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GRAVITATIONAL WAVES AND EFFECTIVE FIELD THEORY METHODS TO MODEL BINARIES

XIX ESCOLA DE VERÃO JORGE ANDRÉ SWIECA DE PARTÍCULAS E CAMPOS, MARESIAS (SP), 6-11 FEBRUARY 2017

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Notations

We use units $c = 1$, which means that 1 light-year (ly)=1 year $\simeq$ 3.16×10^7 sec $=9.46 \times 10^{15}$ m.

Another useful unit is the parsec $\simeq 3.09 \times 10^{16}$ m $\simeq 3.26$ ly.

The mass of the sun is $M_\odot \simeq 1.99 \times 10^{33}$ g and its Schwarzschild radius $2G_N M_\odot \simeq 2.95$ km $\simeq 9.84 \times 10^{-6}$ sec.

We adopt the "mostly plus" ignature, i.e. the Minkowski metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)\,,$$

the Christoffel symbols is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \right)\,,$$

the Riemann tensor is

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}\,,$$

with symmetry properties

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\sigma\rho} = -R_{\mu\nu\sigma\rho}\,, \\ R^\mu_{\nu\rho\sigma} + R^\mu_{\rho\sigma\nu} &+ R^\mu_{\sigma\rho\nu} = 0\,, \end{aligned}$$

which gives $(4 \times 3/2)^2 - 4 \times 4 = 20$ independent components, which in $d+1$ dimensions turn out to be $\left(\frac{(d+1)d}{2}\right)^2 - \frac{(d+1)^2 d(d-1)}{6} = \frac{(d+1)^2 d(d+2)}{12}$.

The Ricci tensor is

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}\,,$$

the Ricci scalar

$$R = R_{\mu\nu} g^{\mu\nu}\,,$$

the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,,$$

the Weyl tensor

$$C^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} - 2\delta^{[\alpha}_{[\mu} R^{\beta]}_{\nu]} + \frac{1}{3}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]} R\,,$$

which has 10 components in $3+1$ dimensions or $\frac{(d+1)^2 d(d+2)}{12} - \frac{(d+1)(d+2)}{2} = (d+2)(d+1)\frac{d^2+d-6}{12}$ in $d+1$ dimensions.

Greek indices $\alpha, \ldots \omega$ run over $d+1$ dimensions, Latin indices $a, \ldots i, j, \ldots$ over d spatial dimensions.

Fourier transform in d dimensions are defined as

$$
\begin{aligned}
F(\mathbf{k}) &= \int d^d x F(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \,, \\
F(\mathbf{x}) &= \int \frac{d^d k}{(2\pi)^d} F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \int_{\mathbf{k}} F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \,.
\end{aligned}
$$

We will denote the modulus of a generic 3-vector $\mathbf{w}$ by $w \equiv |\mathbf{w}|$.

Fourier transform over time is defined as

$$
\begin{aligned}
F(\omega) &= \int dt F(t) e^{i\omega t} \,, \\
F(t) &= \int \frac{d\omega}{2\pi} F(\omega) e^{-i\omega t} \,.
\end{aligned}
$$

Introduction

The existence of gravitational waves (GW) is an unavoidable prediction of General Relativity (GR): any change to a gravitating source must be communicated to distant observers no faster than the speed of light, c, leading to the existence of *gravitational radiation*, or GWs.

Before direct gravitational wave detection the more precise evidence of a system emitting GWs comes from the celebrated "Hulse-Taylor" pulsar [1], where two neutron stars are tightly bound in a binary system with the observed decay rate of their orbit being in agreement with the GR prediction to about one part in a thousand [2], see also [3] for more examples of observed GW emission from pulsar and white dwarf binary systems and sec. 6.2 of [4] for a pedagogical discussion.

Earth-based, kilometer-sized gravitational wave observatories, are currently taking data or under development: the two Laser Interferomenter Gravitational-Wave Observatories (LIGO) in the US (see *www.ligo.org*), which soon will be joined by the Virgo interferometer in Italy (*www.virgo.infn.it*). Another smaller detector, with reduced sensitivity, belonging to the network is the German-British Gravitational Wave Detector GEO600, *www.geo600.uni-hannover.de*). The gravitational detector network is planned to be increased by the Japanese KAGRA (*http://gwcenter.icrr.u-tokyo.ac.jp/en/*) detector by the end of this decade and by an additional interferometer in India (*http://www.gw-indigo.org*) by the beginning of the next decade.

In fig. 1 the sectrum of the two detected gravitational wave events are displayed.

The output of such observatories is particularly sensitive to the phase Φ of GW signals and focusing on coalescing binary systems, it is possible to predict it via

$$\Phi(t) = 2\int_{t_i}^{t} \omega(t')dt' , \qquad (1)$$

where ω is the angular velocity of the individual binary component and t_i stands for the time the signal with increasing frequency enters

[1] R A Hulse and J H Taylor. Discovery of a pulsar in a binary system. *Astrophys. J.*, 195:L51, 1975

[2] J. M. Weisberg and J. H. Taylor. Observations of post-newtonian timing effects in the binary pulsar psr 1913+16. *Phys. Rev. Lett.*, 52:1348, 1984

[3] M. Burgay, N. D'Amico, A. Possenti, R. N. Manchester, A. G. Lyne, B. C. Joshi, M. A. McLaughlin, and M. Kramer *et al.* An increased estimate of the merger rate of double neutron stars from observations of a highly relativistic system. *Nature*, 426:531, 2003; M. Kramer and N. Wex. The double pulsar system: A unique laboratory for gravity. *Class. Quant. Grav.*, 26:073001, 2009; A. Wolszczan. A nearby 37.9-ms radio pulsar in a relativistic binary system. *Nature*, 350:688, 1991; I. H. Stairs, S. E. Thorsett, J. H. Taylor, and A. Wolszczan. Studies of the relativistic binary pulsar psr b1534+12: I. timing analysis. *Astrophys. J.*, 581:501, 2002; and J.J. Hermes, Mukremin Kilic, Warren R. Brown, D.E. Winget, Carlos Allende Prieto, et al. Rapid Orbital Decay in the 12.75-minute WD+WD Binary J0651+2844. 2012. DOI: 10.1088/2041-8205/757/2/L21

[4] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

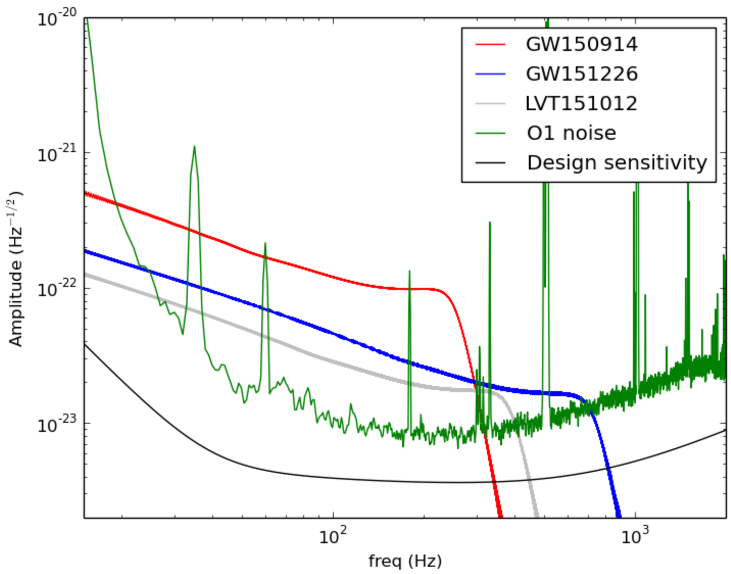

Figure 1: The spectrum of the two detected gravitational events compared to the real O1 noise and the Advanced LIGO design sensitivity. The third most significant trigger O1, not loud enough to be considered an event, is also displayed.

the detector band-width. Note the factor 2 between the *GW* phase Φ and the *orbital* angular velocity ω.

Binary orbits are in general eccentric, but at the frequency we are interested in ($> 10Hz$) binary system will have circuralized, see sec. 4.1.3 of [5] for a quantitative analysis of orbit circularization. For circular orbit the binding energy can be expressed in terms of a single parameter, say the relative velocity of the binary system components v_r, which by the virial theorem $v_r^2 \simeq G_N M/r$, being G_N the standard Newton constant, M the total mass of the binary system, and r the orbital separation between its constituents. Note that the virial relationship, or its equivalent (on circular orbits) Kepler law $\omega^2 \simeq G_N M/r^3$, are not exact in GR, but only at Newtonian level.

[5] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

For spin-less binary constituents the energy E of circular orbits can be expressed in terms of a series in $v \equiv (\pi G_N M f_{GW})^{1/3}$:

$$E(v) = -\frac{1}{2}\eta M v^2 \left(1 + e_{v^2}(\eta)v^2 + e_{v^4}(\eta)v^4 + \dots\right) , \tag{2}$$

where $\eta \equiv m_1 m_2/M^2$ is the symmetric mass ratio and the $e_{v^n}(\eta)$ coefficients stand for GR corrections to the Newtonian formula, and only even power of v are involved for the conservative Energy. For the radiated flux $F(v)$ the leading term is the Einstein quadrupole formula, which we will derive in sec. , that in the circular orbit case reduces to

$$F(v) = \frac{32\eta^2}{5G_N} v^{10} \left(1 + f_{v^2}(\eta)v^2 + f_{v^3}(\eta)v^3 + \dots\right) . \tag{3}$$

Note that using our definition of v we have $v^3 = \omega M$ allowing to re-write (note that during the coalescence v increases monotonically) eq. (1) as

$$\Phi(v) \simeq \frac{2}{G_N M}\int_{v_i}^{v} v^3 \frac{dE/dv}{dE/dt} dv = \frac{5}{16\eta}\int_{v_i}^{v} \frac{1}{v^6}\left(1 + p_{v^2}v^2 + p_{v^3}v^3 + \dots\right) dv\,, \quad (4)$$

where we used $dE/dt = -F$. Since the phase has to be matched with $O(1)$ precision, corrections at least $O(v^6)$ must be considered.

The aim of this course is to show how to compute the E, F functions at required perturbative order. We thus have to treat the binary problem perturbatively, the actual expansion parameter will be $(G_N M \pi f_{GW})^{1/3} = (G_N M\omega)^{1/3} = v$, which represents an expansion around the Minkowski space. Such perturbative expansion of GR has been proven very useful to treat the binary problem and it goes under the name of post-Newtonian (PN) expansion to GR.

The approach to solving for the dynamics of the two body problem adopted here relies on an *effective field theory methods*, originally proposed in [6]. The two body problem is a system which exhibits a clear separation of scales: the size of the compact objects r_s, like black holes and/or neutron stars, the orbital separation r and the gravitational wave-length λ. Using again the virial theorem the hierarchy $r_s < r \sim r_s/v^2 < \lambda \sim r/v$ can be established.

[6] Walter D. Goldberger and Ira Z. Rothstein. An Effective field theory of gravity for extended objects. *Phys.Rev.*, D73:104029, 2006

The author of the present notes recommends the following reviews: for a review of the PN theory see [7], for a pedagogical book on GWs see [8], for an astrophysics oriented review on GWs see [9], for a data-analysis oriented review see [10], for reviews on effective field theory methods for GR see [11] and [12].

These are the notes of the course held at the XIX Jorge André Swieca Summer School on Particle Physcs and Field Theory at Maresias (SP) in February 2017, and they are not meant in any way to replace or improve the extensive literature existent on the topic, but rather to collect in single document the material relevant for this course, which could otherwise be found scattered in different places.

[7] Luc Blanchet. Gravitational radiation from post-newtonian sources and inspiralling compact binaries. *Living Reviews in Relativity*, 9(4), 2006. URL http://www.livingreviews.org/lrr-2006-4

[8] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

[9] B.S. Sathyaprakash and B.F. Schutz. Physics, Astrophysics and Cosmology with Gravitational Waves. *Living Rev.Rel.*, 12:2, 2009

[10] Alessandra Buonanno. Gravitational waves. 2007. URL http://arxiv.org/abs/arXiv:0709.4682

[11] W. D. Goldberger. Les houches lectures on effective field theories and gravitational radiation. In *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime*, 2007

[12] Stefano Foffa and Riccardo Sturani. Effective field theory methods to model compact binaries. *Class.Quant.Grav.*, 31 (4):043001, 2014. DOI: 10.1088/0264-9381/31/4/043001

General Theory of GWs

Expansion around Minkowski

We start by recalling the Einstein eqations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}\,, \tag{5}$$

however it will be useful for our purposes to work also at the level of the action

$$\begin{aligned} S_{EH} &= \frac{1}{16\Pi G_N} \int dt d^d x \sqrt{-g}\, R\,, &(6)\\ S_m \to \delta S_m &= \frac{1}{2} \int dt d^d x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}\,. &(7) \end{aligned}$$

Here we focus on an expansion around the Minkowski space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1\,, \tag{8}$$

and we are interested in a systematic expansion in powers of $h_{\mu\nu}$. As in the binary system case the metric perturbation $|h_{\mu\nu}| \sim G_N m/r \sim v^2$, a suitable velocity expansion will have to be considered.

GR admits invariance under general coordinate transformations

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)\,, \tag{9}$$

which change the metric according to

$$\begin{aligned} g_{\mu\nu}(x) \to g'_{\mu\nu}(x') &= g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \\ h_{\mu\nu}(x) \to h'_{\mu\nu}(x') &= h_{\mu\nu}(x) - \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right)\,. \end{aligned} \tag{10}$$

At linear order around a Minkoswki background we have ($h \equiv \eta_{\mu\nu} h^{\mu\nu}$)

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{1}{2}\left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}\right)\,, \\ R_{\mu\nu} &= \frac{1}{2}\left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h\right)\,, \\ R &= \partial_\mu \partial_\nu h^{\mu\nu} - \Box h\,, \\ G_{\mu\nu} &= \frac{1}{2}\left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \Box h\right)\,, \end{aligned} \tag{11}$$

and we remind that with the metric signature adopted here $\Box = \partial_i\partial^i - \partial_t^2$.

The formula for $G_{\mu\nu}$ can be used to write the Einstein equations as

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} - \partial_\mu\partial_\rho\bar{h}^\rho_\nu - \partial_\nu\partial_\rho\bar{h}^\rho_\mu = -16\pi G_N T_{\mu\nu} \quad (12)$$

whre $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ which transforms as

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha_\alpha \,.$$

Let us be specific about the matter action and assume that it is given by the by the world-line action

$$\begin{aligned} S &= -m\int d\tau \\ &= -m\int d\tau\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \\ &= -m\int dt\left[-g_{00} - 2g_{0i}\frac{dx^i}{dt} - g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right]^{1/2} . \end{aligned} \quad (13)$$

We will need to work at the level of the action, so it will be convenient to expand the "bulk" dynamics of the gravitational degrees of freedom is given by the standard Einstein-Hilbert action, to quadratic order

$$S_{EH} = -\frac{1}{64\pi G_N}\int dtd\mathbf{x}\left[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h\partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu\right] \quad (14)$$

Schematically the the Lagrangean density in Fourier space up to quadratic order is of the type

$$\mathcal{L} = -\frac{1}{64\pi G_N}\left(A^{\mu\nu\rho\sigma}(k_\alpha)h_{\mu\nu}(k)h_{\rho\sigma}(-k)\right) - \frac{1}{2}h_{\mu\nu}(k)T^{\mu\nu}(-k) \quad (15)$$

with

$$\begin{aligned} A_{\mu\nu\rho\sigma} = \; & \frac{1}{2}k^2\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right) \\ & -k^2\eta_{\mu\nu}\eta_{\rho\sigma} \\ & +k_\mu k_\nu\eta_{\rho\sigma} + \eta_{\mu\nu}k_\rho k_\sigma \\ & -\frac{1}{2}\left(k_\mu k_\rho\eta_{\nu\sigma} + k_\mu k_\sigma\eta_{\nu\rho} + k_\nu k_\rho\eta_{\mu\sigma} + k_\nu k_\sigma\eta_{\mu\rho}\right) , \end{aligned} \quad (16)$$

so that it leads to the equation of motions

$$A^{\mu\nu\rho\sigma}(k)h_{\rho\sigma} = 16\pi G_N T^{\mu\nu} ,$$

which represent an intricated system of differential equation. Components of the gravitational field can be disentangled if and only if A woould be invertible, i.e. if existed an operator $B = A^{-1}$ such that

$$B_{\alpha\beta\gamma\delta}A^{\gamma\delta\rho\sigma} = \frac{1}{2}\left(\eta_{\alpha\rho}\eta_{\beta\sigma} + \eta_{\alpha\sigma}\eta_{\beta\rho}\right) .$$

It turns out that A is not ivertible (if it was, ther would be no coordinate transformation invariance!) so analogously to the electromagnetic case we can *add a Gauge-fixing* term to the Lagrangean, which correspond to finding the GR solutions in a specific gauge (class). Here we add the gauge-fixing *Lorentz term*

$$S_{GF} = -\frac{1}{32\pi G_N} \int d^4x \left(\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h \right)^2 , \tag{17}$$

to the Einstein-Hilbert action to obtain

$$\begin{aligned} A'_{\mu\nu\rho\sigma} &= A_{\mu\nu\rho\sigma} - \left(k_\mu k_\nu \eta_{\rho\sigma} + \eta_{\mu\nu} k_\rho k_\sigma \right) \\ &\quad + \frac{1}{2} \left(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho} \right) + \\ &\quad + \frac{1}{2} k^2 \eta_{\mu\nu} \eta_{\rho\sigma} \\ &= \tfrac{1}{2} k^2 \left(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma} \right) \end{aligned} \tag{18}$$

which makes A' easily invertible by noting that

$$A_{\mu\nu\alpha\beta} A^{\alpha\beta}{}_{\rho\sigma} = \frac{1}{2} \left(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} \right) , \tag{19}$$

so that the equation of motion derived by the new Lagrangean are

$$\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu} , \tag{20}$$

(where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$) showing that *in this class of gauges* in which $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$ all polarization of the gravity tensor satisfy a wave-like equation, see later in this section for an analysis of this statement.

Note that in this way we have not completely fixed the gauge as we can always perform a transformation with $\Box \xi_\mu = 0$ and still remain in the Lorentz gauge class.

It will be useful to solve eq. (20) by using the Green function G with appropriate boundary conditions, where the G is defined as

$$\Box_x G(x-y) = \delta^{(4)}(x-y) . \tag{21}$$

The explicit form of the Green functions with time-retarded and and time-advanced boundary conditions are in the direct space (see e.g. sec. 6.44 of [13], with a pre-factor different by -4π becasue of a different definition in eq. (21)

[13] John David Jackson. *Classical Electrodynamics*. John Wiley & Sons, iii edition, 1999

$$\begin{aligned} G_{ret}(t, \mathbf{x}) &= -\delta(t-r) \frac{1}{4\pi r} , \\ G_{adv}(t, \mathbf{x}) &= -\delta(t+r) \frac{1}{4\pi r} , \end{aligned} \tag{22}$$

where $r \equiv |\mathbf{x}| > 0$ (note that $G_{ret}(t, \mathbf{x})) = G_{adv}(-t, \mathbf{x})$. By solving ex. 6,7 one can show that these are indeed Green functions for the eq. (21). For the Feynman prescription of the Green function G_F see ex. 8, where the motivated student is asked to demonstrate that the G_F ensures pure incoming wave at past infinity and pure outgoing wave at future infinity.

We have shown that in the Lorentz gauge class all graviton field polarization satisfy a wave equation: we now show that the correct physical statement is that only 2 degrees of freedom (polarizations) are physical and *radiative*, additional 4 are physical and *non-radiative* and finally 4 are pure gauge, and can be killed by the Lorentz condition in eq. (17), for instance.

Let us now focus on the wave eq. (20) in the vacuum case

$$\Box \bar{h}_{\mu\nu} = 0 \tag{23}$$

and define

$$\xi_{\mu\nu} \equiv \partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu} \partial^\alpha \xi_\alpha \,. \tag{24}$$

Now $\Box \xi_\mu = 0 \implies \Box \xi_{\mu\nu} = 0$, with $\delta \bar{h}_{\mu\nu} = \xi_{\mu\nu}$. Since now the free wave equation is satisfied by both the gravitational field $\bar{h}_{\mu\nu}$ and by the residual gauge transformation parametrized by ξ_μ that preserves the Lorentz condition, we can use the four available ξ_μ to set four conditions on $\bar{h}_{\mu\nu}$. In particular ξ_0 can be used to make h vanish (so that $\bar{h}_{\mu\nu} = h_{\mu\nu}$) and the three ξ_i can be used to make the three $\bar{h}_{0i}$ vanish. The Lorentz condition eq. (17) for $\mu = 0$ will now look like $\partial^0 h_{00} = 0$, which means that h_{00} is constant in time, hence not contributing to any GW. In conclusion, in vacuum one can set

$$h_{0\mu} = h = \partial^i h_{ij} = 0 \,, \tag{25}$$

defining the *transverse traceless*, or TT gauge, which then describe only the physical GW propagating in vacuum. For a wave propagating along the $\mu = 3$ axis, for instance, the wave eq. (20) admits the solution $h_{\mu\nu}(t-z)$ and the gauge condition $\partial^j h_{ij} = 0$ reads $h_{i3} = 0$. In terms of the tensor components we have

$$h^{(TT)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{26}$$

Given a plane wave solution $h^{(GW)}_{kl}$ propagating in the generic $\hat{n}$ direction outside the source, which is in the Lorentz gauge, but not yet TT-ed, its TT form can be obtained by applying the projector $\Lambda_{ij,kl}$ defined as

$$\begin{aligned} \Lambda_{ij,kl}(\hat{n}) &= \frac{1}{2}\left[P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}\right] \,, \\ P_{ij}(\hat{n}) &= \delta_{ij} - n_i n_j \,, \end{aligned} \tag{27}$$

according to

$$h_{ij}^{(TT-GW)} = \Lambda_{ij,kl} h_{kl}^{(GW)} \,. \tag{28}$$

The Λ projector ensures transversality and tracelessness of the resulting tensor (starting from a tensor in the Lorentz gauge!).

The TT gauge cannot be imposed there where $T_{\mu\nu} \neq 0$, as we cannot set to 0 any component of $\bar{h}_{\mu\nu}$ which satisfies a $\Box \bar{h}_{\mu\nu} \neq 0$ equation by using a ξ_μ which satisfies a $\Box \xi_\mu = 0$ equation (and hence $\Box \xi_{\mu\nu} = 0$).

Radiative degrees of freedom of $h_{\mu\nu}$

Following sec. 2.2 of [14] we show that the gravitational perturbation $h_{\mu\nu}$ has 6 physical degrees of freedom: 4 constrained plus 2 radiative. The argument is based on a Minkowski equivalent of Bardeen's gauge-invariant cosmological perturbation formalism.

[14] Eanna E. Flanagan and Scott A. Hughes. The basics of gravitational wave theory. *New J. Phys.*, 7:204, 2005

We begin by defining the decomposition of the metric perturbation $h_{\mu\nu}$, in any gauge, into a number of irreducible pieces. Assuming that $h_{\mu\nu} \to 0$ as $r \to \infty$, we decompose $h_{\mu\nu}$ into a number of irreducible quantities ϕ, β_i, γ, H, ε_i, λ and $h_{ij}^{(\mathrm{TT})}$ via the equations

$$h_{00} = 2\phi \,, \tag{29}$$

$$h_{0i} = \beta_i + \partial_i \gamma \,, \tag{30}$$

$$h_{ij} = h_{ij}^{(\mathrm{TT})} + \frac{1}{3} H \delta_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \lambda \,, \tag{31}$$

together with the constraints

$$\partial_i \beta_i = 0 \quad \text{(1 constraint)} \tag{32}$$

$$\partial_i \varepsilon_i = 0 \quad \text{(1 constraint)} \tag{33}$$

$$\partial_i h_{ij}^{(\mathrm{TT})} = 0 \quad \text{(3 constraints)} \tag{34}$$

$$\delta^{ij} h_{ij}^{(\mathrm{TT})} = 0 \quad \text{(1 constraint)} \tag{35}$$

and boundary conditions

$$\gamma \to 0, \quad \varepsilon_i \to 0, \quad \lambda \to 0, \quad \nabla^2 \lambda \to 0 \tag{36}$$

as $r \to \infty$. Here $H \equiv \delta^{ij} h_{ij}$ is the trace of the *spatial* portion of the metric perturbation. The spatial tensor $h_{ij}^{(\mathrm{TT})}$ is transverse and traceless, and is the TT piece of the metric discussed above which contains the physical radiative degrees of freedom. The quantities β_i and $\partial_i \gamma$ are the transverse and longitudinal pieces of h_{ti}. The uniqueness of this decomposition follows from taking a divergence of Eq. (30) giving $\nabla^2 \gamma = \partial_i h_{ti}$, which has a unique solution by the boundary condition (36). Similarly, taking two derivatives of Eq. (31) yields the equation $2\nabla^2\nabla^2\lambda = 3\partial_i\partial_j h_{ij} - \nabla^2 H$, which has a unique solution by Eq. (36). Having solved for λ, one can obtain a unique ε_i by solving $3\nabla^2\varepsilon_i = 6\partial_j h_{ij} - 2\partial_i H - 4\partial_i\nabla^2\lambda$.

