Black holes

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An elementary introduction to Black Holes is given. Some knowledge of Special Relativity and the Lagrangian formulation of Mechanics is assumed. These lectures were given at the Inverno Astrofísico School, Sítio Vista Verde, Domingos Martins, Brasil, 2022.

CONTENTS

I. THE METRIC IN SPECIAL RELATIVITY

In this section I give a quick introduction to the notion of a manifold with a metric (a smooth set where geometric measurements can be performed). I motivate with surfaces in $\mathbb{R}^3$ and the extrapolate ideas to higher dimensions. In particular, I introduce Minkowski spacetime and the notion of spacetime used in General Relativity. The use of the metric to measure proper time along worldline, and of geodesics as free fall motion are given. The calculus of geodesics is explained using the Lagrangian analogue with Classical Mechanics.

A. Surfaces, manifolds, metrics and curvature

Consider a curve on a sphere of radius R. This can be given as as

$$\vec{r}(u) = R\big(\sin(\theta(u))\cos(\phi(u)), \sin(\theta(u))\sin(\phi(u)), \cos(\theta(u))\big). \tag{1}$$

Define $\dot{\vec{r}}(u) = \frac{d\vec{r}}{du}$, and similarly $\dot{\theta} = \frac{d\theta(u)}{du}$ etc, then (Exercise 1)

$$\dot{\vec{r}} \cdot \dot{\vec{r}} = R^2\,\dot{\theta}^2 + R^2\,\sin^2(\theta)\,\dot{\phi}^2, \tag{2}$$

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The length s of the piece $u_1 < u < u_2$ of the curve is

$$s = \int_{u_1}^{u_2} \sqrt{\dot{\vec{r}}(u) \cdot \dot{\vec{r}}(u)}\, du = \int_{u_1}^{u_2} \sqrt{R^2(\dot{\theta}^2 + \sin^2(\theta)\, \dot{\phi}^2)}\, du \tag{3}$$

The square length of a small piece of curve corresponding to du is

$$ds^2 = R^2(\dot{\theta}^2 + \sin^2(\theta)\, \dot{\phi}^2) du^2 \tag{4}$$

When expressed in terms of the variations $d\theta$ and $d\phi$, ds^2 is called the LINE ELEMENT of the sphere:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2(\theta)\, d\phi^2 \tag{5}$$

More generally, suppose $S \subset \mathbb{R}^3$ uis a surface parametrized as $\vec{r}(q^1, q^2)$. We call the pair (q^1, q^2) the *coordinates* of S. As in the case of the sphere, where $q^1 = \theta$ and $q^2 = \phi$, a curve on S can be given as $\vec{r}(q^1(u), q^2(u))$, then the square length of its tangent vector will be

$$\frac{d\vec{r}}{du} \cdot \frac{d\vec{r}}{du} = \left(\sum_{i=1}^{2} \frac{\partial \vec{r}}{\partial q^i} \frac{dq^i}{du} \right) \cdot \left(\sum_{j=1}^{2} \frac{\partial \vec{r}}{\partial q^j} \frac{dq^j}{du} \right) = \sum_{i,j=1}^{2} \underbrace{\frac{\partial \vec{r}}{\partial q^i} \cdot \frac{\partial \vec{r}}{\partial q^j}}_{h_{ij}(q)} \frac{dq^i}{du} \frac{dq^j}{du}. \tag{6}$$

The symmetric matrix

$$h_{ij}(q) = \frac{\partial \vec{r}}{\partial q^i} \cdot \frac{\partial \vec{r}}{\partial q^j} \tag{7}$$

is called the METRIC FIELD of S.

As in the case of the sphere, the length of the piece of curve corresponding to $u_1 < u < u_2$ is

$$s = \int_{u_1}^{u_2} \sqrt{\sum_{ij} h_{ij} \dot{q}^i \dot{q}^j} du \tag{8}$$

(compare to (3)) If we replace the upper limit u_2 by the variable u and take a derivative of the resulting function $s(u)$, we find that

$$\frac{ds}{du} = \sqrt{\sum_{ij} h_{ij} \frac{dq^i}{du} \frac{dq^j}{du}} \Rightarrow \left(\frac{ds}{du} \right)^2 = \sum_{ij} h_{ij} \frac{dq^i}{du} \frac{dq^j}{du} \tag{9}$$

which is usually expressed as "for small du, ds^2 is given by

$$ds^2 = \sum_{ij} h_{ij} dq^i dq^j, \tag{10}$$

and is called the *line element* of S."

