (7.1.15)$$

Taking into account (7.1.15), the real part of (7.1.14) in the limit $\nu\longrightarrow 0$ gives

$$\frac{\partial f_0}{\partial t}=\frac{1}{2L^3}\frac{\partial}{\partial\vec{V}}\int_{-\infty}^{\infty}\left|\tilde{g}(\vec{k})\right|^2\frac{\partial f_0}{\partial\vec{V}}\delta(\omega+\vec{k}\cdot\vec{V})\,d^3k \qquad (7.1.16)$$

Defining the diffusion coefficient in the velocity space as

$$D(\vec{V})=\frac{1}{2L^3}\int_{-\infty}^{\infty}\left|\tilde{g}(\vec{k})\right|^2\delta(\omega+\vec{k}\cdot\vec{V})d^3k \qquad (7.1.17)$$

permits to rewrite eq.(7.1.16) as a diffusion equation in the velocity space, namely

$$\frac{\partial f_0}{\partial t}=\frac{\partial}{\partial\vec{V}}\left[D(\vec{V})\frac{\partial f_0}{\partial\vec{V}}\right] \qquad (7.1.18)$$

Note that the equation above is essentially the Fokker-Planck equation without the dynamical viscosity term.

Let us compute the diffusion coefficient from our formula (7.1.17). The acceleration produced by a particle of mass m is $g(\vec{r}) = Gm\vec{r}/r^3$. Then, the Fourier transform of the acceleration must be computed, i.e.,

$$\tilde{g}(\vec{k}) = \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} \sin\theta d\theta \int_0^{\infty} Gm e^{-2\pi i kr\cos\theta} dr \qquad (7.1.19)$$

The integrals are immediate and one obtains

$$\tilde{g}(\vec{k}) = \frac{\pi Gm}{k^2}\vec{k} \qquad (7.1.20)$$

Replace this result into eq.(7.1.17) and recalling that $L^{-3} = n$ one obtains

$$D(\vec{V}) = 2\pi^3 G^2 m^2 n \int_{-\infty}^{\infty} \delta(\omega + \vec{k}\cdot\vec{V}) dk \qquad (7.1.21)$$

Since $\delta(\omega + \vec{k}\cdot\vec{V}) = \delta(\vec{k} - \vec{k}_0)/V$ where $\vec{k}_0 = -(\omega/V^2)\vec{V}$, the diffusion coefficient is simply

$$D(V) = \frac{2\pi^3 G^2 m\rho}{V} \qquad (7.1.22)$$

where the mass density $\rho = nm$ of the medium was introduced. The relaxation time is $T_R = V^2/D$ or explicitly

$$T_R = \frac{V^3}{2\pi^3 G^2 m\rho} \qquad (7.1.23)$$

This last expression should be compared with eq.(2.6), which was derived using the physics of a binary collision under the adiabatic approximation and verify that the ratio between the relaxation time derived using the Boltzmann equation and that derived directly from the collision process is $4\lg\Lambda/\pi^2$. On the other side, using the expressions for the diffusion coefficient and the dynamical viscosity coefficient, one obtains after performing velocity averages

$$\frac{D}{\beta} = \frac{\pi^2}{2}\left(\frac{m}{M}\right)\frac{\langle v^2\rangle}{\lg\Lambda} \qquad (7.1.24)$$

Note that if the mass of the test particle is the same as that of the particles of the ambient medium, the ratio D/β depends only on the mean square velocity, a result already anticipated when we have studied the Langevin approximation.

Continuous stellar encounters may produce a systematic increase of the velocity dispersion of stars but the rate of the energy transfer due to these encounters is quite small. However, Spitzer & Schwarzschild (ApJ 114, 385, 1951 idem 118, 106, 1953) recognized that encounters with massive interstellar clouds may produce an important energy transfer rate. Under these conditions, by solving the F-P equation, they have concluded that the mean square velocity of stars increases as

$$\left\langle v^2 \right\rangle = v_0^2 \left[1 + \left(\frac{t}{t_c} \right) \right]^{2/3} \tag{7.1.25}$$

This result can be easily reproduced if we recall that the diffusion coefficient in velocity space is inversely proportional to the velocity (eq. 7.1.22) and hence

$$\frac{d\left\langle v^2 \right\rangle}{dt} = D(v) = \frac{k}{\left\langle v^2 \right\rangle^{1/2}} \tag{7.1.26}$$

Integration of this equation gives immediately the result derived by Spitzer & Schwarzschild.