The total number of free functions in the parameterization (29) – (31) of the metric is 16: 4 scalars (ϕ, γ, H, and λ), 6 vector components (β_i and ε_i), and 6 symmetric tensor components ($h_{ij}^{(\mathrm{TT})}$). The number of constraints (32) – (35) is 6, so the number of independent variables in the parameterization is 10, consistent with a symmetric 4×4 tensor.

We next discuss how the variables ϕ, β_i, γ, H, ε_i, λ and $h_{ij}^{(\mathrm{TT})}$ transform under gauge transformations ξ^a with $\xi^a \to 0$ as $r \to \infty$. We parameterize such gauge transformation as

$$\xi_\mu = (\xi_t, \xi_i) \equiv (A, B_i + \partial_i C) \;, \tag{37}$$

where $\partial_i B_i = 0$ and $C \to 0$ as $r \to \infty$; thus B_i and $\partial_i C$ are the transverse and longitudinal pieces of the spatial gauge transformation. Decomposing this transformed metric into its irreducible pieces yields the transformation laws

$$\phi \to \phi - \dot{A} \;, \tag{38}$$

$$\beta_i \to \beta_i - \dot{B}_i \;, \tag{39}$$

$$\gamma \to \gamma - A - \dot{C} \;, \tag{40}$$

$$H \to H - 2\nabla^2 C \;, \tag{41}$$

$$\lambda \to \lambda - 2C \;, \tag{42}$$

$$\varepsilon_i \to \varepsilon_i - 2B_i \;, \tag{43}$$

$$h_{ij}^{(\mathrm{TT})} \to h_{ij}^{(\mathrm{TT})} \;. \tag{44}$$

Gathering terms, we see that the following combinations of these functions are gauge invariant:

$$\Phi \equiv -\phi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda} \;, \tag{45}$$

$$\Theta \equiv \frac{1}{3}\left(H - \nabla^2\lambda\right) \;, \tag{46}$$

$$\Xi_i \equiv \beta_i - \frac{1}{2}\dot{\varepsilon}_i \;; \tag{47}$$

$h_{ij}^{(\mathrm{TT})}$ is gauge-invariant without any further manipulation. In the Newtonian limit Φ reduces to the Newtonian potential Φ_N, while $\Theta = -2\Phi_N$. The total number of free, gauge-invariant functions is 6: 1 function Θ; 1 function Φ; 3 functions Ξ_i, minus 1 due to the constraint $\partial_i\Xi_i = 0$; and 6 functions $h_{ij}^{(\mathrm{TT})}$, minus 3 due to the constraints $\partial_i h_{ij}^{(\mathrm{TT})} = 0$, minus 1 due to the constraint $\delta^{ij} h_{ij}^{(\mathrm{TT})} = 0$. This is in keeping with the fact that in general the 10 metric functions contain 6 physical and 4 gauge degrees of freedom.

We would now like to enforce Einstein's equation. Before doing so, it is useful to first decompose the stress energy tensor in a manner similar to that of our decomposition of the metric. We define the quantities ρ, S_i, S, P, σ_{ij}, σ_i and σ via the equations

$$T_{00} = \rho \,, \tag{48}$$
$$T_{0i} = S_i + \partial_i S \,, \tag{49}$$
$$T_{ij} = P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \tag{50}$$

together with the constraints

$$\partial_i S_i = 0 \,, \tag{51}$$
$$\partial_i \sigma_i = 0 \,, \tag{52}$$
$$\partial_i \sigma_{ij} = 0 \,, \tag{53}$$
$$\delta^{ij}\sigma_{ij} = 0, \tag{54}$$

and boundary conditions

$$S \to 0, \quad \sigma_i \to 0, \quad \sigma \to 0, \quad \nabla^2\sigma \to 0 \tag{55}$$

as $r \to \infty$. These quantities are not all independent. The variables ρ, P, S_i and σ_{ij} can be specified arbitrarily; stress-energy conservation ($\partial^a T_{ab} = 0$) then determines the remaining variables S, σ, and σ_i via

$$\nabla^2 S = \dot{\rho} \,, \tag{56}$$
$$\nabla^2 \sigma = -\frac{3}{2}P + \frac{3}{2}\dot{S} \,, \tag{57}$$
$$\nabla^2 \sigma_i = 2\dot{S}_i \,. \tag{58}$$

We now compute the Einstein tensor from the metric (29) – (31). The result can be expressed in terms of the gauge invariant observables:

$$G_{00} = -\nabla^2\Theta \,, \tag{59}$$
$$G_{0i} = -\frac{1}{2}\nabla^2\Xi_i - \partial_i\dot{\Theta} \,, \tag{60}$$
$$G_{ij} = -\frac{1}{2}\Box h_{ij}^{(\mathrm{TT})} - \partial_{(i}\dot{\Xi}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi + \Theta\right) + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi + \Theta\right) - \ddot{\Theta}\right] \,. \tag{61}$$

We finally enforce Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and simplify using the conservation relations (56) – (58); this leads to the following field equations:

$$\begin{aligned} \nabla^2\Theta &= -8\pi\rho \,, && (62)\\ \nabla^2\Phi &= 4\pi\left(\rho + 3P - 3\dot{S}\right) \,, && (63)\\ \nabla^2\Xi_i &= -16\pi S_i \,, && (64)\\ \Box h_{ij}^{(\mathrm{TT})} &= -16\pi\sigma_{ij} \,. && (65) \end{aligned}$$

Notice that **only the metric components $h_{ij}^{(\mathrm{TT})}$ obey a wave-like equation**. The other variables Θ, Φ and Ξ_i are determined by Poisson-type equations. Indeed, in a purely vacuum spacetime, the field equations reduce to five Laplace equations and a wave equation:

$$\begin{aligned} \nabla^2\Theta^{\mathrm{vac}} &= 0 \,, && (66)\\ \nabla^2\Phi^{\mathrm{vac}} &= 0 \,, && (67)\\ \nabla^2\Xi_i^{\mathrm{vac}} &= 0 \,, && (68)\\ \Box h_{ij}^{(\mathrm{TT}),\mathrm{vac}} &= 0 \,. && (69) \end{aligned}$$

This manifestly demonstrates that only the $h_{ij}^{(\mathrm{TT})}$ metric components — the transverse, traceless degrees of freedom of the metric perturbation — characterize the radiative degrees of freedom in the spacetime. Although it is possible to pick a gauge in which other metric components *appear* to be radiative, they will not be: Their radiative character is an illusion arising due to the choice of gauge or coordinates.

The field equations (62) – (65) also demonstrate that, far from a dynamic, radiating source, the time-varying portion of the physical degrees of freedom in the metric is dominated by $h_{ij}^{(\mathrm{TT})}$. If we expand the gauge invariant fields Φ, Θ, Ξ_i and $h_{ij}^{(\mathrm{TT})}$ in powers of $1/r$, then, at sufficiently large distances, the leading-order $O(1/r)$ terms will dominate. For the fields Θ, Φ and Ξ_i, the coefficients of the $1/r$ pieces are combinations of the conserved quantities given by the mass $\int d^3x T_{00}$, the linear momentum $\int d^3 T_{0i}$ and the angular momentum $\int d^3x(x_i T_{0j} - x_j T_{0i})$. Thus, the only time-varying piece of the physical degrees of freedom in the metric perturbation at order $O(1/r)$ is the TT piece $h_{ij}^{(\mathrm{TT})}$.

Although the variables Φ, Θ, Ξ_i and $h_{ij}^{(\mathrm{TT})}$ have the advantage of being gauge invariant, they have the disadvantage of being *non-local* (a part from $h_{ij}^{(\mathrm{TT})}$ computation of these variables at a point requires knowledge of the metric perturbation $h_{\mu\nu}$ everywhere). This non-locality obscures the fact that the physical, non-radiative degrees of freedom are causal, a fact which is explicit in Lorentz gauge. One way to see that the guage invariant degrees of freedom are causal is to combine the vacuum wave equation eq. (23) for the metric perturbation with the expression (11) for the gauge-invariant

Riemann tensor. This gives the wave equation $\Box R_{\alpha\beta\gamma\delta} = 0$. Moreover, many observations that seek to detect GWs are sensitive only to the value of the Riemann tensor at a given point in space (see sec.). For example, the Riemann tensor components R_{itjt} are given in terms of the gauge invariant variables as

$$R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij}^{(\mathrm{TT})} + \Phi_{,ij} + \dot{\Xi}_{(i,j)} - \frac{1}{2}\ddot{\Theta}\delta_{ij}. \tag{70}$$

Thus, at least certain combinations of the gauge invariant variables are locally observable.

Energy of GWs

We have shown that only the TT part of the metric is actually a radiative degree of freedom, i.e. a GW, which is capable of transporting energy, momentum and angular momentum from the source emitting them. In order to derive the expression for such quantities we follow here sec. 1.4 of [15].

[15] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

In principle it is not unambigous to separate the background metric from the perturbation, but a natural splitting between spacetime background and GWs arise when there is a clear separation of scales: if the variation scale of $h_{\mu\nu}$ is λ and the variation scale of the background is $L_B \ll \lambda$ a separation is possible. We can e.g. average over a time scale $\bar{t} >> \lambda$ and obtain

$$\bar{R}_{\mu\nu} = 8\pi G_N \left(\tau_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tau\right) - \langle R^{(2)}\rangle\,, \tag{71}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor computed on the background metric (and then vanishing in an expansion over Minkowski background), and $R^{(2)}_{\mu\nu}$ is the part of the Ricci tensor quadratic in the GW perturbation (no part linear in the perturbation survives after averaging). The $-\langle R^{(2)}\rangle$ in the above equation can be interpreted as giving contribution to an effective energy momentum tensor of the GWs $\tau_{\mu\nu}$ given by

$$\tau_{\mu\nu} = -\frac{1}{8\pi G_N}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}\rangle \tag{72}$$

(and $\tau = \langle R^{(2)}\rangle/(8\pi G_N)$). The expression for $R^{(2)}_{\mu\nu}$ is quite lengthy, see eq. (1.131) of [16], but using the Lorentz condition, $h = 0$ and neglecting terms which vanish on the equation of motion $\Box\bar{h}_{\mu\nu} = 0$ we have

[16] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu h_{\alpha\beta}h^{\alpha\beta}\partial_\nu\rangle\,, \tag{73}$$

$$\tau_{\mu\nu} = \frac{1}{32\pi G_N}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\,, \tag{74}$$

This effective energy-momentum tensor is gauge invariant and thus depend only on $h_{ij}^{(TT)}$, giving

$$\tau^{00} = \frac{1}{16\pi G_N}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

For a plane wave travelling along the z direction we have $\tau_{01} = 0 = \tau_{02}$ and $\partial_z h_{ij}^{(TT)} = \partial^0 h_{ij}^{(TT)}$ and then $\tau^{03} = \tau^{00}$. For a spherical wave $\partial_r h_{ij}^{(TT)} = \partial^0 h_{ij}^{(TT)} + O(1/r^2)$, so similarly $\tau^{0r} = \tau^{00}$.

The time derivative of the GW energy E_V (or energy flux dE_V/dt) can be written as

$$\begin{aligned} \frac{dE_V}{dt} &= \int_V d^3x \partial_0 \tau^{00} \\ &= \int_V d^3x \partial_i \tau^{0i} \\ &= \int_S dA n_i \tau^{0i} = \\ &= \int_S dA \tau^{00} \end{aligned} \tag{75}$$

The ouward propagating GW carries then an energy flux $F = \frac{dE}{dt}$

$$F = \frac{r^2}{32\pi G_N}\int d\Omega \langle \dot{h}_{ij}^{(TT)} \dot{h}_{ij}^{(TT)} \rangle \tag{76}$$

or equivalently

$$F = \frac{1}{16\pi G_N}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle\,. \tag{77}$$

The linear momentum P_V^i of GW is

$$P_V^k = \int d^3x \tau^{0i}\,. \tag{78}$$

Considering a GW propagating radially outward, we have

$$\dot{P}_V^i = -\int_S \tau^{0i}\,, \tag{79}$$

hence the momentum carried away by the outward-propagating wave is $\frac{dP^i}{dAdt} = t^{0i}$. In terms of the GW amplitude it is

$$\frac{dP^i}{dt} = -\frac{r^2}{32\pi G_N}\int d\Omega \langle \dot{h}_{jk}^{(TT)} \partial^i h_{jk}^{(TT)} \rangle\,. \tag{80}$$

Note that, if τ^{0k} is odd under a parity transformation $\mathbf{x} \to -\mathbf{x}$, then the angular integral of eq. (80) vanishes. For the angular momentum of GW we refer to sec. 2.1.3 of [17].

[17] M. Maggiore. *Gravitational Waves.* Oxford University Press, 2008

Interaction of GWs with interferometric detectors

We now describe the interaction of a GW with a simplified exeperimental apparatus, following the discussion of sec. 1.3.3 of [18]. In a typical interferometer lights goes back and forth in orthogonal arms and recombining the photons after different trajectories very precise length measurements can be performed.

[18] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

The dynamics of a massive point particle can be inferred from the world-line action (13), whose variation with respect to the particle trajectory x^μ gives the geodesic equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}g^{\mu\alpha}\left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - \frac{1}{2}g_{\nu\rho,\alpha}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\,. \tag{81}$$

Let us appy the above equation to the motion of a mirror, at rest at $\tau = 0$ in the position $\mathbf{x}_m = (L, 0, 0)$, in a laser interferometer under the influence of a GW propagating along the z direction:

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\,\Gamma^i_{00}\left.\left(\frac{dx^0}{d\tau}\right)^2\right|_{\tau=0}\,. \tag{82}$$

In the TT gauge $\Gamma^i_{00} = 0$, showing that an object initially at rest will remain at rest even during the passage of a GW. This does not mean that the GW will have no effect, as the *physical* distance l between the the mirror and the beam splitter, say, at $\mathbf{x}_{bs} = (0, 0, 0)$ is given by

$$l = \int_0^L \sqrt{g_{xx}}dx \simeq \left(1 + \frac{1}{2}h_+(t)\right)L\,,$$

which shows how physical relative distance chage with time (we have assumed that h_+ does not depend on x, which is correct for $\lambda_{GW} \gg L$).

It is instructive to review this derivation in a different frame, the *proper detector frame*, whose coordinates allow a more transparent interpretation, as all physical results are *independent of frame choices*. Experimentally one has the mirror and the beam splitter, which are "freely-falling", as they are suspended to the ceiling forming a pendulum with very little friction and low typical frequency ($\sim$ few Hz), meaning that over frequency scale $10 - 10^3$ Hz they behave as freely-falling particles, see fig. 2. Standard rulers on the other hand are made of tightly bound objects, endowed with friction and resotring forces. The distance they measue $\xi(t)$ under the influence of a monochromatic wave frequency ω undergoes oscillation satisfying the equation

$$\ddot{\xi}(t) + \gamma\omega_0\dot{\xi}(t) + \omega_0^2\xi(t) = -\frac{\omega^2}{2}h_0\cos(\omega t)L \tag{83}$$

with solution

$$\xi(t) = \frac{1}{2} L h_0 \omega^2 \frac{(\omega^2 - \omega_0^2)\cos(\omega t) - \gamma\omega_0\omega\sin(\omega t)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega_0^2\omega^2}\,, \tag{84}$$

showing that for $\omega \gg \omega_0$ the mirror is indeed freely-falling, as it follows the GW time behaviour.

In the proper detector frame, which is the freely fallig frame for the observer at the origin at the coordinates, a general metric can be written as

$$\begin{aligned} ds^2 \;\simeq\;& -dt^2\left(1 + R_{0i0j}x^i x^j + \ldots\right) - 2dtdx^i\left(\frac{2}{3}R_{0ijk}x^j x^k + \ldots\right) \\ &+dx^i dx^j\left(\delta_{ij} - \frac{1}{3}R_{ijkl}x^k x^l \ldots\right) \end{aligned} \tag{85}$$

where terms of cubic and higher order in x have been omitted. The trajectory $x_0^\mu(\tau) = \delta^{\mu 0}\tau$ is clearly a geodesic, and we can consider the *geodesic deviation* equation, which gives the time evolution separation of two nearby geodesics $x_0^\mu(\tau)$ and $x_0^\mu(\tau) + \xi^\mu(\tau)$

$$\frac{D^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\xi^j\left(\frac{dx^0}{d\tau}\right)^2 . \tag{86}$$

In the proper detector frame, the measure of coordinate distances with respect to the origin gives actually a measure of physical distance (up to terms $\sim hL^3/\lambda^3$). The eq.(86) can be recast into

$$\ddot{\xi} = \frac{1}{2}\ddot{h}_{ij}\xi^j\,, \tag{87}$$

where an overdot stands for a derivative with respect to t and terms of order h^2 have been neglected. In the proper detector frame, the effect of GWs on a point particle of mass m placed at coordinate ξ can be described in terms of a Newtonian force F_i

$$F_i = \frac{m}{2}\ddot{h}_{ij}^{(TT)}(t)\xi^j \tag{88}$$

(we neglect again the space dependence of h_+ as typically GW wavelength $\lambda \gg L$).

The laser light travels in two orthogonal arms of the interferometer and the electric fields are finally recombined on the photo-detector. The reflection off a 50-50 beam splitter can be modeled by multiplying the amplitude of the incoming field by $1/\sqrt{2}$ for reflection on one side and $-1/\sqrt{2}$ for reflection on the other, while transmission multiplies it by $1/\sqrt{2}$ and reflection by the end mirrors by -1, see sec. 2.4.1 of [19] for details.

[19] Andreas Freise and Kenneth A. Strain. Interferometer techniques for gravitational-wave detection. *Living Reviews in Relativity*, 13(1), 2010. DOI: 10.12942/lrr-2010-1. URL http://www.livingreviews.org/lrr-2010-1

Setting the mirrors at positions $(L_x, 0, 0)$ and $(0, L_y, 0)$ and the beam splitter at the origin of the coordinates, we can compute the

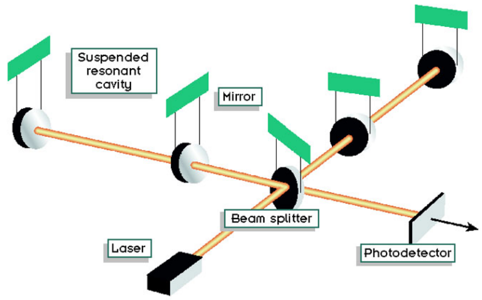

Figure 2: Interferometer scheme. Light emitted from the laser is shared by the two orthogonal arms after going through the beam splitter. After bouncing at the end mirrors it is recombined at the photo-detector.

phase change of the laser electric field moving in the x and y cavity which are eventually recombined on the photo-multiplier. For the light propagating along the x-axis, using the TT metric (26) for a $z-$propagating GW, one has

$$\begin{aligned} L_x &= (t_{1x} - t_{0x}) - \frac{1}{2}\int_{t_{0x}}^{t_{1x}} h_+(t')dt' \,, \\ L_x &= (t_{2x} - t_{1x}) - \frac{1}{2}\int_{t_{1x}}^{t_{2x}} h_+(t')dt' \,, \end{aligned} \tag{89}$$

where $t_{0,1,2}$ stand respectively for the time when the laser leaves the beam splitter, bounces off the mirror, returns to the beam splitter. The time at which laser from the two arms is combined at the beam splitter is common: $t_{2x} = t_{2y} = t_2$, and using $h_+(t) = h_0 \cos(\omega_{GW} t)$ we get

$$\begin{aligned} t_2 &= t_{0x} + 2L_x + \frac{h_0}{2\omega_{GW}}\left[\sin\left(\omega_{GW}(t_0+2L)\right) - \sin\left(\omega_{GW} t_0\right)\right] \\ &= t_{0x} + 2L_x + h_0 L_x \frac{\sin\left(\omega_{GW} L_x\right)}{\omega_{GW} L_x}\cos\left(\omega_{GW}(t_0 + L_x)\right) \end{aligned} \tag{90}$$

where the trigonometric identity $\sin(\alpha + 2\beta) = \sin(\alpha) + 2\cos(\alpha + \beta)\sin\beta$ has been used. Using that the phase of the laser field x (y) at recombination time t_2 is the same it had at the time it left the beam splitter at time t_{0x} (t_{0y}), we can write

$$\begin{aligned} E^{(x)}(t_2) &= -\frac{1}{2}E_0 e^{-i\omega_L t_{0x}} = \\ &= -\frac{1}{2}E_0 e^{-i\omega_L (t_2 - 2L) + i\phi_0 + i\Delta\phi_x} \end{aligned} \tag{91}$$

where

$$\begin{aligned} L &\equiv \frac{L_x + L_y}{2} \\ \phi_0 &= \omega_L (L_X - L_y) \\ \Delta\phi_x &= h_0 \omega_L L \frac{\sin(\omega_{GW} L)}{\omega_{GW} L}\cos(\omega_{GW}(t_2 - L))\,. \end{aligned} \tag{92}$$

where in the terms $O(h)$ we have set $L_x \simeq L_y \simeq L$. Analogously for the field that traveled through the $y-$ arm

$$\begin{aligned} E^{(y)}(t_2) &= \frac{1}{2}E_0 e^{-i\omega_L t_{0y}} = \\ &= \frac{1}{2}E_0 e^{-i\omega_L(t_2-2L)-i\phi_0+i\Delta\phi_y} \end{aligned} \tag{93}$$

with $\Delta\phi_y = -\Delta\phi_x$. The fields E_{pd} recombined at the photo-detector gives

$$\begin{aligned} E_{pd}(t) &= E^{(x)}(t) + E^{(y)}(t) \\ &= -iE_0 e^{-i\omega_L(t-2L)} \sin(\phi_0 + \Delta\phi_x) \ , \end{aligned} \tag{94}$$

with a total power $P = P_0 \sin^2(\phi_0 + \Delta\phi_x(t))$. Note that at the other output of the beam splitter, towards the laser, $E_L = E^{(x)} - E^{(y)}$ so that energy is conserved. The optimal length giving the highest $\Delta\phi_x$ is

$$L = \frac{\pi}{2\omega_{GW}} \simeq 750\text{km}\left(\frac{f_{GW}}{100Hz}\right)^{-1} . \tag{95}$$

Actually real interferometers include Fabry-Perot cavities, where the laser beam goes back and forth several times before being recombined at the beam splitter, allowing the *actual* of the photon travel path to be $\sim$ 100 km. For discussion of real interferometers with Fabry-Perot cavities see e.g. sec. 9.2 of [20], with the result that the sensitivity is enhanced by a factor $2F/\pi$ where F is the finesse of the Fabry-Perot cavity and typcally $F \sim O(100)$, giving a measured phase-shift of the order

[20] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

$$\Delta\phi_{FB} \sim \frac{4F}{\pi}\omega_L L h_0 \simeq \frac{8F}{\pi}\omega_L \Delta L \tag{96}$$

The typical amplitude h_0 that can be measured is of order 10^{-20} which at the best sensitive frequency gives for the Michelson interferometer $\delta L \sim 10^{-15}$ km!

The typical GW amplitude emitted by a binary system is

$$h \sim G_N M v^2/r \simeq 2.4 \times 10^{-22}\left(\frac{M}{M_\odot}\right)\left(\frac{v}{0.1}\right)^2\left(\frac{r}{Mpc}\right)^{-1} \tag{97}$$

many order of magnitudes small than the earth gravitational field. It is its peculiar time oscillating behaviour that makes possible its detection. We will see however in sec. that rather then the instantaneous aamplitude of the signal, its integrated vallue will be of interest for GW detection.