Surfaces in $\mathbb{R}^3$ are two dimensional: the minimum number of coordinates required to locate its points is two (our choice for the sphere was θ and ϕ, but this choice is not unique). More generally, the DIMENSION of a MANIFOLD ($\sim$ a smooth set) is the *minimum* number of coordinates required to locate its points. An $n-$dimensional manifold can be locally described with n coordinates $\{x^1, ..., x^n\}$ and be given a metric. There is a 1-1 relation between metric (symmetric $n \times n$ matrix field $g_{ab}(x)$) and line element: since the line element is quadratic in the dx^a, associated to it there is, as in the case of surfaces, a unique symmetric $n \times n$ matrix $g_{ab}(x)$, called the METRIC. This is defined by the equation

$$ds^2 = \sum_{a=1}^{n} \sum_{b=1}^{n} g_{ab}(x) dx^a\, dx^b \tag{11}$$

From now on we will use Einstein's convention: we will assume there is an implicit sum over *repeated* indices (over their full range). Equation (11), for example, gets simplified to

$$ds^2 = g_{ab}(x) dx^a\, dx^b \tag{12}$$

Note that, under a change of coordinates $x^a \to \tilde{x}^a(x)$ (we will write $\tilde{x}(x)$ for short, and $x(\tilde{x})$ for its inverse) we have $dx^a = \frac{\partial x^a}{\partial \tilde{x}^b} d\tilde{x}^b$, then

$$ds^2 = g_{ab}(x) dx^a\, dx^b = \left[g_{ab}(x(\tilde{x})) \frac{\partial x^a}{\partial \tilde{x}^c} \frac{\partial x^b}{\partial \tilde{x}^d} \right] d\tilde{x}^v d\tilde{x}^d \tag{13}$$

then the metric matrices are related by (Exercise 5)

$$\tilde{g}_{cd}(\tilde{x}) = g_{ab}(x(\tilde{x})) \frac{\partial x^a}{\partial \tilde{x}^c} \frac{\partial x^b}{\partial \tilde{x}^d} \tag{14}$$

A simple example of a *three dimensional manifold* with a metric is $\mathbb{R}^3 = \{(x, y, z)\}$. A curve in this manifold can be parametrized as $\vec{r}(u) = (x(u), y(u), z(u))$. Since $\dot{\vec{r}} \cdot \dot{\vec{r}} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, the length of the curve is $\int_{u_1}^{u_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du$ and thus the line element Cartesian coordinates is

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{15}$$

so the metric in these coordinates is diag$(1, 1, 1)$. We may choose any other set of coordinates (in 1-1, smooth relation to the Cartesian ones) to cover a piece of $\mathbb{R}^3$, such as the spherical coordinates (r, θ, ϕ) in $\mathbb{R}^3$, which are defined by

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r\cos\theta \end{aligned} \tag{16}$$

After some calculations we find that

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \tag{17}$$

so the metric in (r, θ, ϕ) coordinates is diag$(1, r^2, r^2 \sin^2\theta)$. (To check this you can either use (14) or use (16) to calculate $dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi$ and then insert in $dx^2 + dy^2 + dz^2$ (Exercise 2)). Note in pass that, for curves in $\mathbb{R}^3$ which are restricted to the 2-dimensional sphere $r = R$, satisfy $r = R$ and $dr = 0$ which, inserted in the right side of (14), gives back the line element (5), as expected.

The standard metric on $\mathbb{R}^3$ in Cartesian coordinates (x, y, z), spherical coordinates (r, θ, ϕ), and the metric of a sphere of radius R in (θ, ϕ) coordinates are respectively given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \qquad \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} \tag{18}$$

A *Riemannian manifold* is a manifold that comes with a metric (equivalently: a line element). The metric encodes all the relevant geometric information: it allows to measure lengths of curves and also tells us how the manifold is curved. The line element (5) is that of a *curved* two dimensional space, the line element (15) or its equivalent form in spherical coordinates (16) describe 3-dimensional *flat* space. The concept of *curvature*, informally slipped in here, can, in the case of a two dimensional manifold, be formally defined as the failure of the quotient perimeter/radius for a small a circle to match the Euclidean Geometry result of 2π.
More precisely: the DISTANCE between points A and B is the length of the shortest path connecting A and B; a CIRCLE of radius r centered at p is the set of points whose distance to p is r.