8. Dispersion of Stellar Clusters

Gravitational systems are frequently imagined as thermodynamic systems. In this case, we would expect that in an equilibrium state the entropy of the system is an extreme. In order to maximize entropy we have also to impose constraints concerning the conservation of the total number of particles *N* and of the total energy *E* of the system, which are defined respectively by

$$N = \int f d\Gamma \tag{8.1}$$

and

$$E = \int \varepsilon f d\Gamma \tag{8.2}$$

In the above equations $d\Gamma = d^3 r d^3 V$ is the phase space volume element and $\varepsilon = (V^2/2) + \phi$. The entropy functional subjected to the aforementioned constraints is

$$S = \int f \lg f d\Gamma + \lambda \int f d\Gamma + \beta \int \varepsilon f d\Gamma \tag{8.3}$$

where λ and β are Lagrange multipliers. Performing the variation of (8.3) one obtains

$$\delta S = \int (\lg f + 1)(\delta f) d\Gamma + \lambda \int (\delta f) d\Gamma + \beta \int \varepsilon (\delta f) d\Gamma = 0 \tag{8.4}$$

Since the variation δf is arbitrary, the equation above implies

$$\lg f + 1 + \lambda + \beta\varepsilon = 0 \tag{8.5}$$

or, solving for the distribution function

$$f = \frac{1}{A} e^{-\beta\varepsilon} \tag{8.6}$$

where we have defined $\lg A = 1 + \lambda$. Rigorously, a system with a velocity distribution given by eq.(8.6) cannot be in equilibrium. The reason is that particles with velocities higher than the escape velocity V_{esc} leave the system. In this situation, the system searches for a new equilibrium situation through collisions that redistribute the energy of the particles. Solutions to this problem have been proposed by including an energy cutoff in (8.6). Systems built with such a cutoff have mass profiles known as "King profiles" (I.R. King, AJ 71, 64, 1966; idem ApJ 222, 1, 1978), which are able to describe quite well the light distribution of globular clusters and some E-galaxies.

Let us estimate the mean square escape velocity of a system like a globular cluster. If the cluster is in dynamical equilibrium, the virial must be satisfied, i.e.,

$$M \left\langle V^2 \right\rangle + W = 0 \tag{8.7}$$

where M is the total mass of the system, W is the gravitational potential energy and $\left\langle V^2 \right\rangle$ is the mean square velocity of the particles. Escape of a particle occurs when its kinetic energy is equal to the potential energy due to the remaining particles, this means

$$\frac{1}{2} m_i V_{i,esc}^2 = G m_i \sum_j \frac{m_j}{r_{ij}} \tag{8.8}$$

On the other side, the mean square escape velocity is defined by the relation

$$\left\langle V_{esc}^2 \right\rangle = \frac{1}{M} \sum_i m_i V_{i,esc}^2 \tag{8.9}$$

Now sum both sides of eq.(8.8) over all *i-particles* and use (8.9) to obtain

$$M \left\langle V_{esc}^2 \right\rangle = 2G \sum_i \sum_j \frac{m_i m_j}{r_{ij}} = -4W \tag{8.10}$$

Combining eqs.(8.7) and (8.10) one obtains finally

$$\left\langle V_{esc}^2 \right\rangle = 4 \left\langle V^2 \right\rangle \tag{8.11}$$

In other words, the mean square escape velocity is equal to four times the mean square velocity.

Assuming, as a first approximation, a Maxwellian velocity distribution for stars in a globular cluster, the fraction of stars with velocities equal or greater than the escape velocity is

$$Q = \int_{V_{esc}}^{\infty} f(V) d^3V = 0.00738 \quad (8.12)$$

where we have used the previous result (eq. 8.11) and the fact that for a Maxwellian velocity distribution the mean square velocity of the particles is equal to three times the square of the velocity dispersion.
Defining the mean relaxation timescale by

$$T_R = \int_0^{\infty} T_R(V) f(V) d^3V = \frac{\langle V^2 \rangle^{3/2}}{8\pi G^2 m^2 n \lg \Lambda} \quad (8.13)$$