Numerology

Interferometric detectors are very precise and rapidly responsive ruler, they can detect the change of an arm length down to values of

10^{-15} m, however not at all frequency scales. At very low frequency ($f_{GW} \lesssim 10$ Hz) the noise from seismic activity and generic vibrations degrade the sensitivity of the instrument, whereas at high frequency laser shot noise does not allow to detect signal with frequency larger than few kHz. Considering binary systems, which emit according to the flux given in eq. (3), what is the typical lentgth, mass, distance scale of the source? Using

$$v \equiv (\pi f_{GW} G_N M)^{1/3} \,, \tag{98}$$

we obtain

$$\begin{aligned} v = \quad & (G_N M \pi f_{GW})^{1/3} & \simeq \quad & 0.054 \left(\frac{M}{M_\odot}\right)^{1/3} \left(\frac{f_{GW}}{10\text{Hz}}\right)^{1/3} , \\ r = \quad & G_N M (G_N M \pi f_{GW})^{-2/3} & \simeq \quad & 6.4\text{Km} \left(\frac{M}{M_\odot}\right)^{1/3} \left(\frac{f_{GW}}{10\text{Hz}}\right)^{-2/3} . \end{aligned} \tag{99}$$

It can also be interesiting to estimate how long it will take to for a coalescence to take place. Using the lowest order expression for energy and flux, one has

$$\begin{aligned} -\eta M v \frac{dv}{dt} &= -\frac{32}{5G_N} \eta^2 v^{10} \implies \\ \frac{dv}{v^9} &= \frac{32\eta}{5G_N M} dt \implies \\ \frac{1}{v_i^8} - \frac{1}{v_f^8} &= \frac{256\eta}{5G_N M} \Delta t \,, \end{aligned} \tag{100}$$

for the time Δt taken for the inspiralling system to move from v_i to v_f. If $v_i \ll v_f$ we can estimate

$$\Delta t \simeq \frac{5G_N M}{256\eta} v_i^{-8} \simeq 1.4 \times 10^4 \text{sec} \frac{1}{\eta} \left(\frac{M}{M_\odot}\right)^{-5/3} \left(\frac{f_{iGW}}{10Hz}\right)^{-8/3} . \tag{101}$$

Note that v_f can be comparable to v_i for very massive systems, which enter the detector sensitivity band when $v_i \lesssim 1$. For an estimate of the maximum relative binary velocity during the inspiral, we can take the inner-most stable circular orbit v_{ISCO} of the Schwarzschild case, which gives

$$v_{ISCO} = \frac{1}{\sqrt{6}} \simeq 0.41 \,. \tag{102}$$

The number of cycles N the GW spends in the detector sensitivity band can be derived by noting that

$$E = -\frac{1}{2} \eta M \left(G_N M \pi f_{GW}\right)^{2/3} \tag{103}$$

and from eq. (4)

$$
\begin{aligned}
N(t) &= \int_{t_i}^{t} f_{GW}(t)dt' \implies \\
N(f_{GW}) &\simeq \int_{f_{iGW}}^{f_{GW}} f\frac{dE/df}{dE/dt}df \\
&\simeq \frac{5G_N M}{96\eta}\int_{f_{iGW}}^{f_{GW}} (G_N M\pi f)^{-8/3}\, df \\
&= \frac{1}{32\pi\eta}(G_N M\pi)^{-5/3}\left(\frac{1}{f_{iGW}^{-5/3}} - \frac{1}{f_{GW}^{-5/3}}\right) \\
&\simeq 1.5\times 10^5\frac{1}{\eta}\left(\frac{M}{M_\odot}\right)^{-5/3}\left(\frac{f_{iGW}}{10\text{Hz}}\right)^{-5/3}
\end{aligned}
\tag{104}
$$

We can finally obtain the time evolution of the GW frequency

$$
\dot{f}_{GW} = \frac{96}{5}\pi^{8/3}\eta\,(G_N M)^{5/3}\, f_{GW}^{11/3} \tag{105}
$$

which has solution

$$
\begin{aligned}
f_{GW}(t) &= \frac{1}{\eta^{3/8}\pi}\left(\frac{5}{256}\frac{1}{|t|}\right)^{3/8} G_N M^{-5/8} \\
&= 151\text{Hz}\frac{1}{\eta^{3/8}}\left(\frac{M}{M_\odot}\right)^{-5/8}\left(\frac{|t|}{1\text{sec}}\right)^{-3/8},
\end{aligned}
\tag{106}
$$

which can be inverted to give

$$
|t_*(f)| = \frac{5}{256\pi\eta}\left(\frac{1}{\pi G_N M\pi}\right)^{5/3}. \tag{107}
$$

Starting from the GW expression in term of the source quadrupole

$$
h_{ij}^{TT}(t,\mathbf{x}) = \frac{2G_N}{d}\Lambda_{ij;kl}(\hat{\mathbf{n}})\ddot{M}_{kl}(t) \tag{108}
$$

For $\hat{\mathbf{n}} = \hat{\mathbf{z}}$

$$
\Lambda_{ij;kl}M_{kl} = \begin{pmatrix} (M_{xx}-M_{yy})/2 & M_{12} & 0 \\ M_{12} & (M_{yy}-M_{xx})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{109}
$$

we have

$$
\begin{aligned}
h_+ &= G_N\frac{\ddot{M}_{xx}-\ddot{M}_{yy}}{d}, \\
h_+ &= \frac{2G_N\ddot{M}_{xy}}{d}.
\end{aligned}
\tag{110}
$$

Assuming the sources are in circular motion, their relative distance can be parametrized as for

$$
\begin{aligned}
x(t) &= r\sin(\omega_s t), \\
y(t) &= -r\cos(\omega_s t), \\
z(t) &= 0,
\end{aligned}
\tag{111}
$$

hence yielding to

$$\begin{array}{rcl}\ddot{M}_{xx} & = & -\ddot{M}_{yy} = 2\mu r^2\omega_S^2\cos(2\omega_s t)\\ \ddot{M}_{xy} & = & 2\mu r^2\omega_S^2\sin(2\omega_s t)\end{array} \tag{112}$$

When the orbital planes is inclined by an angle ι with respect to the propagation direction (conventionally kept along the z-axis) one has to compute the rotated projected quadrupole tensor according to

$$M'_{ij} = R^{(y)}(\iota)_{ii'}M_{i'j'}\left(R^{(y)}\right)^{-1}_{j'j}(\iota) \tag{113}$$

and then project it with Λ to obtain

$$\Lambda_{ij,kl}M'_{kl} = \begin{pmatrix}\cos^2\iota M_{xx}/2 - M_{yy}/2 & \cos\iota M_{xy} & 0\\ \cos\iota M_{xy} & M_{yy}/2 - \cos^2\iota M_{xx}/2 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{114}$$

Now using the explicit expressions eqs. (112) one finds

$$\begin{array}{rcl}h_+ & = & \dfrac{1}{d}4G_N\mu\omega_S^2R^2\left(\dfrac{1+\cos^2\iota}{2}\right)\cos(2\omega_s t)\,,\\ h_\times & = & \dfrac{1}{d}4G_N\mu\omega_S^2R^2\cos\iota\sin(2\omega_s t)\,.\end{array} \tag{115}$$

Note that the rotation (113) is not the most generic rotation setting the orbital plane at an angle ι with the view direction $(0,0,1)$: an additional rotation by ψ around the z axis is permitted.

Using $f_{GW} = \omega_s/\pi$ and the explicit expression $f_{GW}(t)$ eq. (106) one has

$$\begin{array}{rcl}h_+(t) & = & \dfrac{1}{d}\left(G_N M_c\right)^{5/4}\left(\dfrac{5}{\tau}\right)^{1/4}\left(\dfrac{1+\cos^2\iota}{2}\right)\cos\Phi(\tau)\\ h_\times(t) & = & \dfrac{1}{d}\left(G_N M_c\right)^{5/4}\left(\dfrac{5}{\tau}\right)^{1/4}(\cos\iota)\sin\Phi(\tau)\end{array} \tag{116}$$

For data analysis we actually need this expression in frequency space, so let us compute it for the + polarization

$$\tilde{h}_+(f) = \frac{1}{2}\int dt A(t)\left(e^{i(2\pi f t+\Phi(\tau(t)))} + e^{i(2\pi f t-\Phi(\tau(t)))}\right)\,, \tag{117}$$

where $A(t)$ is defined by comparison with the (116). The above integral can be computed in the *stationary phase approximation* by expanding the exponent in the integrand around the stationary point t_* characterized by

$$2\pi f - \dot{\Phi}(\tau(t_*)) = 2\pi f + \left.\frac{d\Phi(\tau)}{d\tau}\right|_{\tau=t_c-t_*} = 0 \tag{118}$$

as follows:

$$e^{2\pi i f t - i\Phi(t)} \to e^{2\pi i f t_* - i\Phi(t_*)}\exp\left(-i\ddot{\Phi}(t_*)\frac{(t-t_*)^2}{2}\right) \tag{119}$$

then performing the resulting Gaussian integral as

$$\begin{aligned}\tilde{h}_+(f) &= \frac{1}{2}A(t_*)e^{i(2\pi f t_* - \Phi(t_*))}\int dt\, e^{-i\ddot{\Phi}(t_*)(t-t_*)^2/2} \\ &= \frac{1}{2}A(t_*)e^{i(2\pi f t_* - \Phi(t_*)-\pi/4)}\left(\frac{2\pi}{\ddot{\Phi}(t_*)}\right)^{1/2}.\end{aligned} \tag{120}$$

Time t_* can be expressed in terms of f_{GW} by inverting (106) and eq. (118) enable the identification between the Fourier transform argument f and f_{GW}, thus allowing to write

$$\tilde{h}_+(t_*(f)) = \left(\frac{5}{24}\right)^{1/2}\frac{\pi^{-2/3}}{d}(G_N M_c)^{5/6} f^{-7/6}\left(\frac{1+\cos^2\theta}{2}\right)e^{i(2\pi i t_* f - \Phi(t_*)-\pi/4)} \tag{121}$$

It is useful to introduce

$$v \equiv (\pi G_N M f_{GW})^{1/3} = (G_N M\omega)^{1/3}\,. \tag{122}$$

The most commonly used approximant is defined in the frequency domain as *TaylorF2*:

$$\begin{aligned}\psi(f) &= 2\pi f t_* + \phi_{ref} - 2\int^f \omega\frac{dt}{df}df \\ &= \phi_{ref} + \frac{2}{G_N M}\int^f (v_f^3 - v^3(f'))\frac{v(f')}{f'}\frac{dE/dv}{dE/dt}\frac{f'}{v(f')}\frac{dv}{df'}df' \\ &= 2\pi\int^f (v_f^3 v^{-2} - v)\left[\frac{-\eta M v}{-32/(G_N 5)\eta^2 v^{10}}\right]\frac{1}{3}df' \\ &= \frac{5\pi G_N M}{48\eta}\int^f (v_f^3 v^{-11} - v^{-8})\left(1 + c_{1PN}v^2 + \dots\right)df' \\ &= \frac{5}{48\eta}(\pi G_N M f)^{-8/3}\left[\left(-\frac{3}{8}+\frac{3}{5}\right) + \left(-\frac{1}{2}+1\right)c_{1PN}\dots\right] \\ &= \frac{3}{128\eta v^5(f)}\left(1 + \frac{20}{9}c_{1PN}v^2(f) + \dots\right)\end{aligned} \tag{123}$$

It may also be useful to have an expression of the phase in time-domain. Let us now relate the phase of the waveform to the dynamics of the sources by defining

$$\begin{aligned}\Delta\phi(t) &= 2\pi\int_{t_0}^t f_{GW}(t')dt' = 2\int_{v(t_0)}^{v(t)}\omega(v)\frac{dE/dv}{dE/dt}dv \\ &= \frac{2}{G_N M}\int_{v(t_0)}^{v(t)} v^3\frac{dE/dv}{dE/dt}dv \\ &= \frac{5}{16\eta}\int_{v(t_0)}^{v(t)}\frac{1}{v^6}\frac{1 + e_{v^2}v^2 + \dots}{1 + f_{v^3}v^3 + f_{v^4}v^4 + \dots}dv\,,\end{aligned} \tag{124}$$

where we have inserted the formal Taylor expansion of the energy and flux as functions of v and we have substituted $v = (G_N M\omega)^{1/3}$ (that is valid for circular orbits). We now see that we have different possibilities to compute the phase

- Truncate the v-series at some order both in the numerator and the denominator gives rise to the *TaylorT1* expression of the phase

- Expand the fraction in the above formula and truncate at some finite v-order $\to$ *TaylorT4*
- expand the inverse of the fraction appearing in the integrand $\to$ *TaylorT1*.

Elements of data analysis

The output of the detector $o(t)$ is a scalar time domain function, which in general will result of the addition of a part $h(t)$ linear in the impinging GW $h_{ij}(t, \mathbf{x}_D)$ at the location detector $\mathbf{x}_D$ and the instrumental noise $n(t)$. Usually the output of the detector is linearly related to the GW amplitude locally in the frequency space, i.e.

$$\tilde{h}(f) = \tilde{T}_{ij}(f)\tilde{h}_{ij}(f) \qquad (125)$$

where $T_{ij}(f)$ is the *transfer function* of the system and

$$\tilde{o}(f) = \tilde{h}(f) + \tilde{n}(f)\,. \qquad (126)$$

If the noise is *stationary* then different Fourier components are uncorrelated (see derivation below) and we can write

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \delta(f-f')\frac{1}{2}S_n(f)\,, \qquad (127)$$

which defines the *noise correlation function* $S_n(f)$, or *noise power spectral density*, with dimensions Hz^{-1}. Note also that $S_n(-f) = S_n(f)$.

The average in eq. (127) is taken over many noise realizations, but we have only one detectors, so it should be actually replaced over different time span averages:

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \frac{1}{N}\sum_{i=1}^{N}\tilde{n}_i(f)\tilde{n}_i^*(f')\,,$$

where

$$\tilde{n}_i(f) = \int_{t_i-T/2}^{t_i+T/2} n(t)e^{2\pi i f t}dt\,,$$

and $t_i = (\ldots, -2T, -T, 0, T, 2T, \ldots)$. We define the Fourier Transform of a function defined on an interval as

$$\tilde{n}(f) = \int_{t_i-T/2}^{t_i+T/2} n(t)e^{2\pi i f t}\,dt \qquad \text{with } n(t+T) = n(t) \qquad (128)$$

with the periodicity requirement implying the Fourier transform $\tilde{n}(f)$ be discrete, with support at $f_n = n/T$, i.e. with frequency resolution $\Delta f = 1/T$. This discreteness condition on the frequency automatically ensure that the inverse Fourier transform returns a periodic function[21]:

[21] Note that the definition (128) for a function defined on an interval is *not* equivalent to

$$\tilde{n}(f) \neq \int_{-\infty}^{\infty} n(t)e^{2\pi i f t}\theta(t-t_i+T/2)\theta(t_i+T/2-t)dt\,.$$

Had we used this definition one would have dealt with a non-analytic integrand in the time integral, that would have had discontinuities in his derivatives. As the discontinuities in the direct space functions are related to

$$n(t) = \frac{1}{T} \sum_{n\in\mathbb{N}} \tilde{n}(f) e^{-2\pi i n t/T} . \tag{130}$$

A very welcome by-product of the definition (128), restricting the integration domain to a finite interval and imposing periodicity condition on $n(t)$, is

$$\frac{1}{T}\int_{t_i-T/2}^{t_i+T/2} e^{2i\pi f t} dt = \frac{e^{2i\pi f t_i}}{2\pi i f T}\left(e^{2\pi i f T/2} - e^{-2\pi i f T/2}\right) = e^{2\pi i f t_i}\frac{\sin(\pi f T)}{\pi f T} = \delta_{f,0} \tag{131}$$

which is reminiscent of the standard $\int e^{2i\pi f t} dt = \delta(f)$ valid for functions defined on the entire real axis. The only difference between the finite segment and the entire real axis case is that here we deal with a dimension-less, discrete Kronecker delta rather than a delta function with dimension of time, so that the identification

$$\delta(f) \to T\delta_{f,0} \tag{132}$$

can be made.

Let us consider than the definition (127) and exploit the stationarity property of the noise:

$$\begin{aligned}
\langle \tilde{n}(f)\tilde{n}(f')\rangle &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \langle n(t) n(t')\rangle e^{2\pi i (f t + f' t')} \\
&= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \langle n(t+T_s) n(t'+T_s)\rangle e^{2\pi i (f t + f' t' + T_s(f+f'))} \\
&= \delta_{f+f',0} \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \langle n(t) n(t')\rangle e^{2\pi i f (t-t')}
\end{aligned} \tag{133}$$

where we have averaged over a time coordinate T_s and used stationarity to substitute $n(t+T_s) \to n(t)$ inside the average. Short-circuiting the above with the definition (127) and the correspondence (132) we obtain

$$S(f) = 2\frac{\langle |\tilde{n}(f)|^2\rangle}{T} = 2\langle |\tilde{n}(f)|^2\rangle \Delta f . \tag{134}$$

We can make the further connection to the noise auto-correlation function

$$R(\tau) \equiv \langle n(t+\tau) n(t)\rangle , \tag{135}$$

white noise corresponding to $R(\tau) \propto \delta(\tau)$. By noting that

$$\begin{aligned}
&\langle \int df' \int df\, n(f) n(f') e^{-2\pi i (f(t+\tau) + f' t)} \rangle \\
=\;& \frac{1}{2}\int df S_n(f) e^{-2\pi i f \tau}
\end{aligned} \tag{136}$$

we are enabled to interpret the noise spectral density function as the Fourier transform of the noise correlation function

$$\frac{1}{2} S_n(f) = \int d\tau R(\tau) e^{2\pi i f \tau} \tag{137}$$

and hence

$$R(\tau) = \frac{1}{T} \sum_{n\in\mathbb{N}} e^{-2\pi i f\tau} \frac{S_n(f)}{2} \,. \tag{138}$$

The factor $1/2$ in the definition is conventionally inserted as

$$\begin{aligned} \langle n^2(t)\rangle &= \frac{1}{T^2} \sum_n \sum_{n'} \langle n(f) n(f')\rangle e^{2\pi i (ft + f't')} \\ &= \frac{1}{2T} \sum_{n\in\mathbb{N}} S_n(f) = \frac{1}{T} \sum_{n\geq 0} S_n(f) \end{aligned} \tag{139}$$

and the factor $1/2$ disappears when sum is taken over positive frequencies only (we neglect subtleties about the $n = 0$ mode). The power spectral density of white noise is f-independent.

Actually in practice the time domain noise function will be discrete as well, implying that what we'll be really using are

$$\begin{aligned} \tilde{n}(f = k/T) &= \Delta t \sum_{j=0}^{N-1} e^{2\pi i jk\Delta t/T} \\ n(t = j\Delta t) &= \Delta f \sum_{j=0}^{N-1} e^{2\pi i jk\Delta t/T} \end{aligned} \tag{140}$$

with $\Delta f \Delta t = 1/N$ with $T/\Delta t = N$. With a finite sampling size there has to be a maximum frequency, called the *Nyquist frequency*

$$f_{Nyquist} = \frac{1}{2\Delta t} \tag{141}$$

so that we have N points for $n(t)$ and $N/2 + 1$ for $\tilde{n}(f)$ if N is even, as we will assume. Note that $\tilde{n}(f = 0)$ is real as well as $n(f = N/(2T))$ so that the information stored in the N real numbers of $n(t)$ is fully equivalent to the information stored in the $N/2 + 1$ numbers making up $\tilde{n}(f)$, 2 of which are real and $N/2 - 1$ of which are complex.

In particular the Parseval identity has a discrete counterpart

$$\Delta f \sum_{k=0}^{N/2} |\tilde{n}(k/T)|^2 = \Delta t \sum_{j=0}^{N-1} n^2\left(j\Delta t\right) \,. \tag{142}$$

Matched filtering

The signal amplitude is much smaller than the noise, just think of the earth gravitational field that is responsible for $h \sim 10^{-9} \gg 10^{-21}$. However if the signal is known in advance, we can correlate detector's output $o(t)$ with our expectation and dig it out of the noise floor. We thus have to *filter* the detector output to highlight the signal. An important quantity for any experiment is the *signal-to-noise ratio (SNR)* we are going to define now. It must involve a ratio S/N of a quantity linear in the signal h possibly filtered in order to enhance it and a quantity representative of the noise. We want to choose the filter function so to maximize the SNR, i.e. the filter has to *match* the signal. We can assume that by linearly filtering the detector output we can pick only the signal part $h(t)$ in $o(t)$ and we can tentatively define the numerator of the SNR as

$$S = \int dt \langle o(t) \rangle K(t) \,, \tag{143}$$

and since $\langle n(t) \rangle = 0$ we have

$$S = \int dt \langle h(t) \rangle K(t) = \int df \tilde{h}(f) \tilde{K}^*(f) \,, \tag{144}$$

where for simplicity we have moved back to continuum time-frequency space. For the SNR denominator N we want an estimator of the noise. A reasonable guess would be the root mean square of the detector output in absence of the signal, i.e.

$$N^2 \stackrel{?}{=} \langle o^2(t) \rangle - \langle o(t) \rangle^2|_{h=0} = \langle n(t) \rangle^2 \tag{145}$$

but we want the overall filter scale to drop out of the SNR, so we'd better define

$$\begin{aligned} N^2 &= \int dt\, dt'\, K(t) K(t') \langle n(t) n(t') \rangle \\ &= \int df\, df' \langle n(f) n(f') \rangle e^{2\pi i (ft + ft')} \tilde{K}(f) \tilde{K}(f') \\ &= \frac{1}{2} \int df S_n(f) |\tilde{K}(f)|^2 \,. \end{aligned} \tag{146}$$

We have constructed then our SNR as

$$\frac{S}{N} = \frac{2^{1/2} \int_{-\infty}^{\infty} df \tilde{h}(f) \tilde{K}^*(f)}{\left(\int_{-\infty}^{\infty} df S_n(f) |\tilde{K}(f)|^2 \right)^{1/2}} \,. \tag{147}$$

In order to find the filter function maximizing the SNR we define a positive definite scalar product

$$(A|B) \equiv 2 \int df \frac{A(f) B^*(f)}{S_n(f)} \tag{148}$$

which is real if we assume that $A^*(f) = A(-f)$ and $B^*(f) = B(-f)$, as it is for the Fourier transform of real functions. We can now re-write the SNR as

$$\frac{S}{N} = \frac{(\tilde{u}|\tilde{h})}{(\tilde{u}|\tilde{u})^{1/2}} \tag{149}$$

with $\tilde{u}(f) \equiv 1/2 S_n(f) \tilde{K}(f)$. We are thus searching for the normalized vector u that maximizes its scalar product with h, clearly the solution can only be $\tilde{u} \propto \tilde{h}$, i.e.

$$\tilde{K}(f) = \frac{\tilde{h}(f)}{S_n(f)} \tag{150}$$

up to an inessential f-independent constant. Substituting in eq. (147) we finally obtain

$$\frac{S}{N} = \left[2\int_{-\infty}^{\infty} df \frac{|\tilde{h}(f)|^2}{S_n(f)}\right]^{1/2} . \tag{151}$$

So far we have assumed perfect knowledge of the signal. What if we do not know the exact *time location* of the $h(t)$? By considering a function $h(t)$ and its time-shifted $h_{t_0}(t) \equiv h(t-t_0)$ the relationship among their Fourier transform is

$$\tilde{h}(f) = \tilde{h}_{t_0}(f)\, e^{2\pi i f t_0} . \tag{152}$$

If we try to match data $\tilde{h}(f)$ with a *template* signal $\tilde{h}_{tmplt}(f)$ allowing for a generic time-shift t, we obtain an SNR time series

$$\begin{aligned} SNR(t) &= \sqrt{2}\frac{\int_{-\infty}^{\infty} df\, \frac{\tilde{h}(f)\tilde{h}^*_{tmplt}(f)}{S_n(f)} e^{2\pi i f t}}{\left(\int_{-\infty}^{\infty} df\, |\tilde{h}_{tmplt}(f)|^2/S_n(f)\right)^{1/2}} = \\ &= \frac{\int_0^{\infty} df\, \left(\tilde{h}(f)\tilde{h}^*_{tmplt}(f) e^{2\pi i f t} + \tilde{h}^*(f)\tilde{h}_{tmplt}(f) e^{-2\pi i f t}\right)/S_n(f)}{\left(\int_0^{\infty} df\, |\tilde{h}_{tmplt}(f)|^2/S_n(f)\right)^{1/2}} \end{aligned} \tag{153}$$

and the use of a Fast Fourier Transform allows to search for all time values efficiently.

Let us consider now the case of a constant phase offset between the template and the signal and let us concentrate on the simplified situation of a signal $h(t) \propto \cos(2\pi f_0 t)$, i.e. $\tilde{h}(f) \propto \delta(f-f_0) + \delta(f+f_0)$.