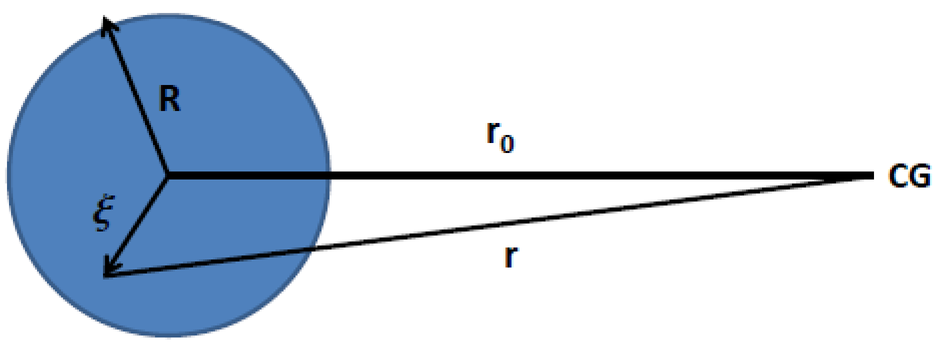
then, in a first approximation, the stellar loss rate is

$$\frac{dN_*}{dt} = -Q\frac{N_*}{T_R} \;\Rightarrow N_*(t) = N_*(0)\exp\left[-\frac{Qt}{T_R}\right] \quad (8.14)$$

The characteristic timescale for stellar "evaporation" is $t_{loss} = T_R / Q$, which for a typical globular cluster is of the order of 200 billion years, much larger than the Hubble timescale.

The situation is rather different for an open cluster, since the tidal field of the Galaxy plays now an important role.

Figure 5



Consider an open cluster, supposed to be spherically symmetric with radius *R*, in a circular orbit around the center of the Galaxy (CG). The distance to the galactic center is r_0. Let ξ be the distance of a given star to the center of mass of the cluster and *r* its distance to the center of the Galaxy (see figure 5).

For the center of mass of the cluster rotating with a tangential velocity V_θ (or, equivalently, with an angular velocity ω_0) around the galactic center, the following condition can be written

$$\frac{V_\theta^2}{r_0} = \omega_0^2 r_0 \quad (8.15)$$

On the other hand, a star located at a distance ξ from the center of the cluster feels a different force field that produces an acceleration given by

$$\gamma = -\frac{V_\theta^2}{r} + \omega_0^2 r \cong -\frac{V_\theta^2}{r_0} - \frac{2V_\theta}{r_0}\left(\frac{\partial V_\theta}{\partial r}\right)\xi + \frac{V_\theta^2}{r_0^2}\xi + \omega_0^2 r_0 + \omega_0^2 \xi + ... \tag{8.16}$$

where a Taylor expansion of the tangential velocity and of the vector position of the star was performed and only linear terms were kept. Combining eqs.(8.15) and (8.16) results for the acceleration

$$\gamma = -2\left[\frac{V_\theta}{r_0}\left(\frac{\partial V_\theta}{\partial r}\right) - \frac{V_\theta^2}{r_0^2}\right]\xi \tag{8.17}$$

Introducing the Oort constants

$$A = \frac{1}{2}\left(\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}\right) \quad and \quad B = \frac{1}{2}\left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\right) \tag{8.18}$$

Then adding and subtracting these relations, one obtains easily

$$A - B = -\frac{V_\theta}{r} \quad and \quad A + B = \frac{\partial V_\theta}{\partial r} \tag{8.19}$$

With the relations above, eq.(8.17) can be recast as

$$\gamma = 4A(A - B)\xi \tag{8.20}$$

The star to be maintained in the cluster must be under the action of the self-gravity of the system, which must produce an acceleration higher than (8.20), i.e.,

$$\frac{4\pi}{3} G\bar{\rho}\xi > 4A(A - B)\xi \tag{8.21}$$

This condition means that the average mass density of the cluster must be higher than

$$\bar{\rho} > \frac{3}{\pi}\frac{A}{G}(A - B) \tag{8.22}$$

which is the stability criterion for an open cluster. Using the values of the Oort constants valid for the solar vicinity (see Section 6.2), one obtains for the critical density $\bar{\rho}_{crit} = 0.089 M_\odot pc^{-3}$. Clusters with mean densities lower than this critical value will be rapidly dissolved. As examples, the Hyades cluster has a mean density of $0.25 M_\odot pc^{-3}$ while the Pleiades cluster has a mean density of $1.7 M_\odot pc^{-3}$. Thus, these clusters are expected to be stable against tidal losses, raison why they still exist.