Taking the correlator with a filter function of the type $K(t) \propto \cos(2\pi f_1 t + \bar{\phi})$ we would have a non-null result in case $f_0 = f_1$ proportional to $\cos(\phi)$. Clearly the filter *is* matching the signal but our ignorance on the right ϕ value may suppress the matched filter output. What one would like is the SNR time series to be the maximum as $\bar{\phi}$ varies. It turns out that it is in fact possible to maximize over $\bar{\phi}$ analytically. In order to show how let us fix the $t = 0$ for simplicity:

$$\begin{aligned} \left(\frac{1}{2}\int_{-\infty}^{\infty} df |h_t(f)|^2/S_n(f)\right)^{1/2} \frac{S}{N}(t=0) &= \int_{-\infty}^{\infty} \tilde{h}(f)\tilde{h}^*_t(f) e^{i\bar{\phi}(f)} df \\ &= \int_0^{\infty} \left(\tilde{h}(f)\tilde{h}_t(-f)e^{i\bar{\phi}} + \tilde{h}(-f)\tilde{h}_t(f)e^{-i\bar{\phi}}\right) df \\ &= \mathcal{R}\cos\bar{\phi} - \mathcal{I}\sin\bar{\phi} , \end{aligned} \tag{154}$$

where the real quantities $\mathcal{R}$ and $\mathcal{I}$ are defined as

$$\begin{aligned} \mathcal{R} &\equiv \int_0^{\infty} (\tilde{h}(f)\tilde{h}^*_t(f) + \tilde{h}^*(f)\tilde{h}^*_t(f))\, df &&= 2Re\int_0^{\infty} \tilde{h}(f)\tilde{h}_t(f) df \\ \mathcal{I} &\equiv i\int_0^{\infty} (h(f)h^*_t(f) - h^*(f)h_t(f))\, df &&= -2Im\int_0^{\infty} \tilde{h}(f)\tilde{h}^*_t(f) df . \end{aligned} \tag{155}$$

The output of the matched filter depends on $\bar{\phi}$, but analytically maximizing over $\bar{\phi}$ is possible:

$$\frac{dS/N}{d\phi_0} = 0 \implies \cos\bar{\phi} = \frac{\mathcal{R}}{\sqrt{\mathcal{R}^2 + \mathcal{I}^2}}, \qquad \sin\bar{\phi} = -\frac{\mathcal{I}}{\sqrt{\mathcal{R}^2 + \mathcal{I}^2}}, \tag{156}$$

which give the *SNR* maximized over $\bar{\phi}$

$$\left({}^{Max}_{\phi_0} \left.\frac{S}{N}\right|_{t=0} \right)^2 = 2 \frac{\mathcal{R}^2 + \mathcal{I}^2}{\int_{-\infty}^{\infty} |h_t(f)|^2 / S_n(f) df}. \tag{157}$$

Note that there is an efficient way to compute both the quantities $\mathcal{R}$ and $\mathcal{I}$: it is by computing the *complex inverse Fourier transform*

$$\rho(t) \equiv \sqrt{2}(\tilde{h}_t|\tilde{h}_t)^{-1/2} \int_0^{\infty} \frac{\tilde{h}(f)\tilde{h}_t^*(f)}{S_n(f)} e^{2\pi i f t} df \tag{158}$$

and we have

$$ {}^{Max}_{\phi_0} \frac{S}{N}(t) = 2\sqrt{2}|\rho(t)|. \tag{159}$$

Templates differing by a phase shift like $\tilde{h}_{tmplt}(f) \to \tilde{h}'_{tmplt}(f) = \tilde{h}_{tmplt}(f)e^{i\phi_0}$ (for $f > 0$ and $\tilde{h}'_{tmplt}(f) = \tilde{h}_{tmplt}(f)e^{-i\phi_0}$ for $f < 0$) will give rise to different SNRs, but the modulus of ρ obtained with them will be the same.

Exercise 1 ** Coordinate transformation

Derive the second of eq. (10) by assuming (9).

Exercise 2 * Newtonian gravitational waveform**

Using eqs. (2,3,4) compute numerically the "Newtonian" gravitational waveform (for semplicity assume constant amplitude).

Exercise 3 ** Linearized Riemann, Ricci and Einstein tensors

Using that the Christoffel symbols at linear level are

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^\alpha_\nu + \partial_\nu h^\alpha_\mu - \partial^\alpha h_{\mu\nu}\right)$$

derive eqs.(11)

Exercise 4 * Retarded Green function I

Show that the Green functions in eqs. (21) satisfy eq. (22) Hint: use that in spherical coordinates

$$\nabla^2\psi(r) = \frac{1}{r}\partial_r^2(r\psi(r))\,.$$

Exercise 5 * Retarded Green function I**

Show that the two representation of the retarded Green function given by eq. (22) and

$$G_{ret}(t,x) = -i\theta(t)\left(\Delta_+(t,x) - \Delta_-(t,x)\right)\,,$$

where

$$\Delta_\pm(t,x) \equiv \int_{\mathbf{k}} e^{\mp ikt}\frac{e^{i\mathbf{kx}}}{2k}$$

are equivalent. Hint: use that

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx} = \delta(x)\,,$$

and that

$$\theta(t)\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(t+r)} = 0 \quad \text{for } r \geq 0\,.$$

Exercise 6 * Retarded Green function II**

Use the representation of the G_{ret} obtained in the previous exercise to show that

$$G_{ret}(t,\mathbf{x}) = -\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega t + i\mathbf{kx}}}{k^2 - (\omega + i\epsilon)^2}\,,$$

where ϵ is an arbitrarily small positive quantity. Hint: use that

$$\theta(\pm t) = \mp\frac{1}{2\pi i}\int\frac{e^{-i\omega t}}{\omega \pm i\epsilon}\,.$$

Show that G_{ret} is real.

Exercise 7 * Advanced Green function**

Same as the two exercises above for G_{adv}, with

$$\begin{aligned} G_{adv}(t,\mathbf{x}) &= i\theta(-t)\left(\Delta_+(t,x)-\Delta_-(t,x)\right), \\ G_{adv}(t,\mathbf{x}) &= -\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega t+i\mathbf{kx}}}{k^2-(\omega-i\epsilon)^2}. \end{aligned}$$

Show that G_{adv} is real.

Exercise 8 ** Feynman Green function I

Show that the G_F defined by

$$G_F(t,\mathbf{x})=-i\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega t+i\mathbf{kx}}}{k^2-\omega^2-i\epsilon}$$

is equivalent to

$$G_F(t,\mathbf{x})=\theta(t)\Delta_+(t,\mathbf{x})+\theta(-t)\Delta_-(t,\mathbf{x}).$$

Derive the relationship

$$G_F(t,\mathbf{x})=\frac{i}{2}\left(G_{adv}(t,\mathbf{x})+G_{ret}(t,\mathbf{x})\right)+\frac{\Delta_+(t,\mathbf{x})+\Delta_-(t,\mathbf{x})}{2}.$$

Exercise 9 * Feynman Green function II**

By integrating over $\mathbf{k}$ the G_F in the $\sim 1/(k^2-\omega^2)$ representation, show that G_F implements boundary conditions giving rise to field h behaving as

$$h(t,\mathbf{x})\sim\int d\omega e^{-i\omega t+i|\omega|r},$$

corresponding to out-going (in-going) wave for $\omega > (<)0$. Since an $\omega < 0$ solution is equivalent to a $\omega > 0$ solution propagating backward in time, this result can be interpreted by saying that using G_F results into having pure out-going (in-going) wave for $t\to\pm\infty$.

Exercise 10 * TT gauge

Show that the projectors defined in eq. 27 satisfy the relationships

$$\begin{aligned} P_{ij}P_{jk} &= P_{ik} \\ \Lambda_{ij,kl}\Lambda_{kl,mn} &= \Lambda_{ij,mn}, \end{aligned}$$

which charecterize projectors operator.

Exercise 11 *** Energy of circular orbits in a Schwarzschild metric**

Consider the Schwarzschild metric

$$ds^2=-\left(1-\frac{2G_NM}{r}\right)dt^2+\frac{dr^2}{\left(1-\frac{2G_NM}{r}\right)}+r^2d\Omega^2. \tag{160}$$

The dynamics of a point particle with mass m moving in such a background can be described by the action

$$S=-m\int d\tau=-m\int d\lambda\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

for any coordinate λ parametrizing the particle world-line. Using $S = \int d\lambda\, L$, we can write

$$L = -m \left[\left(1 - \frac{2G_N M}{r}\right) \left(\frac{dt}{d\tau}\right)^2 - \frac{\left(\frac{dr}{d\tau}\right)^2}{\left(1 - \frac{2G_N M}{r}\right)} - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right]^{1/2} .$$

Verify that L has cyclic variables t and ϕ and derive the corresponding conserved momenta.
(Hint: use $g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = -1$. Result: $E = m(dt/d\tau)(1 - 2G_N M/r) \equiv m\, e$ and $L = mr^2(d\phi/d\tau) \equiv m\, l$).
By expressing $dt/d\tau$ and $d\phi/d\tau$ in terms of e and l, derive the relationship

$$e^2 = (1 - 2G_N M/r)\left(1 + l^2/r^2\right) + \left(\frac{dr}{d\tau}\right)^2 .$$

From the circular orbit conditions ($\frac{de}{dr} = 0 = dr/d\tau = 0$), derive the relationship between l and r for circular orbits.
(Result: $l^2 = Mr/(1 - 3M/r)$).
Substitute into the energy function e and find the circular orbit energy

$$e(x) = \frac{1 - 2x}{\sqrt{1 - 3x}} ,$$

where $x \equiv (G_N M\dot{\phi})^{2/3}$ is an observable quantity as it is related to the GW frequency f_{GW} by $x = (G_N M \pi f_{GW})^{2/3}$.
(Hint: Use

$$\dot{\phi} = \frac{d\phi}{d\tau}\dot{\tau} = \frac{d\phi}{d\tau}\frac{1 - 2M/r}{e}$$

to find that on circular orbits $(G_N M\dot{\phi})^2 = (G_N M/r)^3$, an overdot stands for derivative with respect to t.)
Derive the relationships for the Inner-most stable circular orbit

$$\begin{aligned} r_{ISCO} &= 6G_N M = 4.4\text{km}\left(\frac{M}{M_\odot}\right) \\ f_{ISCO} &= \frac{1}{6^{3/2}}\frac{1}{G_N M \pi} \simeq 8.8\text{kHz}\left(\frac{M}{M_\odot}\right)^{-1} \\ v_{ISCO} &= \frac{1}{\sqrt{6}} \simeq 0.41 \end{aligned}$$

Exercise 12 **   Ruler under GW action

Derive the solution (84) to the eq. (83).

Exercise 13 **   Newtonian force exerced by GWs

Derive eq. (87) from eq. (86)

Exercise 14 **   Energy realeased by GWs**

Derive the work done on the experimental apparatus by the GW Newtonian force of eq. (87).

Exercise 15 ***   Energy realeased by GWs**

Separate the degrees of freedom of the electromagnetic field into gauge, longitudinal and transverse, analogously to the gravitational case in eqs. (62-65).

Hint: split the equation

$$\partial_\nu F^{\nu\mu} = -4\pi J^\mu$$

into its 0 and its 3 spatial components

$$\nabla^2 A^0 + \dot{A}^i_i = -4\pi\rho$$
$$\Box A^i + \dot{A}_{0,i} + A_{k,i}{}^{k} = -4\pi J^i\,.$$

Now decompose spatial vectors into a sum of a pure gradient and a divergence-free part:

$$J^i = \tilde{J}^i + \partial^i \mathcal{J}$$

where $\nabla_i \cdot \tilde{J}^i = 0$ and $\mathcal{J} \equiv \frac{\partial_i J^i}{\nabla^2}$ and analogously for A_i.

Note that under a gauge transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$, hence $A_0 \to A'_0 = A_0 + \dot{\Lambda}$, $\mathcal{A} \to \mathcal{A}' = \mathcal{A} + \Lambda$ and $\tilde{A}_i \to \tilde{A}'_i = \tilde{A}_i$, implying that $A_0 - \dot{\mathcal{A}}$ is invariant and we can rewrite the equations as

$$\nabla^2(A^0 + \dot{\mathcal{A}}) \quad = \quad -4\pi\rho$$
$$-\ddot{\tilde{A}}^i + \nabla^2 \tilde{A}^i = -4\pi \tilde{J}^i$$
$$\ddot{\mathcal{A}} + \dot{A}_0 = 4\pi\mathcal{J}$$

which are compatible $\iff$ $\dot{\rho} + \partial_i J^i = 0$, which is the continuity equation for the electromagnetic source.

Exercise 16 ** Gauge fixing of electromagnetic field

Consider the electromagnetic action

$$-\frac{1}{2}\left(k^2\eta_{\mu\nu} - k^\mu k^\nu\right) A_\mu A_\nu$$

ad try to inver the kinetic operator. What goes wrong?

Add to the Lagrangean an arbitrary term $\frac{1}{\xi} k^\mu k^\nu A_\mu A_\nu$ and derive the inverse of the kinetic operator

Exercise 17 ** Lorentz gauge

Show that a coordinate transformation characterized by $\Box\xi^\mu = 0$ does not invalidate the Lorentz gauge condition $\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h$.

The post-Newtonian expansion in the effective field theory approach

We want to obtain the PN correction to the Newtonian potential due to GR and we want to work at the level of the *equation of motions*. The dynamics for the massive particle (star/black hole) is given by the world-line action (13) and the "bulk" gravitational dynamics by

$$S_{EH+GF\Gamma} = -\frac{1}{64\pi G_N}\int dt d^d x\, h^{\mu\nu} A'_{\mu\nu\rho\sigma}\Box h^{\rho\sigma} \qquad (161)$$

with A' given by eq. (18).

Working at the level of e.o.m. we could solve them perturbatively, by taking as a first approximation [22]

$$\Box\bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}\,, \qquad (162)$$

then using the solution

$$\bar{h}^{(N)}_{\mu\nu} = 4G_N \int dt d^3\mathbf{x}'\; G_{Ret}(t-t',\mathbf{x}-\mathbf{x}')T_{\mu\nu}(t',\mathbf{x}') \qquad (163)$$

and finally pluggin this solution back into the $O(h^2)$ Einstein equation

$$\Box h^{(1PN)}_{\mu\nu} \simeq \partial^2\left(h^{(N)}_{\mu\nu}\right)^2 \qquad (164)$$

but we are going to perform the computation more efficiently.

We can use the Lagrangian construction in order to solve for the h field, as in ex. 20, but here we want to show how powerful the effective action method is in determining the dynamics of the 2-body system, "integrating out" the gravitational degrees of freedom.

Let us pause briefly to introduce some technicalities about *Gaussian integrals* that will be helpful later. The basic formula we will need is

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2+Jx}dx = \left(\frac{2\pi}{a}\right)^{1/2}\exp\left(\frac{J^2}{2a}\right)\,. \qquad (165)$$

or its multi-dimensional generalization

$$\int e^{-\frac{1}{2}x^i A_{ij}x^j + J^i x_i}dx_1\dots dx_n = \frac{(2\pi)^{n/2}}{(det A)^{1/2}}\exp\left(\frac{1}{2}J^t A^{-1}J\right)\,. \qquad (166)$$

[22] Note that left hand side of eq. (20) does not follow from eq. (14), as the gauge fixing term (**??**) is missing.

Other useful formulae are

$$\int x^k e^{-\frac{1}{2}ax^2+Jx}dx = \left(\frac{2\pi}{a}\right)^{1/2}\left(\frac{d}{dJ}\right)^k \exp\left(\frac{J^2}{2a}\right) \tag{167}$$

from which it follows that

$$\begin{aligned}\int x^{2n} e^{-\frac{1}{2}ax^2}dx &= \left(\frac{2\pi}{a}\right)^{1/2}\left(\frac{d}{dJ}\right)^{2n} \exp\left(\frac{J^2}{2a}\right)\Bigg|_{J=0} \\ &= \frac{(2n-1)!!}{a^n}\left(\frac{2\pi}{a}\right)^{1/2},\end{aligned} \tag{168}$$

which also admit a natural generalization in case of x is not a real number but an element of a vector space. Let us see how this will be useful in the toy model of massless, non self-interacting scalar field Φ interacting with a source J:

$$\begin{aligned} S_{toy} &= \int dtd^dx\left[-\frac{1}{2}\left(\partial\Phi(t,\mathbf{x})\right)^2 + J(t,\mathbf{x})\Phi(t,\mathbf{x})\right] \\ &= \int_{\mathbf{k}}\frac{dk_0}{2\pi}\left[\Phi(k_0,\mathbf{k})\Phi^*(k_0,\mathbf{k})\left(k_0^2-k^2\right) + J(k_0,\mathbf{k})\Phi^*(k_0,\mathbf{k})\right]\end{aligned} \tag{169}$$

and apply the above eqs.(165–168), with two differences:

- Here the integration variable is Φ, depending on 2 continous indices $(k_0, \mathbf{k})$, instead of the discrete index $i \in 1 \ldots n$
- the Gaussian integrand is actually turned into a complex one, as we are taking at the exponent

  $$\begin{aligned} Z_0[J] \equiv &\int \mathcal{D}\Phi \exp\left\{ i\int_{\mathbf{k}}\frac{dk_0}{2\pi}\left[\frac{1}{2}\left(k_0^2-k^2\right)\Phi(k_0,\mathbf{k})\Phi(-k_0,-\mathbf{k}) + \right.\right. \\ &\left.\left. J(k_0,\mathbf{k})\Phi(-k_0,-\mathbf{k}) + i\epsilon|\Phi(k_0,\mathbf{k})|^2\right]\right\},\end{aligned} \tag{170}$$

  where the ϵ has been added term ensure convergence for $|\Phi| \to \infty$.

We can now perform the Gaussian integral by using the new variable

$$\Phi'(k_0,\mathbf{k}) = \Phi(k_0,\mathbf{k}) + (k_0^2 - k^2 + i\epsilon)J(k_0,\mathbf{k})$$

that allows to rewrite eq. (170) as

$$\begin{aligned} Z_0[J] = &\exp\left[-\frac{i}{2}\int_{\mathbf{k}}\frac{dk_0}{2\pi}\frac{J(k_0,\mathbf{k})J^*(k_0,\mathbf{k})}{k_0^2-k^2+i\epsilon}\right] \\ &\times \int \mathcal{D}\Phi' \exp\left\{ i\int_{\mathbf{k}}\frac{dk_0}{2\pi}\left[\frac{1}{2}\left(k_0^2-k^2\right)\Phi'(k_0,\mathbf{k})\Phi^{*'}(k_0,\mathbf{k}) + i\epsilon|\Phi|^2\right]\right\}.\end{aligned} \tag{171}$$

The integral over Φ' gives an uninteresting normalization factor $\mathcal{N}$, thus we can write the result of the functional integration as

$$\begin{aligned} Z_0[J] &= \mathcal{N}\exp\left[-\frac{i}{2}\int_{\mathbf{k}}\frac{dk_0}{2\pi}\frac{J(k_0,\mathbf{k})J(-k_0,-\mathbf{k})}{k_0^2-k^2+i\epsilon}\right] \\ &= \mathcal{N}\exp\left[-\frac{1}{2}\int dtd^3x\, G_F(t-t',\mathbf{x}-\mathbf{x}')J(t,\mathbf{x})J(t',\mathbf{x}')\right],\end{aligned} \tag{172}$$

The $Z_0[J]$ functional is the main ingredient allowing to compute the *effective action* describing the dynamics of the sources of the field we are integrating over and the dynamics of the extra field we are *not* integrating over. For instance starting from S_{toy} defined in eq. (169) we would obtain the effective action for the sources J from

$$S_{eff}[J] = -i \log Z_0[J]\,, \tag{173}$$

where we can safely discard the normalization constant $\mathcal{N}$. For instance substituting in S_{toy} $J\phi \to J_0 + J\Phi$, with

$$J_0(t,\mathbf{x}) + J(t,\mathbf{x})\Phi(t,\mathbf{x}) = -\sum_A m_A \delta^{(3)}(\mathbf{x}-\mathbf{x}_A)(1+\Phi(t,\mathbf{x}))\,, \tag{174}$$

one would obtain the effective action

$$\begin{aligned} S_{eff}(x_A) \;\; &= \;\; \sum_A \Big[-m_A \int d\tau_A + \\ &\frac{i}{2}\sum_B m_A m_B \int d\tau_A d\tau_B G_F\left(t_A - t_B, \mathbf{x}_A(t_A) - \mathbf{x}_B(t_B)\right)\Big]\,. \end{aligned} \tag{175}$$

Taking the sources in Fourier domain

$$\tilde{J}_A(k) = \int dt e^{-i\omega t} e^{i\mathbf{k}\mathbf{x}_A(t)} \tag{176}$$

and the quasi static limit of the Green function

$$\begin{aligned} & -i\int dt_A dt_B \int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_A-t_B)+i\mathbf{k}(\mathbf{x}_A-\mathbf{x}_B)}}{k^2-\omega^2+i\epsilon} \\ \simeq \;\; & -i\int dt_A dt_B \int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_A-t_B)+i\mathbf{k}(\mathbf{x}_A-\mathbf{x}_B)}}{k^2}\left(1+\frac{\omega^2}{k^2}+\ldots\right) \\ \simeq \;\; & -i\int dt_A dt_B \delta(t_A-t_B)\int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_A-\mathbf{x}_B)}}{\mathbf{k}^2}\left(1+\frac{\partial_{t_1}\partial_{t_2}}{\mathbf{k}^2}\right) \\ = \;\; & -i\int dt\left[\frac{1}{4\pi|\mathbf{x}|}+\frac{O(v^2)}{|\mathbf{x}|}\right] \end{aligned} \tag{177}$$

one recover the instantaneous $1/r$, Newtonian interaction (plus $O(v^2)$ corrections). Note that we have implemented the substitution $\omega = -i\partial_{t_1} = i\partial_{t_2}$ in order to work out the systematic expansion in v. This is justified by observing that the wave-number $k^\mu \equiv (k^0, \mathbf{k})$ of the gravitational modes mediating this interaction have ($k^0 \sim v/r, k \sim 1/r$), so in order to have manifest power counting it is necessary to Taylor expand the propagator.

The individual particles can also exchange *radiative* gravitons (with $k_0 \simeq k \sim v/r$), but such processes give sub-leading contributions to the effective potential in the PN expansion, and they will be dealt with in sec. . In other words we are not integrating out the entire gravity field, but the specific off-shell modes in the kinematic region $k_0 \ll k$.

Actually there are some more terms we would have obtained, like the $J^2_{A,B}G_F(0,0)$ which are divergent, as they involve the Green function computed at 0 separation in space-time. These corresponds to a source interacting with itself and we can safely discard it, as such term does not contribute in any way to the 2-body potential. Its unobservable (infinite) contribution can be re-absorbed by an (infinite) shift of the value of the mass (any ultraviolet divergence can be reabsorbed by a *local* counter term). Of course we cannot trust our theory at arbitrarily short distance, where this divergence may be regularized by new physics (quantum gravity?) but as we do not aim to *predict* the parameters of the theory, but rather take them as input to compute other quantities like interaction potential, we keep intact the predictive power of our approach. From the technical point of view it is a power-law divergence, which in *dimensional regularization* is automatically set to zero.

If there are interaction terms which cannot be written with terms linear or quadratic in the field, the Gaussian integral cannot be done analytically, so the rule to follow is to separate the quadratic action $S_{quad}[\Phi]$ of the field (its kinetic term) and Taylor expand all the rest: for an action $S = S_{quad} + (S - S_{quad})$ one would write

$$\begin{aligned} Z[J] &= \int \mathcal{D}\Phi e^{iS+i\int J\Phi} \\ &= \int \mathcal{D}\Phi e^{iS_{quad}+i\int J\Phi} \left[1 + i\left(S - S_{quad}\right) - \frac{\left(S - S_{quad}\right)^2}{2} + \ldots \right], \end{aligned} \tag{178}$$

where J is now an auxiliary source, the physical source term will be Taylor expanded in the $S - S_{quad}$ term. As long as the $S - S_{quad}$ contains only polynomials of the field which is integrated over, the integral can be *perturbatively* performed analytically, inheriting the rule from eqs. (167,168): roughly speaking fields have to be paired up, each pair is going to be substitued by a Green function.

Our perturbative expansion admits a nice and powerful representation in terms of Feynman diagrams. Incoming and outgoing particle world-lines are represented by horizontal lines, Green functions by dashed lines connecting points, see e.g. fig. 3 the Feynman diagram accounting for the Newtonian potential, which is obtained by pairing the fields connected in the following expression

$$\begin{aligned} e^{iS_{eff}} &= Z[J, x_A]|_{J=0} = \int \mathcal{D}\Phi e^{iS_{quad}} \times \{1 \\ & -\frac{1}{2}\left[\sum_A m_A \int dt_A \Phi\left(t_A, \mathbf{x}_A(t_A)\right)\right]\left[\sum_B m_B \int dt_B \Phi\left(t_B, \mathbf{x}_B(t_B)\right)\right] + \ldots \Big\} \end{aligned} \tag{179}$$

If following Green function's lines all the vertices can be connected the diagram is said *connected*, otherwise it is said *disconnected*: only

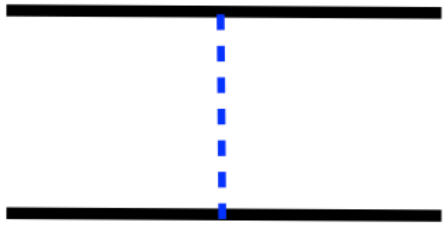

Figure 3: Feynman graph accounting for the Newtonian potential.

connected diagrams contribute to the effective action. We will not demonstrate this last statement, but its proof relies on the following argument. Taking the logarithm of eq. (179) we get

$$S_{eff} = -i \log Z_0[0] - i \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} Z_0^{-1}[0] \left(Z[0] - Z_0[0] \right)^n . \qquad (180)$$

All terms with $n > 1$ describe disconnected diagrams, and some disconnected diagrams can also be present in the $n = 1$ term. However the $n = 1$ disconnected contribution is precisely canceled by the $n = 2$ terms. Beside discarding the disconnected diagrams, we can also discard diagrams involving Green function at vanishing separation $G_F(0,0)$, which give an infinite re-normalization to parameters with dimensions. For instance the diagram in fig. 4, which would arise from the $(S_{quad} - S)^4$ term in eq. (179), is both disconnected and involve a $G_F(0,0)$, cancelling the contribution from the product of disconnected diagrams from $(Z - Z_0)^2$, where in one factor of $Z - Z_0$ one contracts two ϕs at equal point, and in the other two ϕs at different points.

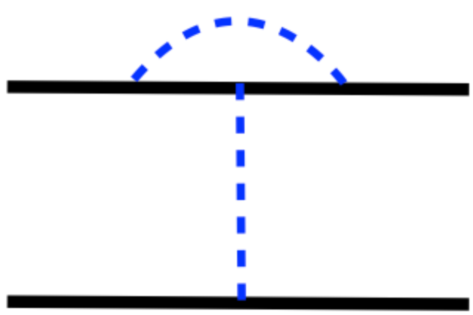

Figure 4: Diagram giving a power-law divergent contribution to the mass.

Having exposed the general method, let us apply it to the comutation of the effective action for the conservative dynamics of binary systems. In order to do that it will be useful to decompose the metric as

$$g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\phi/\Lambda}\gamma_{ij} - A_iA_j/\Lambda^2 \end{pmatrix}, \tag{181}$$

with $\gamma_{ij} = \delta_{ij} + \sigma_{ij}/\Lambda$, $c_d = 2\frac{(d-1)}{(d-2)}$ ($\Lambda = (32\pi G_N)^{-1/2}$ is a constant with dimensions that will allow a simpler normalization of the Green functions). In terms of this parametrization, the Einstein-Hilbert plus gauge fixing action is at quadratic order

$$\begin{aligned} S_{quad} &= \int dt\, d^d x \sqrt{\gamma} \left\{ \frac{1}{4}\left[\left(\vec{\nabla}\sigma\right)^2 - 2\left(\vec{\nabla}\sigma_{ij}\right)^2 - \left(\dot{\sigma}^2 - 2(\dot{\sigma}_{ij})^2\right)\right] \right. \\ & \left. -c_d \left[\left(\vec{\nabla}\phi\right)^2 - \dot{\phi}^2\right] + \left[\frac{F_{ij}^2}{2} + \left(\vec{\nabla}\cdot\vec{A}\right)^2 - \dot{\vec{A}}^2\right]\right\}, \end{aligned} \tag{182}$$

where the gauge fixing term

$$S_{GF} = \frac{1}{32\pi G_N}\int dt d^d x \left(g_{il}\tilde{\Gamma}^i_{jk}\tilde{\Gamma}^l_{mn} g^{jk} g^{mn}\right) \tag{183}$$

with $\tilde{\Gamma}^i_{jk} \equiv \frac{1}{2}\gamma^{il}\left(\gamma_{lj,k} + \gamma_{lk,j} - \gamma_{jk,l}\right)$ has been used, and the source term is

$$\begin{aligned} S_p &= -m\int dt e^{\phi/\Lambda}\sqrt{\left(1 - \frac{A_i v^i}{\Lambda}\right) - e^{-c_d\phi/\Lambda}\left(v^2 + \sigma_{ij}v^i v^j\right)} \\ &\simeq -m\int dt\left\{\sqrt{1-v^2} + \phi\left[1 + \frac{3}{2}v^2\right] + \frac{\phi^2}{2\Lambda}\left[1 + O(v^2)\right] + \right. \\ & \left. -A_i v_i\left[1 + O(v^2)\right] - \frac{\sigma_{ij}}{2}v_i v_j\left[1 + O(v^2)\right] + \ldots\right\}. \end{aligned} \tag{184}$$

The Green functions (inverse of the quadratic terms) of the fields are

$$G_F(t,\mathbf{x}) = -i\int_{\mathbf{k}} \frac{dk_0}{2\pi}\frac{e^{-ik_0 t + i\mathbf{k}\mathbf{x}}}{k^2 - k_0^2} \times \begin{cases} -\frac{1}{2c_d} & \phi \\ \frac{1}{2}\delta_{ij} & A \\ -\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-2}\delta_{ij}\delta_{kl}\right) & \sigma \end{cases} \tag{185}$$

Let us consider the computation of the effective action

$$\begin{aligned} iS_{eff} &= \log\int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{quad}}\left\{ 1 + \ldots - \frac{1}{2}\right. \\ & \times\left[-\frac{m_1}{\Lambda}\int dt_1\left(\phi\left(1 + \frac{3}{2}v_1^2\right) + \frac{\phi^2}{2\Lambda} + A_i v_{1i}\right)\right. \\ & \left. -\frac{m_2}{\Lambda}\int dt_2\left(\phi\left(1 + \frac{3}{2}v_2^2\right) + \frac{\phi^2}{2\Lambda} + A_i v_{2i}\right) + \ldots\right]^2 \\ & \left. + \frac{i^3}{6}\left[-\frac{m_1}{\Lambda}\int dt_1\left(\phi\left(1 + \ldots\right)\right) - \frac{m_2}{\Lambda}\int dt_2\left(\ldots + \frac{\phi^2}{2\Lambda} + \ldots\right)\right]^3 + \ldots\right\}, \end{aligned} \tag{186}$$

We see that we have to pair up, or *contract* the term linear in ϕ of the second line with the analog term in the third line, to give a contribution to the effective action

$$
\begin{aligned}
iS_{eff}|_{fig.\ 3-\phi} &\supset -i^3\frac{m_1m_2}{8\Lambda^2}\int dt_1\,dt_2\delta(t_1-t_2)\int_{\mathbf{k}}\frac{e^{i\mathbf{k}(\mathbf{x}_1(t_1)-\mathbf{x}_2(t_2))}}{k^2}\left[1+\frac{3}{2}\left(v_1^2+v_2^2\right)\right]\left(1+\frac{\partial_{t_1}\partial_{t_2}}{k^2}\right)\\
&= i\frac{m_1m_2}{8\Lambda^2}\int dt\int_{\mathbf{k}}\frac{e^{i\mathbf{k}(\mathbf{x}_1(t)-\mathbf{x}_2(t))}}{k^2}\left[1+\frac{3}{2}\left(v_1^2+v_2^2\right)\right]\left(1+\frac{v_1^iv_2^jk_ik_j}{k^2}\right)\\
&\simeq i\frac{G_Nm_1m_2}{r}\left[1+\frac{3}{2}\left(v_1^2+v_2^2\right)+\frac{1}{2}\left(v_1v_2-(v_1\hat{r})(v_2\hat{r})\right)\right]
\end{aligned}
\tag{187}
$$

where the propagator has been Taylor expanded around $k_0/k\sim 0$ as in eq. (177) and the formulae

$$
\int_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}\frac{1}{k^{2\alpha}} = \frac{\Gamma(d/2-\alpha)}{(4\pi)^{d/2}\Gamma(\alpha)}\left(\frac{r}{2}\right)^{2\alpha-d} \tag{188}
$$

$$
\int_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}\frac{k^ik^j}{k^{2\alpha}} = \left(\frac{1}{2}\delta^{ij}-\left(\frac{d}{2}-\alpha+1\right)\hat{r}^i\hat{r}^j\right)\frac{\Gamma(d/2-\alpha+1)}{(4\pi)^{d/2}\Gamma(\alpha)}\left(\frac{r}{2}\right)^{2\alpha-d-2} \tag{189}
$$

have been used. Considering the contraction of two A fields one gets

$$
\begin{aligned}
iS_{eff}|_{fig.\ 3-A_i} &\supset i^3\frac{m_1m_2}{2\Lambda^2}\int dt\int_{\mathbf{k}}\frac{e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}}{k^2}\delta_{ij}v_1^iv_2^j\\
&= -i\frac{4G_Nm_1m_2}{r}v_1v_2
\end{aligned}
\tag{190}
$$

This is still not the whole story for the v^2 corrections to the Newtonian potential, as we still have to contract two ϕ's from the fourth line of eq. (186) with a ϕ^2 of the same line, getting the contribution

$$
\begin{aligned}
iS_{eff}|_{fig.\ 5} &\supset i^5\frac{m_1^2m_2}{128\Lambda^4}\int dt\left(\int_{\mathbf{k}}\frac{e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}}{k^2}\right)^2\\
&= i\frac{G_N^2m_1^2m_2}{2r^2}
\end{aligned}
\tag{191}
$$

Summing the contributions from eqs.(187,190,191), plus the $1\leftrightarrow 2$ of eq. (191), one obtains the Einstein-Infeld-Hoffman potential (remember that the potential enters the Lagrangian with a minus sign!)

$$
V_{EIH} = -\frac{G_Nm_1m_2}{2r}\left[3\left(v_1^2+v_2^2\right)-7v_1v_2-(v_1\hat{r})(v_2\hat{r})\right]+\frac{G_N^2m_1m_2(m_1+m_2)}{2r^2} \tag{192}
$$

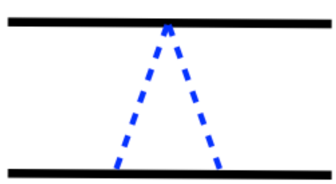

Figure 5: Graph giving a G_N^2 contribution to the 1PN potential via ϕ propagators.

Power counting

We have nevertheless neglected a contribution from the pairing of the two ϕ^2 terms appearing at $(S_{quad} - S)^2$ order expansion in eq. (186), which gives rise to a term proportional to $G_N m_1 m_2 G_F^2(\mathbf{x}_1 - \mathbf{x}_2)$. We have rightfully discarded it as it represents a *quantum* contribution to the potential, and in the phenomenological situation we are considering to apply this theory, quantum corrections are suppressed with respect to classical terms by terms of the order $\hbar/L$, where L is the typical angular momentum of the systems, whic in our case is

$$L \sim mvr \sim 10^{77}\hbar \left(\frac{m}{M_\odot}\right)^2 \left(\frac{v}{0.1}\right)^{-1} . \qquad (193)$$

Intermediate massive object lines, (like the ones in fig. 5) have no propagator associated, as they represent a static source (or sink) of gravitational modes. At the graviton-massive object vertex momentum is *not* conserved, as the graviton momentum is ultra-soft compared to the massive source.

The $\hbar$ counting of the diagrams can be obtained by restoring the proper normalization in the functional action definition eq. (178), implying that in the expansion we have $[(S - S_{quad})/\hbar]^n$ and that each Green function, being the inverse of the quadratic operator acting on the fields, brings a $\hbar^{-1}$ factor. Note that we consider that the classical sources do not recoil when interacting with via "field pairing". This is indeed consistent with neglecting quantum effects, as the wavenumber $\mathbf{k}$ of the exchanged gravitational mode has $k \sim 1/r$ and thus momentum $\hbar/r$.

By inspecting Feynman diagrams we can systematically infer the scaling of their numerical result accordinf to the following rule:

- associate to each n-graviton-particle vertex a factor $m/\Lambda^n dt (d^d k)^n \sim dt m/\Lambda^n r^{-dn}$, and analogously for multiple graviton vertices

- each propagator scales as $\delta(t)\delta^d(k)/k^2 \sim \delta(t) r^{-2+d}$

- each n-graviton internal vertex scale as $(k^2, k_0 k, k_0^2) dt \delta^d(k) (d^d k)^n \sim dt(1, v, v^2) r^{-d(n-1)}$

Figure 6: Vertex scaling: $\frac{m}{\Lambda} dt d^d k \sim dt \frac{m}{\Lambda} r^{-d}$

Figure 7: A Green function is represented by a propagator, with scaling: $\delta(t)\delta^d(k)/k^2 \sim \delta(t)r^{2+d}$

Figure 8: Triple internal vertex scaling: $\frac{(k^2, kk_0, k_0^2)}{\Lambda}\delta^d(k)dt(d^dk)^3 \sim dt\frac{(1,v,v^2)}{r^{2+2d}\Lambda}$

Putting together the previous rule one find for instance that the diagram in fig. 10 scale as dtm (times the appropriate powers of v from the expansion of the vertex and of the propagator)

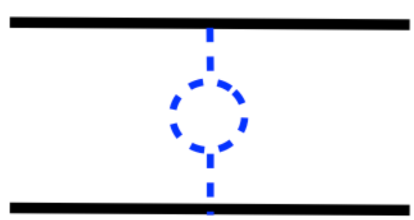

Figure 9: Quantum contribution to the 2-body potential.

Note that instead of computing the efective potential we can also compute the effective energy momentum tensor of an isolated source by considering the the effective action with one external gravitational mode $H_{\mu\nu}(t,x)$ *not* to be integrated over:

$$\begin{aligned} iS_{eff-1g}(m, H_{\mu\nu}) &= \log \int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{EH+GF}(h_{\mu\nu}+H_{\mu\nu})} \\ &\quad \times \left(1 - m\int d\tau(h_{\mu\nu}+H_{\mu\nu}) + \ldots\right)|_{H^1_{\mu\nu}} \\ &= \frac{i}{2}\int dtd^3xT^{(eff)}_{\mu\nu}H_{\mu\nu}\,, \end{aligned} \tag{194}$$

where the computation is made as usual by performing a Gaussian integration on all gravity field variables and the result will be linear in the external field which instead of being integrated over, is the $H_{\mu\nu}$ is the one we want to find what it is coupled to. By Lorentz invariance $H_{\mu\nu}$ must be coupled to a symmetric 2-tensor by which by definition is the energy momentum tensor. Alternatively one can take eq. (179) in the presence of physical sources $J \sim -m \int d\tau$ and compute perturbatively the Feynman integral to obtain

$$\langle H_{\mu\nu} \rangle = \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}\sigma H_{\mu\nu} e^{iS(h+H)} . \tag{195}$$

For instance at lowest order it will give

$$\begin{aligned} \langle H_{\mu\nu}(t,x) \rangle &= \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}\sigma H_{\mu\nu} e^{iS_{quad} - im\int dt \frac{\phi}{\Lambda} + H_{00} \ldots} \\ &\simeq -i\frac{m}{\Lambda^2} \int dt' d^3y G_F(t-t', x-y) \delta^{(3)}(y-x_1) . \end{aligned} \tag{196}$$

By stripping this result by the Green function will give the energy momentum tensor which is coupled to the gravity field, as the solution of Performing this computation at higher perturbative orders will give the higher order corrections to the Newtonian potential.

Exercise 18 * Geodesic Equation**

Derive the geodesic equation from the world line action (13).

Exercise 19 * Gauge fixed quadratic action**

Derive eq. (161) from eq. (14) and the gauge fixing term (**??**).

Exercise 20 * Schwarzschild solution at Newtonian order**

Starting from action (161) derive the eq. (20) (or one can start with (14 and dropping all terms which *quadratically* vanish on for the Lorentz gauge condition). Expand (13) so to obtain

$$S_{m-static} = -m \int dt' \left(1 - \frac{h_{00}(t', \mathbf{x}_p)}{2} \right)$$

and then derive

$$T^{(static)}_{\mu\nu}(t,\mathbf{x}) = 2\frac{\delta S_{static}}{\delta g^{\mu\nu}(t,\mathbf{x})} = m\delta^{(3)}(\mathbf{x}-\mathbf{x}_p)\delta_{\mu 0}\delta_{\nu 0}$$

Use eq. (19) to obtain from the variation of the action eq. (20) and plug in the specific form of the retarded Green function to obtain

$$\begin{aligned} \bar{h}_{\mu\nu} &= (-32\pi G_N) \left(-\frac{1}{4\pi} \right) \frac{\delta_{\mu 0}\delta_{\nu 0}}{2} m \int dt' d\mathbf{x}' \frac{\delta\left(t - t' - |\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|} \delta^{(3)}(\mathbf{x}'-\mathbf{x}_p) \\ &= \frac{4G_N m}{r} \delta_{\mu 0}\delta_{\nu 0} . \end{aligned}$$

Using that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}_{\mu\nu}$$

and that on the above solution

$$\bar{h} = -4\frac{G_N m}{r} ,$$

find the final result

$$\begin{aligned} h_{00} &= \bar{h}_{00} + \frac{1}{2}\bar{h} = 2\frac{G_N m}{r} \\ h_{xx} &= \bar{h}_{xx} - \frac{1}{2}\bar{h} = 2\frac{G_N m}{r}\,. \end{aligned}$$

Exercise 21 * Gaussian integrals

Derive eq. (165) from

$$\int e^{-\frac{1}{2}ax^2}dx = \left(\frac{2\pi}{a}\right)^{1/2}.$$

Derive the above from

$$\left[\int dx e^{-\frac{1}{2}ax^2}\right]^2 = 2\pi \int \rho e^{-\frac{1}{2}a\rho^2} d\rho\,.$$

Exercise 22 **** ϕ^3 vertex contribution to the 2PN potential**

Compute the contribution of fig. 10 to S_{eff}. The ϕ^3 term in the Einstein-Hilbert Lagrangian is $-c_d\phi\dot{\phi}^2/\Lambda$, see [23]. Hint: this diagram is originated from the $(S - S_{quad})^4$ term which is understood in (186).

[23] Stefano Foffa and Riccardo Sturani. Effective field theory calculation of conservative binary dynamics at third post-Newtonian order. *Phys.Rev.*, D84: 044031, 2011

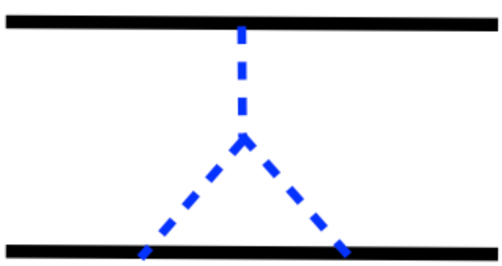

Figure 10: Sub-leading correction to the 2 particle scattering process due to gravity self-interaction. The ϕ^3 vertex brings two time derivative, making this diagram contribute from 2PN order on.

Exercise 23 ** Power $\hbar$

Derive the $\hbar$ scaling of graphs: $\hbar^{I-V}$, where I stands for the number of internal lines (propagators) and V for the numer of vertices. All classical diagrams are homogeneous in $\hbar^{-1}$

Exercise 24 ** Quantum correction to classical potential**

Derive the quantum correction to the classical potential given by the term proportional to $G_N m_1 m_2 G_F^2(\mathbf{x}_1 - \mathbf{x}_2)$ originated from the expansion of $S_{quad} - S$ at linear order.

Exercise 25 *** Newtonian potential as graviton exchange**

In non-relativistic quantum mechanics a one-particle state with momentum $\hbar\mathbf{p}$ in the coordinate representation is given by a plane wave

$$\psi_{\mathbf{p}}(\mathbf{x}) = C \exp\left(i\mathbf{p}\mathbf{x}\right) \ ,$$

where the normalization constant C can be found by imposing

$$\int_V |\psi_{\mathbf{p}}|^2 = 1 \implies C = \frac{1}{\sqrt{V}} \, .$$

Since we want to trace the powers of M and $\hbar$ in the amplitude, it is necessary to avoid ambiguities: variables $\mathbf{p}_i$ are wave-numbers and ω is a frequency, so that $\hbar\omega$ is an energy and $\hbar\mathbf{p}$ is a momentum.

We define the non-relativistic scalar product

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle^{\mathrm{NR}} = \int d^3x \, \psi^*_{\mathbf{p}_1}(\mathbf{x}) \psi_{\mathbf{p}_2}(\mathbf{x}) = \delta_{\mathbf{p}_1,\mathbf{p}_2}$$

which differs from the relativistic normalization used in quantum field theory for the scalar product $\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle^R$ according to:

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle^{\mathrm{R}} = \frac{2\omega_{\mathbf{p}_1}}{\hbar} V \langle \mathbf{p}_1 | \mathbf{p}_2 \rangle^{\mathrm{NR}} ,$$

which implies

$$|\mathbf{p}\rangle^{NR} = \left(\frac{\hbar}{2\omega_{\mathbf{p}} V} \right)^{1/2} |\mathbf{p}\rangle^R \, .$$

Note that a scalar field in the relativistic normalization has the *relativistic* expansion in terms of creator and annihilator operators

$$\psi(t, \mathbf{x}) = \int_{\mathbf{k}} \frac{1}{\sqrt{\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i\omega_k t + i\mathbf{k}\mathbf{x}} + a^\dagger_{\mathbf{k}} e^{i\omega_k t - i\mathbf{k}\mathbf{x}} \right] \tag{197}$$

with

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = i\hbar (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \, .$$

We want to study the relativistic analog of the process described in fig. 3, assuming the coupling of gravity with a scalar particle given by the second quantized action

$$S_{pp-quant} = \frac{1}{2} \int dt d^3x \left\{ \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} \eta_{\mu\nu} \left[(\partial_\alpha \psi \partial^\alpha \psi) + \frac{M^2}{\hbar^2} \right] \right\} \frac{h^{\mu\nu}}{\Lambda} \, ,$$

where the Λ factor has been introduced to have a canonically normalized gravity field.

Derive the quantum amplitude $A(p_1, p_2, p_1 + k, p_2 - k)$ for the 1-graviton exchange between two particles with incoming momenta $p_{1,2}$ and outgoing momenta $p_{1,2} \pm k$

$$\begin{aligned} iA(p_1, p_2, p_1 + k, p_2 - k) &= A'_{\alpha\beta\mu\nu} \frac{i\hbar}{k^2 - k_0} \\ &\times \frac{i^2}{\hbar^2 \Lambda^2} \langle \mathbf{p}_1 + \mathbf{k} | T_{\mu\nu} | \mathbf{p}_1 \rangle^{NR} \langle \mathbf{p}_2 - \mathbf{k} | T_{\alpha\beta} | \mathbf{p}_2 \rangle^{NR} \end{aligned}$$

where $A'_{\alpha\beta\mu\nu}$ is defined in eq.(18).

Hint: from $S_{p-quant}$ above derive

$$
\begin{aligned}
T_{00} &= \tfrac{1}{2}\left[\dot\psi^2 + (\partial_i\psi)^2 + \tfrac{M^2}{\hbar^2}\psi^2\right] , \\
T_{0i} &= \partial_0\psi\partial_i\psi , \\
T_{ij} &= \partial_i\psi\partial_j\psi - \tfrac{1}{2}\delta_{ij}\left[-\dot\psi^2 + \partial_i\psi\partial^i\psi + \tfrac{M^2}{\hbar^2}\psi^2\right] ,
\end{aligned}
$$

show that in the non-relativistic limit the term the A_{0000} term dominates and use the above equation to express $|\mathbf{q}\rangle^R$ in terms of $|\mathbf{q}\rangle^{NR}$. Use also that in the NR limit $p^\mu = \delta^{\mu 0}\omega_p = \delta^{\mu 0}M/\hbar$.

Exercise 26 ****** Classical corrections to the Newtonian potential in a quantum set-up**

Consider the process in fig. 10, write down its amplitude

$$
\begin{aligned}
iA_{fig.10} &= \int dt \left(\frac{i}{\hbar}\right)^3 \langle \mathbf{p}_1+\mathbf{k}|T_{00}|\mathbf{p}_1\rangle^{NR}\langle \mathbf{p}_2+\mathbf{q}|T_{00}|\mathbf{p}_2\rangle^{NR}\langle \mathbf{p}_2-\mathbf{k}|T_{00}|\mathbf{p}_2\rangle^{NR} \\
&\times \int_{\mathbf{k},\mathbf{q}} \frac{d\omega_2}{2\pi} e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)} \frac{i\hbar}{k^2}\frac{i\hbar}{(\mathbf{k}+\mathbf{q})^2}\frac{i\hbar}{q^2}\frac{i\hbar}{(\mathbf{p}_2+\mathbf{q})^2+M^2/\hbar^2-\omega_2^2-i\epsilon} .
\end{aligned}
$$

After writing the massive propagator as

$$
\frac{1}{\sqrt{(\mathbf{p}_2+\mathbf{q})^2+M^2/\hbar^2}+\omega_2}\frac{1}{\sqrt{(\mathbf{p}_2+\mathbf{q})^2+M^2/\hbar^2}-\omega_2)} \sim \frac{\hbar/M}{\sqrt{(\mathbf{p}_2+\mathbf{q})^2+M^2/\hbar^2}-\omega_2}
$$

Perform the integral first in ω_2 and take the limit $\hbar \to 0$ and $M \to \infty$ to recover the result in the non-relativistic classical theory.

Exercise 27 ** Correction to Newtonian equation of motions**

Derive the Newtonian-like e.o.m. from the effective action

$$
S_{1PN} = \int dt \frac{1}{8}\left(m_1 v_1^4 + m_2 v_2^4\right) - V_{EIH} ,
$$

with V_{EIH} given by eq. (192).

Verify that a result of the type

$$
a_1^i = -\omega^2(\mathbf{r},\mathbf{v})(\mathbf{x}_1^i - \mathbf{x}_2^i) + A(\mathbf{r},\mathbf{v})(v_1^i - v_2^i)
$$

is obtained, for appropriate functions ω and A. Take the circular orbit limit ($\mathbf{r}\cdot\mathbf{v}_{1,2} = 0$) and substitute $x \equiv (G_N M\omega)^{2/3}$, with $M \equiv m_1 + m_2$, to express r in terms of x. Once obtained the formula

$$
\frac{G_N M}{r} = x\left[1 + \left(1 - \frac{\eta}{3}\right)x\right] ,
$$

where $\eta \equiv \frac{m_1 m_2}{M^2}$. Having M/r expressed as a function of ω, verify that $v_N \equiv |\dot{\vec{x}}_1 - \dot{\vec{x}}_2| = \omega r$ can be written as

$$
v_N^2 = x\left[1 + 2\left(\frac{\eta}{3} - 1\right)\right] .
$$

Exercise 28 ***** Lorentz invariance in the non-relativistic limit**

Derive the non-relativistic limit of the Lorentz transformation

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = (1-w^2)^{-1/2} \begin{pmatrix} 1 & -\mathbf{w} \\ -\mathbf{w} & 1 \end{pmatrix} \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$$

Result:

$$\begin{array}{ccccc} \mathbf{x}_a & \to & \mathbf{x}'_a & = & \mathbf{x} - \mathbf{w}t + \mathbf{v}_a(\mathbf{w}\cdot\mathbf{x}_a) \\ t & \to & t' & = & t \end{array}$$

where $\mathbf{w}$ is the boost velocity and $\mathbf{w}_a = \dot{\mathbf{x}}_a$. From the Lagrangian $L(x_a, v_a)$, let us define

$$\frac{\delta L}{\delta x_a^i} \equiv -\frac{d}{dt}\left(\frac{\partial L}{\partial v_a^i}\right) + \frac{\partial L}{\partial x_a}\,.$$

and on the equations of motion $\delta L/\delta x_a^i = 0$. Show that for *any* transformation $\delta x_a^i = x_a^{i'} - x_a^i$ one can write the Lagrangian variation as

$$\delta L = \frac{dQ}{dt} + \sum_a \frac{\delta L}{\delta x_a^i}\delta x_a^i + O(\delta x_a^2)\,,$$

with

$$Q \equiv \sum_a \frac{\partial L}{\partial v_a^i}\delta x_a^i = \sum_a p_a^i \delta x_a^i\,.$$

If the transformation δx_a^i is a symmetry, the Lagrangian transforms as a total derivative, that is, in the case of boosts, $\delta L = w^i dZ^i/dt + O(w^2)$ for some function Z^i. Apply the above equation for δL in terms of Q to derive that invariance under boosts implies conservation of the quantity

$$G^i - \sum_a p_a^i t \equiv -Z^i + \sum_a x_a^i(\mathbf{p}_a\cdot\mathbf{v}_a) - \sum_a p_a^i t$$

and find the specific form of Z^i at Newtonian level and at 1PN level. Interpret G^i as the center of mass position, by imposing $G^i = 0$ and $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{r}$ find the expressions of $x_{1,2}^i$ in terms of the relative coordinate r^i.

Find the energy E of the system

$$E = \sum_a v_a^i p_a^i - L\,.$$

at Newtonian and 1PN level. Use the result of the previous step to express v_a in terms of the relative velocity v. Use the result of the previous exercise to find the energy of the circular orbit in terms of x and compare with the Schwarzschild result of ex. 11, the result being

$$E(x) = -\mu\frac{x}{2}\left[1 + x\left(-\frac{3}{4} - \frac{1}{12}\eta)\right)\right]\,,$$

with $\eta \equiv m_1 m_2/(m_1+m_2)^2$.

Exercise 29 *** Graviton loop**

Diagrams like the one in fig. 9 do not affect the classical potential. How many powers of $\hbar$ does this diagram contains? Estimate quantitatively its contribution to the effective action. Is it correct to expand the G_F in its quasi-static limit ($\omega^2 \ll k^2$)?

Exercise 30 *** Field solution**

Derive the expression of the potential of an isolated source at 1PN. Hint: you need to perform the path integral by setting an extra field in the integrand:

$$\langle h_{\mu\nu} \rangle = \int \mathcal{D}h_{\mu\nu}\, h_{\mu\nu} e^{iS_{eff}(J,h_{\mu\nu})} \tag{198}$$

which by virtue of eq. (167), is equivalent to taking a derivative with respect to the J source. Substitute the source-field coupling $J\Phi$ in eq. (170) with $J_\phi \phi + J_A A + J_\sigma \sigma$ given by $-m_a \int dt d^3x \delta(\mathbf{x} - X_a)\left[(1+\frac{3}{2}v_a^2)\phi + A^i v_{ai} + \sigma^{ij} v_{aj} v_{ai}\right]$. Find the solution to the equation of motion of ϕ, A and σ_{ij} at Newtonian order by using

$$(\phi, A, \sigma_{ij}) = \frac{\delta Z_0[J]}{\delta\left(J_\phi, J_A, J_\sigma\right)}\,.$$

Use the presence of the term $c_d/2\partial_i\phi\partial_j\phi(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \delta^{ij}\delta^{kl})\sigma_{kl}$ in the Einstein-Hilbert action to derive $G_N m/r$ correction to the above solution.

Exercise 31 **** 2PN potential of an isolated source**

Similarly to the previous exercise, derive the 2PN potential of a static, isolated source. Note that in this case one can neglect the A and σ coupling in the world-line action, but *not* in the Einstein-Hilbert action.

Hint: According to the ansatz in eq. (181) ϕ couples to $T_{00} + T_{kk}$, σ_{ij} to T_{ij} and A_i to T_{0i}. Use the "bulk" 3-linear interaction, which can be deduced from the Einstein-Hilbert + gauge fixing:

$$\begin{aligned} S_{EH-h^3} \supset & \int dt d^d x \frac{\phi}{\Lambda}\left[-4\dot{\phi}^2 + 4\dot{A}_i^2 + A_{i,j}A_{i,j}\right. \\ & \left. +\dot{\sigma}_{ij}\dot{\sigma}_{kl}\left(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)\right] - 8\dot{\phi}\phi_k A_k \end{aligned}$$

Result: one should derive the Schwarzschild metric in harmonic gauge

$$d\tau^2 = \frac{r - G_N M}{r + G_N M} dt^2 + \frac{(r+G_N M)^2}{r^2}\left[\delta_{ij} - \frac{x_i x_j}{r^2}\right] dx^i dx^j + \frac{r + G_N M}{r - G_N M}\frac{x_i x_j}{r^2} dx^i dx^j\,.$$

Exercise 32 **** 2PN correction to the Energy momentum tensor of an isolated point particle**

We now want to derive an expression for the "dressing" of the energy-momentum tensor because of the gravitational interacion. In quantum mechanics this correspond to evaluate the 1-point function of $T_{\mu\nu}$, that is $\langle T_{\mu\nu} \rangle$. In order to compute this it is useful to use the "background field method", consisting in substituting in the integrand of the Feyman integral

$$h_{\mu\nu} \to h_{\mu\nu} + H_{\mu\nu} ,$$

perform the Gaussian integration and then keep only the terms linear in $H_{\mu\nu}$, which by definition describe the energy momentum tensor. Explain why this computation is the same as the one of the previous exercise, it just needs the use of 1 Green function less.

GW Radiation

In the previous chapters we have shown how to obtain an effective action describing the dynamics of a binary system at the orbital scale r in which gravitational degrees of freedom have been integrated out, resulting in a series expansion in v^2, as in a conservative system odd powers of v are forbidden by invariance under time reversal.

The gravitational tensor in 3+1 dimensions has 6 physical degrees of freedom (10 independent entries of the symmetric rank 2 tensor in 3+1 dimensions minus 4 gauge choices): 4 of them are actually constrained, non radiative physical degrees of freedom, responsible for the gravitational potential, and the remaining 2 are radiative, or GWs.

In order to compute interesting observables, like the average energy flux emitted by or the radiation reaction on the binary system, we have to consider processes involving the emission of GW, i.e. on-shell gravitational modes escaping to infinity.

In order then to obtain an effective action for the sources alone, including the back-reaction for the emissioin of GWs, it will be useful to "integrate out" also the radiative degrees of freedom, with characteristic length scale $\lambda = r/v$, as it will be shown in the next sections. s We aim now at writing the coupling of an extended source in terms of the energy momentum tensor $T^{\mu\nu}(t,x)$ moments, i.e. we want to introduce the *multipole expansion*. Here we use $T^{\mu\nu}$, as in [24], to denote the term relating the effective action $\mathcal{S}_{ext}$ relative to the external graviton emission

[24] W. D. Goldberger and A. Ross. Gravitational radiative corrections from effective field theory. *Phys. Rev. D*, 81: 124015, 2010

$$\mathcal{S}_{ext} = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^d x\, T^{\mu\nu}(t,x) h_{\mu\nu}(t,x)\,, \tag{199}$$

to the gravitational mode generically denoted by $h_{\mu\nu}$. With this definition $T^{\mu\nu}$ receives contribution from both matter and the gravity *pseudo-tensor* appearing in the traditional GR description of the emisson processes.

Given that the variation scale of the energy momentum tensor and of the radiation field are respectively r_{source} and λ, by Taylor-expanding the wave solution[25]

[25] Note that substituing T_{kl} with $\int_{\mathbf{k}} d\omega/(2\pi)\tilde{T}_{kl}(\omega,\vec{k})$ one gets

$$h_{ij}^{(TT)} = \frac{4G_N}{D}\Lambda_{ij,kl}(\hat{n})\int\frac{d\omega}{2\pi}\tilde{T}_{kl}(\omega,\omega\hat{n})e^{-i\omega(t-r)}$$

$$h_{ij}^{(TT)} = \frac{4G_N}{D}\Lambda_{ij;kl}\int d^3x' T_{kl}\left(t - |\vec{r} - \vec{r}'|, x'\right) . \qquad (200)$$

one has

$$T_{kl}(t - r + \vec{x'} \cdot \hat{n}, \vec{x}') \simeq T_{kl}(t - r, \vec{x}') + \vec{x}' \cdot \hat{n}\dot{T}_{kl} + \frac{1}{2}(\vec{x}' \cdot \hat{n})^2\ddot{T}_{kl} + \ldots , \qquad (201)$$

i.e. we obtain a series in r_{source}/λ, which for binary systems gives $r_{source} = r \ll \lambda = r/v$.

Inserting the expansion (201) into The results of the integral in eq. (200) are source moments that, following standard procedures not exclusive of the effective field theory approach described here, are traded for mass and velocity multipoles. For instance, the integrated moment of the energy momentum tensor can be traded for the mass quadrupole

$$Q^{ij}(t) \equiv \int \mathrm{d}^d x\, T^{00}(t, x) x^i x^j , \qquad (202)$$

by repeatedly using the equations of motion under the form $T^{\mu\nu}_{\ \ ,\nu} = 0$:

$$\begin{aligned} \int \mathrm{d}^d x \left[T^{0i}x^j + T^{0j}x^i\right] &= \int \mathrm{d}^d x\, T^{0k}\left(x^i x^j\right)_{,k} \\ &= -\int \mathrm{d}^d x\, T^{0k}_{\ \ ,k}\, x^i x^j \\ &= \int \mathrm{d}^d x\, \dot{T}^{00}\, x^i x^j = \dot{Q}^{ij} \end{aligned} \qquad (203)$$

$$\begin{aligned} 2\int \mathrm{d}^d x\, T^{ij} &= \int \mathrm{d}^d x \left[T^{ik}\, x^j_{\ ,k} + T^{kj}\, x^i_{\ ,k}\right] \\ &= \int \mathrm{d}^d x \left[\dot{T}^{0i}x^j + \dot{T}^{0j}x^i\right] \\ &= \int \mathrm{d}^d x\, \ddot{T}^{00} x^i x^j = \ddot{Q}^{ij} . \end{aligned} \qquad (204)$$

The above equations also show that as for a composite binary system $T_{00} \sim O(v^0)$, then $T_{0i} \sim O(v^1)$ and $T_{ij} \sim O(v^2)$.

Taking as the source of GWs the composite binary system, the multipole series is an expansion in terms of $r/\lambda = v$, so when expressing the multipoles in terms of the parameter of the individual binary constituents, powers of v have to be tracked in order to arrange a consistent expansion. At lowest order in the multipole expansion and at v^0 order (we remined that $h_{00} = -2\phi$)

$$\mathcal{S}_{ext}|_{v^0} = -\frac{1}{\Lambda}\int \mathrm{d}t\, \mathrm{d}^d x\, T^{00}|_{v^0}\, \phi = -\frac{M}{\Lambda}\int \mathrm{d}t\, \phi , \qquad (205)$$

where in the last passage the explicit expression

$$T^{00}(t, x)|_{v^0} = \sum_A m_A \delta^{(3)}(x - x_A(t)) , \qquad (206)$$

has been inserted. At order v the contribution from the first order derivative in ϕ have to be added the contribution of $T^{\mu\nu}|_v$, which gives

$$\mathcal{S}_{ext}|_v = -\frac{1}{\Lambda}\int \mathrm{d}t\,\mathrm{d}^d x\,\left(T^{00}|_{v^0}\,x^i\phi_{,i} - T^{0i}|_v\,A_i\right)\,, \tag{207}$$

with

$$T^{0i}(t,x)|_v = \sum m_A v_{Ai}\delta^{(3)}(x - x_A(t))\,, \tag{208}$$

and neither T_{00} nor T_{ij} contain terms linear in v. Since the total mass appearing in eq. (205) is conserved (at this order) and given that in the center of mass frame $\sum_A m_A x_{Ai} = 0 = \sum_A m_A v_{Ai}$, there is no radiation up to order v. From order v^2 on, following a standard procedure, see e.g. [26], it is useful to decompose the source coupling to the gravitational fields in irreducible representations of the $SO(3)$ rotation group, to obtain

[26] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

$$\begin{aligned} S_{ext}|^{\mathbf{1}}_{v^2} &= \frac{1}{2\Lambda}\int \mathrm{d}t\,\mathrm{d}^d x\,T^{0i}|_v\,x^j(A_{i,j} - A_{j,i})\,, \\ S_{ext}|^{\mathbf{0+2}}_{v^2} &= \frac{1}{4\Lambda}\int \mathrm{d}t\,Q^{ij}|_{v^0}\left(\ddot{\sigma}_{ij} - 2\phi_{,ij} - \frac{2}{d-2}\ddot{\phi}\,\delta_{ij} - \dot{A}_{i,j} - \dot{A}_{j,i}\right)\,, \end{aligned} \tag{209}$$

were eqs. (203,204) and integration by parts have been used, $\mathbf{0},\mathbf{1},\mathbf{2}$ stand for the scalar, vector and symmetric-traceless representations of $SO(3)$, and

$$Q^{ij}\Big|_{v^n} = \int \mathrm{d}^d x\,T^{00}|_{v^n}\,x^i x^j\,. \tag{210}$$

The $\mathbf{1}$ part matches the second term in eq. (211), and it is not responsible for radiation as it couples A_i to the conserved angular momentum. Let us pause for a moment to overlook our procedure and identify the $\mathbf{0}$ and $\mathbf{2}$ terms. We want to describe a composite system, the coalescing binary made of two compact objects. At the scale of GW radiation $\lambda = r/v \gg r$ the binary system can be seen as a single object characterized by its coupling to the background gravitational fields. The point action (13) describe point particles, but extended objects experiences tidal field due to their non-zero size. It is possible to parametrize such effects by effective terms that take into account finite size in a completely general way, via the expression

$$\mathcal{S}_{ext} \supset \int \mathrm{d}\tau\left(-M - \frac{1}{2}S_{ab}\omega_\mu^{ab}u^\mu \;+\frac{c_Q}{2}I^{ij}E_{ij} + \frac{c_J}{2}J^{ij}B_{ij} + \frac{c_O}{2}I^{ijk}\partial_i E_{jk} + \ldots\right), \tag{211}$$

where ω_μ^{ab} is the spin connection coupling to the total angular momentum, while the electric (magnetic) tensor E_{ij} (B_{ij}) is defined by

$$\begin{aligned} E_{ij} &= C_{\mu i \nu j}u^\mu u^\nu\,, \\ B_{ij} &= -\frac{1}{2}\epsilon_{i\mu\nu\rho}\,u^\rho C^{\mu\nu}{}_{j\sigma}\,u^\sigma\,, \end{aligned} \tag{212}$$

decomposing the Weyl tensor $C_{\mu\alpha\nu\beta}$ analogously to the electric and magnetic decomposition of the standard electromagnetic tensor $F_{\mu\nu}$. The $\mathbf{0}+\mathbf{2}$ term in eq. (209) reproduces at linear order the coupling $Q^{ij}R^0{}_{i0j}$ term in eq. (211), allowing to indetify I_{ij} with Q_{ij} at linear order and to impose $c_Q = 1$.

This amounts to decompose the source motion in terms of the world-line of its center of mass and moments describing its internal dynamics. The I^{ij}, I^{ijk}, J^{ij} tensors are the lowest order in an infinite series of source moments, the 2^{nth} electric (magnetic) moment in the above action scale at leading order as mr^n_{source} (mvr^n_{source}), and they couple to the Taylor expanded E_{ij} (B_{ij}) which scales as $L^{-(1+n)}$, showing that the above multipole expansion is an expansion in terms of r_{source}/L. Note that the world-line of the composite object cannot couple to R and $R_{\mu\nu}$, as by the Einstein equation they vanish where sources are not present, thus such couplings can be neglected if we are interested in describing how a composite obect respond to the gravity field of other objects. [27]. As linear terms in the Ricci tensor and Ricci scalar cannot appear, the terms involving the least number of derivatives are the ones written above in eq. (211), in terms of the (traceless part of the) Riemann tensor. Out of the Riemann tensor, we select only its trace-free part (10 components out of 20 in $3+1$ dimensions, which are then re-arranged in the 5 components of the electric Weyl tensor E_{ij} and 5 components in the magnetic Weyl tensor B_{ij}).

[27] Equivalently, it can be shown that the field redefinition $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ with

$$\delta g_{\mu\nu} = \int d\tau \frac{\delta(x^\alpha - x^\alpha(\tau))}{\sqrt{-g}} \left[\left(-c_R + \frac{c_V}{2}\right) g_{\mu\nu} - c_V u_\nu u_\nu \right]$$

can be used to set to zero the above terms linear in the curvature, see for details.
W. D. Goldberger. Les houches lectures on effective field theories and gravitational radiation. In *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime*, 2007

Note that the multipoles, beside being intrinsic, can also be induced by the tidal gravitational field or by the intrinsic angular momentum (spin) of the source. For quadrupole moments the tidal induced quadrupole moments $I_{ij}, J_{ij}|_{tidal} \propto E_{ij}, B_{ij}$ give rise to the following terms in the effective action

$$\mathcal{S}_{tidal} = \int d\tau \left[c_E E_{ij} E^{ij} + c_B B_{ij} B^{ij} \right] . \tag{213}$$

This is also in full analogy with electromagnetism, where for instance particles with no permanent electric dipole experience a quadratic coupling to an external electric field. Eq. (213) can be used to describe a single, spin-less compact object in the field of its binary system companion. Considering that the Riemann tensor generated at a distance r by a source of mass m goes as m/r^3, the finite size effect given by the $E_{ij}E^{ij}$ term goes as $c_E m^2/r^6$. For dimensional reasons $c_E \sim G_N r^5_{source}$ [28], thus showing that the finite size effects of a spherical symmetric body in the binary potential are $O(Gm/r)^5$ times the Newtonian potential, a well known result which goes under the name of *effacement principle* [29] (the coefficient c_E actually vanishes for black holes in $3+1$ dimensions [30]).

[28] Walter D. Goldberger and Ira Z. Rothstein. An Effective field theory of gravity for extended objects. *Phys.Rev.*, D73:104029, 2006

[29] T. Damour. *Gravitational radiation and the motion of compact bodies*, pages 59–144. North-Holland, Amsterdam, 1983b

[30] T. Damour. Gravitational radiation and the motion of compact bodies. In N. Deruelle and T. Piran, editors, *Gravitational Radiation*, pages 59–144. North-Holland, Amsterdam, 1983a; and Barak Kol and Michael Smolkin. Black hole stereotyping: Induced gravito-static polarization. *JHEP*, 1202:010, 2012. DOI: 10.1007/JHEP02(2012)010

In order to simplify the calculation, we work from now on in the transverse-traceless (TT) gauge, in which the only relevant radiation field is the traceless and transverse part of σ_{ij}. The presence of the other gravity polarizations is required by gauge invariance.

Discarding all fields but the TT-part of the σ_{ij} field, at order v^3 one has

$$S_{ext}|_{v^3} = \frac{1}{2}\int \mathrm{d}t\, \mathrm{d}^d x\, T^{ij}|_{v^2}\, x^k\, \sigma_{ij,k} \tag{214}$$

and using the decomposition see ex. 34

$$\begin{aligned}\int \mathrm{d}^d x\, T^{ij} x^k = \;& \frac{1}{6}\int \mathrm{d}^d x\, \ddot{T}^{00} x^i x^j x^k \\ & +\frac{1}{3}\int \mathrm{d}^d x\, \left(\dot{T}^{0i} x^j x^k + \dot{T}^{0j} x^i x^k - 2\dot{T}^{0k} x^i x^j\right)\,,\end{aligned} \tag{215}$$

we can re-write, see ex.35,

$$S_{ext}|_{v^3} = \int \mathrm{d}t \left(\frac{1}{6} Q^{ijk} E_{ij,k} - \frac{2}{3} P^{ij} B_{ij}\right) \tag{216}$$

where

$$Q^{ijk} = \int \mathrm{d}^d x\, T^{00}\, x^i x^j x^k\,, \tag{217}$$

and

$$P^{ij} = \int \mathrm{d}^d x\, \left(\epsilon^{ikl} T^0_{\ k} x_l x^j + \epsilon^{jkl} T^0_{\ k} x_l x^i\right)\,, \tag{218}$$

allowing to identify $J^{ij} \leftrightarrow P^{ij}$ and $I^{ijk} \leftrightarrow Q^{ijk}$ at leading order, with $c_J = -4/3$ and $c_O = 1/3$. In order to derive eq. (218) see ex. 35. Note that from now on it is understood that the GW field is the TT one, so it couple only to the trace-free part of the source

At v^4 order the $T^{ij} x^k x^l \sigma_{ij,kl}$ term, beside giving the leading hexadecapole term (or 2^{4th}-pole) and v corrections to the leading magnetic quadrupole and electric octupole, also gives a v^2 correction to the leading quadrupole interaction $I^{ij} E_{ij}$, which can be written as (we give no demonstration here, see sec. 3.5 of [31] for the general method)

[31] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

$$\begin{aligned} S_{ext}|_{v^4} \;\supset\; & \int dt d^3 x \left[T^{00}|_{v^2} + T^{kk}|_{v^2} - \frac{4}{3}\dot{T}^{0k}|_v\, x^k + \frac{11}{42}\ddot{T}_{00}|_{v^0} x^2\right] \\ \times & \left(x^i x^j - \frac{\delta^{ij}}{d} x^2\right) E_{ij}\,. \end{aligned} \tag{219}$$

For the systematics at higher orders we refer to the standard textbook [32].

[32] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

Matching between the radiation and the orbital scale

In the previous subsection we have spelled out the general expression of the effective multipole moments in terms of the energy-momentum tensor moments. However we have only used two ingredients from the specific binary problem

- $T_{00} \sim mv^0$
- the source size is r and the length variation of the background is $\lambda \sim r/v$.

Now we are going to match the coefficients appearing in eq. (211) with the parameters of the specific theory at the orbital scale.

At leading order $Q_{ij}|_{v^0} = \sum_A m_A x_{Ai} x_{Aj} = \mu r^i r^j$ and the v^2 corrections to T_{00} can be read from diagrams in figs. 11. As ϕ couples to $T_{00} + T_{kk}$ and σ_{ij} to T_{ij}, from the diagrams one obtains[33]

[33] Note that since only $\int T_{kk}$ is needed, and not T_{kk} itself, it could have been computed from eq. (204) instead of from the diagram in fig. 11b.

$$
\begin{aligned}
\int d^3x \, (T_{00} + T_{kk})\Big|_{v^2} &= \sum_A \frac{3}{2} m_A v_A^2 - 2\frac{G_N m_1 m_2}{r}\,, \\
\frac{1}{2}\int d^3x T_{kk}|_{v^2} &= \sum_A \frac{1}{2} m_A v_A^2 - \frac{1}{2}\frac{G_N m_1 m_2}{r}\,.
\end{aligned}
\tag{220}
$$

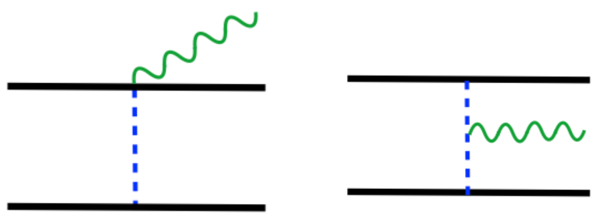

Figure 11: Graph dressing T_{00} at v^2 order and T_{ij} at leading order. The external radiation graviton does not carry momentum but it is Taylor expanded.

These diagrams are obtained by performing the Gaussian path integral with an external gravitational mode $H_{\mu\nu}$ on which one does not have to integrate over. This correspond to adopting the "background-field method", in which every graviton in the original Lagrangian is splitted according to $h_{\mu\nu} \to h_{\mu\nu} + H_{\mu\nu}$ and then integration is performed on $h_{\mu\nu}$ alone. The term linear in $H_{\mu\nu}$ in the resulting effective action is by definition (proportional to) the energy-momentum tensor.

Integrating out the radiating graviton and mass renormalization

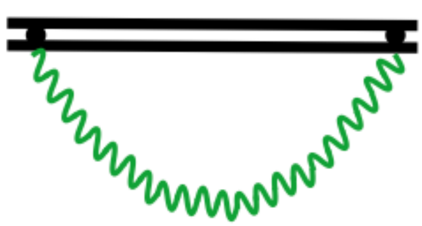

Figure 12: Diagram giving the leading term of the amplitude describing radiation back-reaction on the sources.

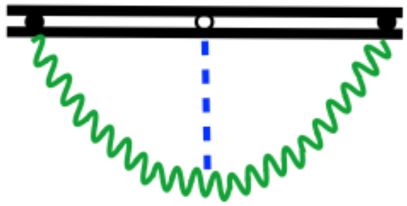

Figure 13: Next-to-leading order term in the back-reaction amplitude.

We have now built an effective theory for extended objects in terms of the source moments and we also shown how to match the orbital scale with the theory describing two point particles experiencing mutual gravitational attraction. We can further use the extended object action in eq. (211) to integrate out the gravitational radiation to obtain an effective action S_{mult} for the source multipoles alone.

In order to perform such computation, boundary conditions *asymmetric in time* have to be imposed, as no incoming radiation at past infinity is required. Using the standard *Feynman* propagator, which ensures a pure in-(out-)going wave at past (future) infinity, would lead to a non-causal evolution as it can be shown by looking at the following toy model [34], which is defined by a scalar field Ψ coupled to a source J:

[34] M. Tiglio C. R. Galley. Radiation reaction and gravitational waves in the effective field theory approach. *Phys. Rev.*, D79:124027, 2009

$$S_{toy} = \int \mathrm{d}^{d+1}x \left[-\frac{1}{2}(\partial\psi)^2 + \psi J\right] . \qquad (221)$$

We may recover the field generated by the source J as

$$\psi(t,x) = \int \mathrm{d}^{d+1}x \, G_F(t-t', x-x') J(t', x') . \qquad (222)$$

In a causal theory ψ would be given by the same eq. (222) but with the Feynman propagator replaced by the retarded one $G_{Ret}(t,x)$, however it is not possible to naively use the retarded propagator in the action (221), as it would still yield non-causal equations of motions as

$$\begin{aligned}&\int dtd^3xdt'd^3x'\psi(t,x)G^{-1}_{ret}(t-t',x-x')\psi(t',x')\\&=\frac{1}{2}\int dtd^3xdt'd^3x'\psi(t,x)\left(G^{-1}_{ret}(t-t',x-x')+G^{-1}_{adv}(t-t',x-x')\right)\psi(t',x')\end{aligned} \qquad (223)$$

This problem was not present in the conservative dynamics described in sec. as there we had a closed system with no leak of energy: there we actually had the Feynman Green function which is symmetric in its arguments, see ex. 8.

However there is a consistent way to define an action for non-conservative system with asymmetric time boundary condition: by adopting a generalization of the Hamilton's variational principle similar to the closed-time-path, or in-in formalism (first proposed in [35], see [36] for a review) as described in [37], which requires a *doubling* of the field variables. For instance the toy model in eq. (221) is modified so that the generating functional for connected correlation functions in the in-in formalism has the path integral representation

[35] J. S. Schwinger. Brownian motion of a quantum oscillator. *J. Math. Phys.*, 2: 407–432, 1961

[36] B. DeWitt. Effectice action for expectation values. In R. Penrose and C. J. Isham, editors, *Quantum concepts in Space and Time*. Clarendon Press, Oxford, 1986

[37] C. R. Galley. The classical mechanics of non-conservative systems. *Phys. Rev. Lett.*, 110:174301, 2013

$$\begin{aligned} e^{i\mathcal{S}_{eff}[J_1,J_2]} &= \int \mathcal{D}\psi_1\mathcal{D}\psi_2 \\ &\times \quad \exp\left\{i\int d^{3+1}x\,\left[-\frac{1}{2}(\partial\psi_1)^2+\frac{1}{2}(\partial\psi_2)^2-J_1\psi_2+J_1\psi_2\right]\right\}\,. \end{aligned} \tag{224}$$

In this toy example the path integral can be performed exactly, and using the Keldysh representation [38] defined by $\Psi_- \equiv \Psi_1 - \Psi_2$, $\Psi_+ \equiv (\Psi_1+\Psi_2)/2$, one can write

[38] L. V. Keldysh. Diagram technique for nonequilibrium processes. *Zh. Eksp. Teor. Fiz.*, 47:1515–1527, 1964

$$\mathcal{S}_{eff}[J_+,J_-] = \frac{i}{2}\int \mathrm{d}^{d+1}x\, d^{d+1}y\, J_B(x)G^{BC}(x-y)J_C(y)\,, \tag{225}$$

where the B, C indices take values $\{+,-\}$ and

$$G^{BC}(t,\mathbf{x}) = \begin{pmatrix} 0 & G_{adv}(t,\mathbf{x}) \\ G_{ret}(t,\mathbf{x}) & 0 \end{pmatrix}\,. \tag{226}$$

In our case, the lowest order expression of the quadrupole in terms of the binary constituents world-lines x_A, i.e.

$$Q_{ij}|_{v^0} = \sum_{A=1}^{2} m_A\left(x_{Ai}x_{Aj} - \frac{\delta_{ij}}{d}x_{Ak}x_{Ak}\right)\,, \tag{227}$$

is doubled to

$$\begin{aligned} Q_{-ij}|_{v^0} &= \sum_{A=1}^{2} m_A\left[x_{-Ai}x_{+Aj} + x_{+Ai}x_{-Aj} - \frac{2}{d}\delta_{ij}x_{+Ak}x_{-Ak}\right] \\ Q_{+ij}|_{v^0} &= \sum_{A=1}^{2} m_A\left[x_{+Ai}x_{+Aj} - \frac{1}{d}\delta_{ij}x_{+A}^2 + O(x_-^2)\right]\,. \end{aligned} \tag{228}$$

The word-line equations of motion that properly include radiation reaction effects are given by

$$0 = \left.\frac{\delta S_{eff}[x_{1\pm}, x_{2\pm}]}{\delta x_{A-}}\right|_{\substack{x_{A-}=0 \\ x_{A+}=x_A}}\,. \tag{229}$$

At lowest order, by integrating out the radiation graviton, i.e. by computing the diagram in fig. 12, one obtains the Burke-Thorne [39]

[39] W. L. Burke and K. S. Thorne. In S. I. Fickler M. Carmeli and L. Witten, editors, *Relativity*, pages 209–228. Plenum, New York, 1970

potential term in the effective action S_{mult}

$$
\begin{aligned}
iS_{mult}|_{fig.\ 12} &= i^2 \int dt\,dt'\, Q_{-ij}(t)Q_{+kl}(t')\underbrace{E_{+ij}(t)E_{-kl}}(t') \\
&= \frac{-i^3}{\Lambda^2}\int dt\,dt'\, Q_{-ij}(t)Q_{+kl}(t')\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{k^2-(\omega+i\epsilon)^2} \\
\times &\left[\frac{\omega^4}{8}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-2\delta_{ij}\delta_{kl}\right)\right. \\
&+\frac{\omega^2}{8}\left(k_ik_k\delta_{jl}+k_ik_l\delta_{jk}+k_jk_k\delta_{il}+k_jk_l\delta_{ik}\right) \\
&\left.-\frac{1}{8}\left(\omega^4\delta_{ij}\delta_{kl}+\omega^2\delta_{ij}k_kk_l+\omega^2k_ik_j\delta_{kl}+k_ik_jk_kk_l\right)\right]
\end{aligned}
\tag{230}
$$

After performing the angular integration

$$
\begin{aligned}
\int d\Omega\, k_ik_j &= \frac{4}{3}\pi k^2\delta_{ij}\,, \\
\int d\Omega\, k_ik_jk_kk_l &= \frac{4}{15}\pi k^4\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)
\end{aligned}
\tag{231}
$$

and using

$$
\int_0^\infty dk\frac{k^a}{k^2-(\omega+i\epsilon)^2}=\frac{\pi}{\cos\left(a\frac{\pi}{2}\right)}\left[\frac{1}{-(\omega+i\epsilon)^2}\right]^{\frac{1-a}{2}}\,, \tag{232}
$$

one finds the result

$$
iS_{mult}|_{fig.\ 12} = i\frac{G_N}{5}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^5 Q_{-ij}(\omega)Q_{+ij}(\omega) \tag{233}
$$

$$
= \frac{G_N}{5}\int dt Q_{-ij}(t)\frac{d^6Q_{+ij}(t)}{dt} \tag{234}
$$

and using eq. (229) the equation of motion follows:

$$
\ddot{x}_i|_{fig.\ 12} = -\frac{2}{5}Q^{(5)}_{ij}x^j \tag{235}
$$

Corrections to the leading effect appears when considering as in the previous subsection higher orders in the multipole expansion: the 1PN correction to the Burke Thorne potential were originally computed in [40] and re-derived with effective field theory methods in [41].

The genuinely non-linear effect, computed originally in [42] and within effective field theory methods in [43], appears at relative 1.5PN order and it is due to the diagram in fig. 13. The contribution to the effective action can be obtained by considering the contribution respectively from the $\sigma^2\psi$, $\sigma\phi A$, $\sigma\phi^2$, $A^2\phi$, $A\phi^2$, ϕ^3 vertices, which give the effective action contribution

[40] B. R. Iyer and C. M. Will. Postnewtonian gravitational radiation reaction for two-body systems. *Phys. Rev. Lett.*, 70: 113, 1993

[41] C. R. Galley and A. K. Leibovich. Radiation reaction at 3.5 post-newtonian order in effective field theory. *Phys. Rev. D*, 86:044029, 2012

[42] Luc Blanchet and Thibault Damour. Tail transposrted temporal correlations in the dynamics of a gravitating system. *Phys.Rev.*, D37:1410, 1988; and L. Blanchet. Time asymmetric structure of gravitational radiation. *Phys.Rev.*, D47: 4392–4420, 1993

[43] S. Foffa and R. Sturani. Tail terms in gravitational radiation reaction via effective field theory. *Phys. Rev. D*, 87: 044056, 2013

$$
\begin{aligned}
iS_{eff}|_{fig.\ 12} \quad = \quad & -i\frac{M}{16\Lambda^4}\int\frac{d\omega}{2\pi}\left(Q_{-ij}(\omega)\,Q_{+kl}(-\omega)+Q_{+ij}(-\omega)\,Q_{-kl}(\omega)\right)\\
& \int\frac{d^dk}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\,\frac{1}{\mathbf{k}^2-(\omega-ia)^2}\,\frac{1}{(\mathbf{k}+\mathbf{q})^2-(\omega-ia)^2}\,\frac{1}{\mathbf{q}^2}\\
& \Big\{-\frac{1}{8}\omega^6\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{d-2}\delta_{ij}\delta_{kl}\right)\\
& +\frac{1}{2}\omega^4\left(q_iq_k\delta_{jl}\right)\\
& +\frac{1}{2c_d}\omega^2\left(-\frac{k_0^2q_iq_j}{d-2}\delta_{kl}-q_iq_jk_kk_l-2k_iq_jk_kq_l-3q_iq_jq_kk_l-q_iq_jq_kq_l\right) \qquad (236)\\
& \frac{1}{2}\omega\left[k_ik_jq_kq_l-q_ik_jk_kq_l+\delta_{ik}\left(k_jk_l\mathbf{k}^2+k_jq_l(\mathbf{kq})+\mathbf{k}^2k_jq_l+\mathbf{kq}k_jk_l\right)\right]+\\
& \frac{1}{2c_d}\omega\left(2q_ik_jq_kk_l-k_ik_jq_kq_l+q_ik_jq_kq_l-k_0^2\delta_{kl}\frac{q_iq_j}{d-2}\right)\\
& -\frac{1}{2c_d}\omega\left[k_ik_jk_kk_l+k_ik_jq_kq_l+2k_ik_jk_kq_l+\right.\\
& \quad\left.\frac{\omega^2}{d-2}\left(\delta_{ij}k_kk_l+\delta_{kl}k_ik_j+\delta_{ij}q_kq_l+2\delta_{ij}k_kq_l+\frac{\omega}{d-2}\delta_{ij}\delta_{kl}\right)\right]\Big\}\,.
\end{aligned}
$$

where we have been careful to work in generic d spatial dimensions: this diagram has a logarithmic divergence in $d=3$, which is cured by performing the computation in generic d dimension. In order to isolate the cause of the divergence, let us focus on the first line of eq. (236) within the curly bracket, which is the simplest integral to perform, has it boils down to integrate the two propagators of the previous line. After using (see eq.(8-7) of [44])

[44] C. Itzinkinson and J. B. Zuber. *Quantum field theory*. Mac Graw-Hill International Book Company, 1980

$$
\begin{aligned}
&\int_{\mathbf{k}}\frac{1}{\left[k^2-\omega_1^2\right]^{2a}}\frac{1}{\left[(\mathbf{k}+\mathbf{q})^2-\omega_2^2\right]^{2b}}=\frac{1}{(4\pi)^d/2}\frac{\Gamma(a+b-d/2)}{\Gamma(a)\Gamma(b)}\\
&\times\int_0^1 dx x^{a-1}(1-x)^{b-1}\left[x(1-x)q^2-x\omega_1^2-(1-x)\omega_2^2\right]^{d/2-a-b}
\end{aligned} \qquad (237)
$$

and the identity

$$
\int_0^1 x^a(1-x)^b=\frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}\,, \qquad (238)
$$

we get a term proportional to

$$
\int_{\mathbf{q}}\frac{1}{q^2}\left[q^2-\frac{(\omega+i\epsilon)^2}{x(1-x)}\right]^{d/2-2} \qquad (239)
$$

which diverges logarithmically as $q\to\infty$ for $d=3$. Performing the full computation, whose details can be found in [45], one obtains

[45] S. Foffa and R. Sturani. Tail terms in gravitational radiation reaction via effective field theory. *Phys. Rev. D*, 87: 044056, 2013

$$
\begin{aligned}
S_{eff}|_{fig.13} \quad = \quad & -\frac{1}{5}G_N^2M\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega^6\left(\frac{1}{d-3}-\frac{41}{30}+i\pi sgn(\omega)-\log\pi+\gamma+\log(\omega^2/\mu^2)\right)\times\\
& \left[Q_{ij-}(\omega)Q_{ij+}(-\omega)+Q_{ij-}(-\omega)Q_{ij+}(\omega)\right]\,.
\end{aligned} \qquad (240)
$$

Note the appearance of the arbitrary scale μ (with dimension of inverse-length) because in generic d dimensions we have $\Lambda^{-2} = 32\pi G_N \mu^{3-d}$, with μ setting the scale of the d-dimensional Newton's constant.

Let us look at the physical process we are considering in order to interpret this result: a GW is emitted from the quadrupole, scattered by the curavture and absorbed by the binary itself. In the limit of very large q, i.e. when the scattering occurs arbitrarily close to the binary system, this amplitude diverges. We cannot trust our effective theory arbitrarily close to the source, where the system cannot be described as a single object with a quadrupole, but we have to match with the fundamental theory of a binary system. The finite numbers we get with the result are then menaingless, as they are obtained by pushing our effective theory farther than it can actually be trusted. However the logarithmic terms *is* physical. By cutting the $\mathbf{q}$ integral at any finite scale $\mathbf{q}_0$, we would have obtained a term $\log(\omega/q_0)$, which would tell that integrating out gravitational interactions up to an arbitrary finite scale, will introduce a logarithmic scaling of the result. Of course it does not make sense that a *physical* result depends on an arbitrary scale, indeed shifting μ to a new value μ' would change the *unphsyical* finite terms in eq. (240) by a finite amount proportional to $\log(\mu'/\mu)$. How to make physical sense out of this result? The source of the Newtonian potential has a logarthmic dependence of the scale μ: it changes if we probe it at different distances, because of the "cloud" of GW which is continously emitted and absorbed by the binary systems, thorugh their quadrupole coupling to the source.

It is possible to derive the correct finite terms to complete the result by using the correct theory to describe the physics very close to the bynary system, which at short distances cannot be treated as a point of zero size, but rather as a binary system with size $0 < r \ll \lambda r/v$.

We note the presence of the logarithmic term which is non-analytic in ω-space and non-local (but causal) in direct space: after integrating out a mass-less propagating degree of freedom the effective action is not expected to be local [46].

[46] T. Appelquist and J. Carazzone. Infrared singularities and massive fields. *Phys. Rev. D*, 11:2856, 1975

In order to make sense of the result (240) we have first to regularize it, which can be done by adding a *local* counter term to the action (211) given by M_{ct} defined by

$$M_{ct} = -\frac{2G_N^2}{5} M \left(\frac{1}{\epsilon} + \gamma - \log \pi\right) Q_{-ij} Q^{(6)}_{+ij} . \qquad (241)$$

The combination of the *bare* monopole mass term M in eq.(211) plus the above M_{ct} give the mass which is the measured parameter.

The scale μ appearing in the logarithm is arbitrary, but the presence of the logarithm is indeed *physical*: the way to avoid an unphysical dependence on μ of the physical effective action is to *assume* that M is μ dependent, that is the monopole mass of the binary system depends on the scale at which we are probing it, see ex. 37 for a thorough way to derive how the mass of the binary system is affected by the GW emission.

According to the standard renormalization procedure, one can define a renormalized mass $M^{(R)}(\mu)$ for the monopole term in the action (211), depending on the arbitrary scale μ in such a way that physical quantities (like the force derived from the effective action of the composite system) will be μ-independent. The force looks like

$$\delta\ddot{x}_{Ai}(t)|_{log} = -\frac{8}{5}x_{aj}(t)G_N^2 M\int_{-\infty}^{t}\mathrm{d}t'\, Q_{ij}^{(7)}(t')\log\left[(t-t')\mu\right]\,, \quad (242)$$

and it can be separated into a t-dependet and a t-independent part, with the t-independent part being:

$$\mu\frac{d\delta\ddot{x}_{Ai}(t)}{d\mu} = \frac{8}{5}x_{aj}(t)G_N^2 M\int_{-\infty}^{t}\mathrm{d}t'\, Q_{ij}^{(7)}(t') = \frac{8}{5}x_{aj}(t)G_N^2 M Q^{(6)}(t)\,. \quad (243)$$

We can see how this effect gives a logarithmic shift δM to the monopole mass:

$$\frac{d\delta M}{dt} = -\sum_A m_A\delta\ddot{x}_{Ai}\dot{x}_{Ai}\,. \quad (244)$$

Substituting eq. (242) into eq. (244) and using the leading order quadrupole moment expression in eq. (227) allows to turn the right hand side of eq. (244) into a total time derivative, enabling to identify the logarithmic mass shift as [47]

$$\mu\frac{d\delta M^{(R)}}{d\mu} = -\frac{2G_N^2 M}{5}\left(2Q_{ij}^{(5)}Q_{ij}^{(1)} - 2Q_{ij}^{(4)}Q_{ij}^{(2)} + Q_{ij}^{(3)}Q_{ij}^{(3)}\right)\,. \quad (245)$$

[47] L. Blanchet, S. L. Detweiler, A. Le Tiec, and B. F. Whiting. High-order post-newtonian fit of the gravitational self-force for circular orbits in the schwarzschild geometry. *Phys. Rev.*, D81, 2010

This classical renormalization of the mass monopole term (which can be identified with the Bondi mass of the binary system, that does not include the energy radiated to infinity) is due to the fact that the emitted *physical* (not virtual!) radiation is scattered by the curved space and then absorbed, hence observers at different distance from the source would not agree on the value of the mass.

The ultraviolet nature of the divergence points to the incompleteness of the effective theory in terms of multipole moments: the terms analytic in ω in eq. (240) are sensitive to the short distance physics and their actual value should be obtained by going to the theory at orbital radius.

Emitted flux

We have now shown how to perform the matching between the theory of extended objects with multipoles and the theory at the orbital scale. Taking the action for extended bodies in eq. (211) as a starting point, the emitted GW-form and the total radiated power can be computed in terms of the source multipoles by using Feynman diagrams with one external radiating gravitational particle. Following the same rules given at the end of the previous chapter for the computation of the effective energy momentum tensor and the effective potential, we can write here the effective action for a composite particle described by the action (211). At leading order the effective action describing the emission/absorption of a GW described in fig. 14, one does not need to perform any computation and it can be just read from the action (211)

$$iS_{eff}(M, Q, \dots, \sigma_{ij}) \supset i \int dt\, \ddot{Q}_{ij} \frac{\sigma_{ij}}{4\Lambda} \tag{246}$$

The GW-form can be computed by convolving the appropriate Green function with the source, which at this order is simply $\ddot{Q}_{ij}$ and it is then possible to derive the actual GW-form

$$\begin{aligned} \sigma_{ij}(t, x) \quad \supset \quad & 8\pi G_N \Lambda_{ij;kl} \int \mathrm{d}t' \mathrm{d}^d x'\, G_R(t - t', x - x') \\ & \times \left[\ddot{I}_{kl} + \frac{4}{3} \epsilon_{lmn} \dot{J}_{mk,n} - \frac{1}{3} \ddot{I}_{klm,m} \right], \end{aligned} \tag{247}$$

where $\Lambda_{ij,kl}$ is the TT-projector $\Lambda_{ij;kl}$ defined in eq. (27).

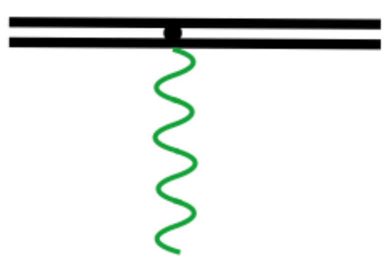

Figure 14: Diagram representing the emission of a GW from a quadrupole source

Analogously to what shown in the previous subsection, we have to take into account the GW interaction with the space time curvature produced by the source itself. Including such effect give rise to a *tail* effect, accounted by the diagram in fig. 15, which gives a contribution to the GW amplitude and phase [48]

$$\begin{aligned} \sigma_{ij}|_{fig.\ 15} \quad \supset \quad & \Lambda_{ij;kl} \pi M G \int_{\mathbf{k}} \frac{d\omega}{2\pi} \int_{\mathbf{k}} e^{-i\omega t + i\mathbf{k}\mathbf{x}} \omega^4 \\ & \frac{1}{k^2 - (\omega + i\epsilon)^2} \int_{\mathbf{q}} \frac{q^{-2}}{(\mathbf{k} + \mathbf{q})^2 - (\omega + i\epsilon)^2} I_{kl}(\omega) \end{aligned} \tag{248}$$

[48] L. Blanchet and Gerhard Schaefer. Gravitational wave tails and binary star systems. *Class.Quant.Grav.*, 10:2699–2721, 1993; and A. Ross R. A. Porto and I. Z. Rothstein. Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 post-newtonian order. *JCAP*, 1209:028, 2012

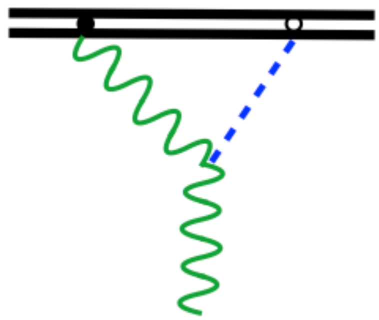

Figure 15: Emission of a GW from a quadrupole source with post-Minkowskian correction represented by the scattering off the background curved by the presence of binary system.

The solution of the above integral is a bit involved, now we just want to underline that it contains an *infra-red* divergence for $q \to 0$: performing first the integral over ω yields a term proportional to

$$\int_{\mathbf{q}} \frac{1}{q^2} \frac{1}{2\mathbf{kq}} \,. \qquad (249)$$

The infra-red singularity in the phase of the emitted wave is un-physical as it can be absorbed in a re-definition of time in eq. (248). Moreover any experiment, like LIGO and Virgo for instance, can only probe phase *differences* (e.g. the GW phase difference between the instants when the wave enters and exits the experiment sensitive band) and the un-physical dependencies on the regulator ϵ and on the subtraction scale μ drops out of any observable.

The total emitted flux can be computed once the amplitude of the GW has been evaluated, via the standard formula, see eq. (76)

$$P = \frac{r^2}{32\pi G_N} \int d\Omega \, \langle \dot{h}_{ij} \dot{h}_{ij} \rangle \,, \qquad (250)$$

but there is actually a shortcut, as the emission energy rate can be computed directly from the amplitude effective action (246) without solving for σ_{ij}. The derivation of the shortcut formula requires the interpretation of

$$A_h \equiv -k^2 Q_{ij} \frac{\epsilon_{ij}(\mathbf{k})}{4\Lambda} \qquad (251)$$

as a *probability amplitude* for emitting a GW with polarization tensor $\epsilon_{ij}(\mathbf{k}, h)$ (the polarization tensor). The differential probability per unit of time for emitting a GW over the full phase space volume is indeed given by

$$dP(k) = \frac{1}{2T} \frac{d^3k}{(2\pi)^3} |k^2 Q_{ij}(k) \epsilon_{ij}(\mathbf{k})|^2 \,. \qquad (252)$$

This formula is usually obtained in quantum mechanics by applying the *optical theorem*, whose demonstration is obtained by noting that $|A_h|$ is obtained by taking the imaginary part of eq. (230), with the retarded propagator replaced by the Feynman one and using

$$\frac{1}{k^2-\omega^2+i\epsilon} = P\frac{1}{k^2-\omega^2} + i\pi\delta\left(k^2-\omega^2\right) . \qquad (253)$$

The imaginary part is then interpreted as a *probability loss* per unit of k space, as eq. 230 can be rewritten as

$$\begin{aligned} Im\, S'_{mult}|_{fig.\ 12} &= \int_{\mathbf{k}} \frac{d\omega}{2\pi} \sum_h |A_h(\omega,\mathbf{k})|^2 \pi\delta(k^2-\omega^2) \\ &= \int_{\mathbf{k}} \frac{1}{2k} \sum_h |A_h(k,\mathbf{k})|^2 , \end{aligned} \qquad (254)$$

where $\sum_h$ indicates sum over the two GW elicities. The imaginary part of the action gives the integrated probability loss, which weighted by one power of k and divided by the total observation time, gives the average energy flux. The last missing ingredient to get to the final formula is the sum over the two polarizations, which depends on $\mathbf{k}$ via the specific form of the Λ projector. For the electric quadrupole term we have

$$\begin{aligned} &\int \frac{d\Omega}{4\pi} \sum_h \epsilon^*_{ij}(\mathbf{k})\epsilon_{kl}(\mathbf{k}) \\ &= \int \frac{d\Omega}{4\pi}\frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ik}n_jn_l - \delta_{jl}n_in_k - \delta_{il}n_jn_k - \delta_{jk}n_in_k + n_in_jn_kn_l \right. \\ &\quad \left. -\delta_{ij}\delta_{kl} + \delta_{ij}n_kn_l + \delta_{kl}n_in_j\right] \\ &= \frac{1}{5}\left(\delta_{ik}\delta_{jl} + \delta_{ik}\delta_{jl} - \frac{8}{3}\delta_{ij}\delta_{kl}\right) . \end{aligned} \qquad (255)$$

For the magnetic quadrupole and electric octupole see ex. 42.

After summing over GW polarizations one gets:

$$\begin{aligned} P &= \frac{1}{32\Lambda^2 T}\int_0^\infty \frac{k^2dk}{2\pi^2}\left[k^4 Q_{ij}(k)Q_{kl}(-k)\frac{2}{5}\left(\delta_{ik}\delta_{jl} - \frac{4}{3}\delta_{ij}\delta_{kl}\right)\right. \\ &\quad + \frac{16}{9}k^4 J_{ij}(k)J_{kl}(-k)\frac{2}{5}\left(\delta_{ik}\delta_{jl} - \frac{4}{3}\delta_{ij}\delta_{kl}\right) \\ &\quad \left. + \frac{k^6}{9} O_{ijk}(k)O_{lmn}(-k)\frac{2}{21}\left(\delta_{il}\delta_{jm}\delta_{kn} + \text{trace terms}\right)\right] \\ &= \frac{2G_N}{5}\int_0^\infty \frac{d\omega}{2\pi}\omega^6\left[|I_{ij}(\omega)|^2 + \frac{16}{9}|J_{ij}(\omega)|^2 + \frac{5}{189}\omega^2|I_{ijk}(\omega)|^2 + \dots\right] \end{aligned} \qquad (256)$$

which, once averaged over time, recovers at the lowest order the standard Einstein quadrupole formula plus magnetic quadrupole and electric octupole contributions

$$P = \frac{G_N}{5}\langle \dddot{Q}^2_{ij}\rangle + \frac{16G_N}{45}\langle \dddot{J}_{ij}\rangle + \frac{G_N}{189}\langle \ddddot{O}_{ijk}\rangle \qquad (257)$$

where we have used that

$$\int_0^\infty \frac{d\omega}{2\pi} |\tilde{A}(\omega)|^2 = \frac{1}{2} \int_\infty^\infty dt A^2(t) \,.$$

There are however corrections to this result for any given multipole, due to the scattering of the GW off the curved space-time because of the presence of the static potential due to the presence of the massive binary system. The first of such corrections scale as $G_N M \int (\mathrm{d}^4 k \frac{1}{k}^2)^2 \delta^3(k) \sim G_N M k \sim v^3$ (for radiation $k \sim v/r$), that is a 1.5PN correction with respect to the leading order. The tail amplitude is described by the diagram in fig. 15 and it adds up to the leading order to give a contribution to the flux going as

$$\left| \frac{A_h|_{v^3}}{A_h|_{v^0}} \right|^2 = 1 + 2\pi G_N M \omega + O(v^6) \,. \tag{258}$$

Finally one could consider the scattering of the emitted GW wave off another GW, as in fig. 16. This process is known as *non-linear memory* effect, it represents a 2.5PN correction with respect to the leading emission amplitude [49] and it has not yet been computed within the effective field theory formalism.

Combined tail and memory effects enter at 4PN order in the emitted radiation, i.e. double scattering of the emitted radiation off the background curvature *and* off another GW. The divergences describing such process have been analyzed in [50], leading to the original derivation of the mass renormalization described in subsec. . The renormalization group equations allow a resummation of the logarithmic term making a non-trivial prediction for the pattern of the leading UV logarithms appearing at higher orders [51].

[49] D. Christodoulou. Nonlinear nature of gravitation and gravitational wave experiments. *Phys.Rev.Lett.*, 67:1486–1489, 1991. DOI: 10.1103/PhysRevLett.67.1486; Luc Blanchet and Thibault Damour. Hereditary effects in gravitational radiation. *Phys.Rev.*, D46:4304–4319, 1992. DOI: 10.1103/PhysRevD.46.4304; and Luc Blanchet. Gravitational wave tails of tails. *Class.Quant.Grav.*, 15:113–141, 1998. DOI: 10.1088/0264-9381/15/1/009

[50] A. Ross W. D. Goldberger and I. Z. Rothstein. Black hole mass dynamics and renormalization group evolution. arXiv:1211.6095 [hep-th], 2012

[51] W. D. Goldberger and A. Ross. Gravitational radiative corrections from effective field theory. *Phys. Rev. D*, 81: 124015, 2010; and A. Ross W. D. Goldberger and I. Z. Rothstein. Black hole mass dynamics and renormalization group evolution. arXiv:1211.6095 [hep-th], 2012

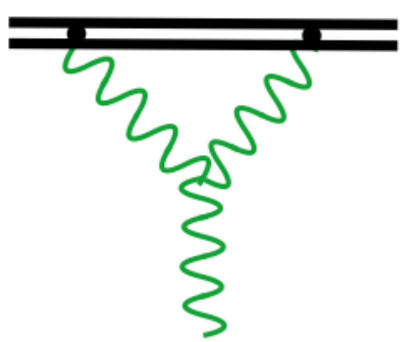

Figure 16: Memory diagram: GW emitted from a source scattered by another GW before reaching the observer.

Hints of data analysis

Using that for a binary system in circular orbit the relevant quadrupole moment components are

$$\begin{aligned} M_{11} &= \frac{\mu}{2}\omega_{GW}^2 R^2 \cos\left(\omega_{GW} t\right) , \\ M_{12} &= \frac{\mu}{2}\omega_{GW}^2 R^2 \sin\left(\omega_{GW} t\right) , \end{aligned} \tag{259}$$

where the reduced mass $\mu \equiv m_1 m_2 / m_1 + m_2$, we have now gathered all the elements to compute the GW-form, which is

$$\begin{aligned} h_+ &= \frac{1}{r} G_N \mu \left(2\pi R f_{GW}\right)^2 F_+(\iota) \cos(\Phi(t)) , \\ &= \frac{4}{r}\eta \left(G_N M\right)^{5/3} \left(\pi f_{GW}\right)^{2/3} F_+(\iota) \cos(\Phi(t) \end{aligned} \tag{260}$$

where the Kepler's law $(\omega_{GW}/2)^2 = G_N M/R^3$. Note that both $h(t)$ as increasing amplitude and instantaneous frequency. Its Fourier transform can be computed via the stationary phase approximation, see ex. 41 to give

$$\tilde{h}_+(f) = \left(\frac{5}{24\pi^{4/3}}\right)^{1/2} e^{\Psi(f)} \frac{1}{r} \eta^{1/2} \frac{\left(G_N M\right)^{5/6}}{f^{7/6}} F_+(\iota) \tag{261}$$

Given the noise characteristic of the detector, quantified by the noise spectral density $S_n(f)$ defined by the average over noise realiations

$$\langle n(f) n(f') \rangle \equiv S_n(f) \delta(f - f') \tag{262}$$

one can define the Signal-to-Noise Ratio (SNR) of a signal as

$$\begin{aligned} SNR^2 &= \int df \frac{|h(f)|^2}{S_n(f)} \\ &= \int d\ln f \frac{f|h(f)|^2}{S_n(f)} \end{aligned} \tag{263}$$

Exercise 33 *** Multipole decomposition**

Derive eq. (215).

Hint: use the trick

$$\int d^3x \left(T_{ij} x_k + T_{ik} x_j\right) = \int d^3x T_{il} \left(x_j x_k\right)_{,l}$$

to derive, via $\dot{T}_{0i} = -T_{il,l}$, that

$$\int d^3x \left(T_{ij} x_k + T_{ik} x_j\right) = \int d^3x \dot{T}_{i0} x_j x_k .$$

Now use the same trick to derive

$$\int d^3x \dot{T}_{i0} x_j x_k + \dot{T}_{j0} x_k x_i + \dot{T}_{k0} x_i x_j = \int d^3x \ddot{T}_{00} x_i x_j x_k .$$

Exercise 34 ** Multipole reduction**

Derive eq. (215).

Hint: use a trick analog to eq. (204) to derive

$$\int d^3x \left[T_{ij}x_k + T_{ki}x_j + T_{jk}x_i\right] = \frac{1}{2}\int d^3x \left[\dot{T}_{i0}x_jx_k + \dot{T}_{k0}x_ix_j + \dot{T}_{k0}x_ix_j\right]$$

and then use

$$\begin{aligned}
\int d^3x \left[T_{ij}x_k - T_{ik}x_j\right] &= \int d^3x \left[T_{il}x_{j,l}x_k - T_{kl}x_{i,l}x_j\right] \\
&= \int d^3x \left[\dot{T}_{i0}x_jx_k - T_{ik}x_j - \dot{T}_{k0}x_ix_j + T_{kj}x_i\right] \\
\Longrightarrow \int d^3x \left[T_{ij}x_k - T_{jk}x_i\right] &= \int d^3x \left[\dot{T}_{i0}x_jx_k - \dot{T}_{k0}x_ix_j\right] \\
\Longrightarrow \int d^3x \left[T_{ij}x_k - T_{ki}x_j\right] &= \int d^3x \left[\dot{T}_{j0}x_ix_k - \dot{T}_{k0}x_ix_j\right]
\end{aligned}$$

Now combine the 3 equations involving $T_{ij}x_k$ to derive the result.

Exercise 35 Velocity quadrupole coupling to the Weyl tensor

From eq. 212 derive the formulae

$$\begin{aligned}
E_{ij} &= \frac{1}{2}\ddot{\sigma}_{ij} \\
B_{ij} &= \frac{1}{4}\epsilon_{ikl}\left(\dot{\sigma}_{kj,l} - \dot{\sigma}_{lj,k}\right)
\end{aligned}$$

valid at linear order in the TT gauge. From the expression of B_{ij} derive eq. (216).

Hint: express the contraction of the second line of eq. (215) with $\sigma_{ij,k}$ as

$$\begin{aligned}
&\left(\dot{T}_{0i}x_jx_k + \dot{T}_{0j}x_ix_k - 2\dot{T}_{0k}x_ix_j\right) = \\
&T_{0i}x_jx_k\left(\delta^{im}\delta^{kn} - \delta^{in}\delta^{km}\right) + T_{0j}x_ix_k\left(\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km}\right).
\end{aligned}$$

Now substitute $\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} = \epsilon^{abi}\epsilon^{cdi}$ to obtain the last term in eq. (216).

Exercise 36 *** Alternative derivation of the Burke-Thorne potential**

Go through eq.(230) and contract the two Weyl tensor using their expression in the TT gauge:

$$R^0_{i0j0} = \frac{1}{2}\ddot{\sigma}^{(TT)}_{ij}.$$

Hint: Remember that the propagator (185) for σ is not the same as the one for $\sigma_{(TT)}$, as

$$\sigma^{(TT)}_{ij} = \Lambda_{ij,kl}\sigma_{kl},$$

with $\Lambda_{ij,kl}$ given by eq.(27).

Exercise 37 ******** Time dependence of total mass of a binary system**

In order to compute the mass of a binary system in terms of its binary constituents properties check how the h_{00} components of the gravitational field couple to it. At leading order the diagram we have the first diagram in fig. 17 where the field ϕ couple to the monopole mass. Neglecting the angular momentum (spin) of the composite object, the next diagram comes from the second and third graphs in fig. 17. For computing the last diagram it will be useful the expansion of $R^0{}_{i0j}$ at linear order in ϕ and σ, i.e.

$$\begin{aligned} R^0{}_{i0j} \;=\;& \frac{1}{2}\ddot{\sigma}_{ij} - \delta_{ij}\ddot{\phi} - \phi_{ij} - \frac{1}{2}A_{i,j} - \frac{1}{2}A_{j,i} \\ &+\frac{1}{2}\phi_k\left(\sigma_{ik,j} + \sigma_{jk,i} - \sigma_{ij,k}\right) - \frac{3}{2}\dot{\phi}\dot{\sigma}_{ij} - 2\phi\ddot{\sigma}_{ij} - \ddot{\phi}\sigma_{ij} \\ &+\delta_{ij}\left(A_k\dot{\phi}_k + \phi_k\dot{A}_k\right) - 2A_j\dot{\phi}_i - 2A_i\dot{\phi}_j - 2\phi_i\dot{A}_j - 2\phi_j\dot{A}_i - \frac{1}{2}(A_{i,j} + A_{j,i})\dot{\phi} \\ &+\delta_{ij}\phi_k\phi_k + 2\delta_{ij}\dot{\phi}^2 - 3\phi_i\phi_j + 4\phi\ddot{\phi} + O(h^3) \end{aligned}$$

and using the "bulk" vertices of ex. 31.

Result: see eq.(8) of [52].

[52] A. Ross W. D. Goldberger and I. Z. Rothstein. Black hole mass dynamics and renormalization group evolution. arXiv:1211.6095 [hep-th], 2012

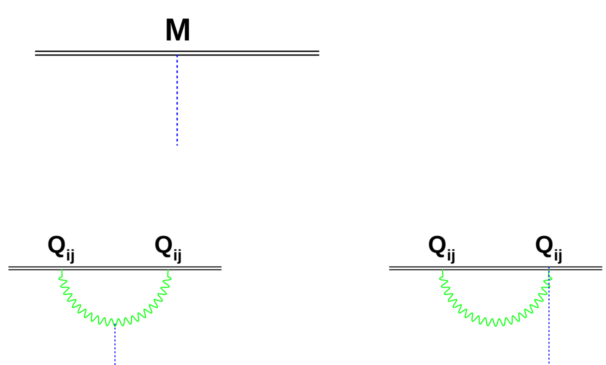

Figure 17: Coupling of gravity to a binary system at leading order and at Q^2 order.

Exercise 38 *   Scaling of GW emission diagrams**

Derive the scaling of the diagram in fig. 14.

Result: $d\omega d^3k\omega^2\tilde{Q}(\omega)H(\omega,\mathbf{k})$.

Derive the scaling of the diagram in fig. 15. Hint: use

- the scaling of the vertex at the GW emission above (divided by Λ)
- the scaling of the GW propagator $\delta^{(4)}(k)/k^2$
- the scaling of the Newtonian graviton emission $dtd^3qM\phi(t,\mathbf{q})/Lambda$
- the scaling of the ϕ propagator $\delta(t)/q^2$
- the scaling of the $\sigma^2\phi$ vertex: $dtd^3xH(t,\mathbf{x})\int d\omega d^3kd^3q\omega^2$.

Result: the amplitude scales as $G_N H(\omega,\mathbf{k})M\tilde{Q}(\omega)d\omega d^3k\omega^3$, i.e. as $G_N M\omega \sim v^3$ times the above one.

Exercise 39 **   Memory effect**

The amplitude for emission of GWs receives corrections from the diagram in fig. 16, where GWs are emitted from two different quadrupole-gravity vertices and then interact together. Adapt the scaling rules described in ex. 38 to this case, in which circulating momenta have $k_0 \sim k \sim v/r$ and the sources emitting the GW $\ddot{Q} \sim Mv^2 \sim d\omega\omega^2\tilde{Q}(\omega)$.

Result: $G_N H(\omega, \mathbf{k}) d\omega d^3k\omega^5\tilde{Q}(\omega)d\omega\tilde{Q}(\omega)$, i.e. of order v^5 with respect leading order diagram in fig. 14.

Exercise 40 **   Energy from propability amplitude

Check that eq. (252) has the correct dimensions

Exercise 41 ***   Fourier transform of the GW-form**

From eq. (260) perform the Fourier integral

$$\tilde{h}_+(f) = \int df h_+(t) e^{2\pi i f t}$$

to obtain eq. (261).

Hint: the integral can be done using the stationary phase approximation, i.e.

$$\begin{aligned}\tilde{h}_+(f) &\propto \frac{1}{2}\int e^{2\pi i f t - i\Phi(t)} dt \\ &\simeq \frac{1}{2} e^{2\pi i f t_* - i\Phi(t_*)} \int e^{-i\ddot{\Phi}(t_*)(t-t_*)^2/2} dt \\ &= \frac{1}{2} e^{2\pi i f t_* - i\Phi(t_*)} \left(\frac{2\pi}{\ddot{\Phi}(t_*)}\right)^{1/2} e^{-i\pi/4}\end{aligned}$$

where t_* is the solution of $2\pi f = \dot{\Phi}(t_*)$ and in the last passage the complext Gaussian integral has been performed. Substituting the expression for the phase as a function of $t_*(f)$ as it can be inferred from eq. (104), one obtains eq. (261).

Exercise 42 ***   Polarization sum**

For the magnetic quadrupole, the polarization tensor sum boils down to:

$$\begin{aligned}&\int \frac{d\Omega}{4\pi} \sum_h \epsilon^{imn} n_n \epsilon^{kpq} n_q \epsilon^*_{ij}(\mathbf{k}, h)\epsilon_{kl}(\mathbf{k}, h) \\ &= \int \frac{d\Omega}{4\pi} \frac{n_n n_q}{2} \Big[\left(\delta_{mp}\delta_{jl} + \delta_{ml}\delta_{jp} - \delta_{mj}\delta_{pl}\right) + n_j n_l n_m n_p \\ &- \left(\delta_{jl} n_m n_p + \delta_{mp} n_j n_l + \delta_{ml} n_j n_p + \delta_{jp} n_m n_l - \delta_{mj} n_p n_l - \delta_{pl} n_m n_j\right)\Big] \\ &= \frac{1}{5}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right) .\end{aligned} \tag{264}$$

For the octupole the analog sum is given by (symmetrization under $i \leftrightarrow j$ and $l \leftrightarrow m$ is understood, however since it is going to multiply tensors which are symmetric under such exchange, we do not need to woeey about that):

$$
\begin{aligned}
&\int d\Omega 4\pi \sum_h n^k n^n \epsilon^*_{ij}(\mathbf{k},h)\epsilon_{lm}(\mathbf{k},h) \\
&= \int d\Omega n_m n_n \left(\delta_{il}\delta_{jm} - n_i n_l \delta_{jm} - n_j n_m \delta_{il} + \frac{1}{2} n_i n_j n_l n_m - \frac{1}{2}\delta_{ij}\delta_{lm} + \frac{1}{2}\left(n_i n_j \delta_{lm} + n_l n_m \delta_{ij}\right)\right) \\
&= \frac{2}{21}\left[\delta_{il}\delta_{jm}\delta_{kn} - \frac{1}{5}\left(\delta_{il}\delta_{jk}\delta_{mn} + \delta_{im}\delta_{ln}\delta_{jk} + \delta_{in}\delta_{lm}\delta_{jk}\right)\right]
\end{aligned} \tag{265}
$$

Derive from the sums above the relative term in eq. (256).

Exercise 43 * Derivation of the flux formula in terms of binary velocity)**

Use the leading quadrupole term in eq.(256) to derive the leading order of the flux formula eq. (3) for circular orbits.

Hint: use eq. (259) and that

$$
v = (G_N M \pi f_{GW})^{1/3}.
$$

Exercise 44 ** Expressing the gravitational wave in terms of the sources in Fourier space**

Find the gravitational wave in terms of the Fourier transform of the energy momentum tensor

$$
\tilde{h}^{TT}_{ij}(t,\vec{x}) = \frac{4G_N}{r}\Lambda_{ij,kl}\int d\omega \tilde{T}_{kl}(\omega,\omega\hat{n})e^{i\omega t - r}
$$

in the case of a scattering of two particles:

$$
T_{\mu\nu}(t,x) = \theta(-t)\left(m_1\delta^{(3)}(\vec{x}-\vec{v}_1 t) + m_2\delta^{(3)}(\vec{x}-\vec{v}_2 t)\right) + \theta(t)\left(m_1\delta^{(3)}(\vec{x}-\vec{v}'_1 t) + m_2\delta^{(3)}(\vec{x}-\vec{v}'_2 t)\right) +
$$

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