Problem 1: Consider a circle of radius r centered on the "north pole" $\theta = 0$ of a sphere of radius R.

a) Prove that the circle r is the set of points satisfying $\theta = r/R$, which can be parametrized as $\theta = r/R, \phi = u$, $0 \leq u \leq 2\pi$.

b) Prove that the perimeter of this circle is $P = 2\pi R \sin(\theta_o) = 2\pi R \sin(r/R)$ (use (3))

c) Prove that for small r we have

$$P = 2\pi r - \frac{\pi}{3} \frac{r^3}{R^2} + \mathcal{O}(r^5) \tag{19}$$

Note that, in the limit $r \to 0$, $P/(2\pi r) \to 1$, as in Euclidean Geometry (that is, circles in a plane), but that P fails to equal $2\pi r$ as in flat space.

In general, a circle of radius r centered at a point p of a surface S has a perimeter that satisfies

$$P = 2\pi r - \frac{\pi}{3}\mathcal{K} r^3 + \mathcal{O}(r^5). \tag{20}$$

Here

$$\mathcal{K} = \lim_{r\to 0} \frac{3}{\pi}\frac{2\pi r - P}{r^3} \tag{21}$$

defines the Gaussian curvature of S at p. From (19) we learn that the curvature of the sphere of radius R at the north pole (and thus at any other point, since they are all equivalent) is $\mathcal{K} = R^{-2}$. Note that the $\mathcal{K} \to 0$ for large spheres ($R \to \infty$)

Note from equation (21), which holds for generic surfaces, that:
i) when the curvature is positive the perimeter is smaller than that of the Euclidean flat case.
ii) when the curvature is negative the perimeter is larger than that of the Euclidean flat case.
iii) when the curvature is zero the perimeter tends to that of the Euclidean flat case as $r \to 0$.
In particular, if a circle is centered at a point of a surface with *negative* curvature, its perimeter to radio ratio will be larger than the Euclidean rate 2π.

Another characterization of curvature of a surface at a point p, illustrated in Figure 1, is the following:

i) Intersect the surface with planes containing the normal through p. Each intersection defines a curve (called a *(normal) section*).

ii) Give a sign to the radius of curvature of every section according to the side of the tangent plane towards the curve bends (say, positive/negative if in the direction opposite/same to the chosen normal).

iii) There are sections with maximum and minimum values of the signed curvature: $1/R_{max}$ and $1/R_{min}$.

iv) The Gaussian curvature is $\mathcal{K} = 1/(R_{max}R_{min})$. In particular, the curvature of the surface at p is a) positive if all sections through p have center of curvature on the same side of the tangent plane-; b) negative if different sections bend towards different directions of the tangent plane; and c) null at p when all but one curve bends towards the same size, (the exceptional curve has zero curvature radius at p)

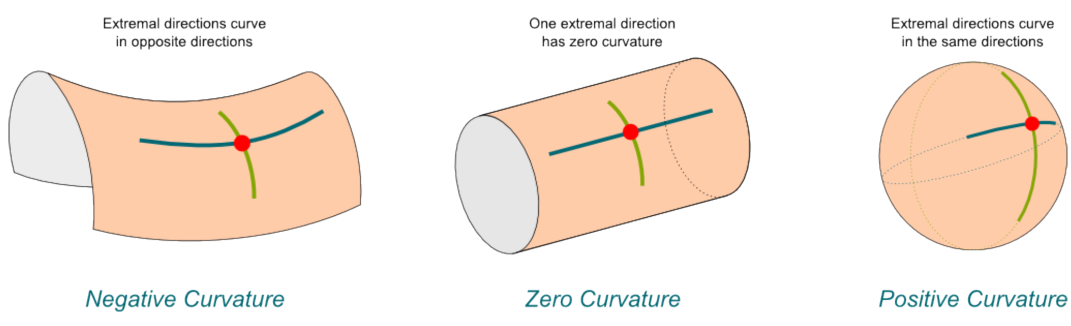

FIG. 1. Example of points p where the curvature is positive (all sections bend towards the same side of the tangent plane), negative (some sections bend towards one side and others towards the opposite) and null (the limit situation). This is an *extrinsic* characterization of curvature for surfaces embedded in an ambient space. An intrinsic characterization (that is, done by measurements restricted to the surface) is possible by defining curvature as in equation (21). Only the surface metric is required in this case.

Note that this second characterization of curvature is extrinsec (requires measurements in the ambient three dimensional space where the surface is embedded), whereas the definition using the excess/deficit of circle perimeter is intrinsic. It was Gauss (1820's) who realized the two definitions agree (*Egregium* theorem).

In higher dimensions, (intrinsic) curvature is not characterized by a single number, but with $n^2(n^2-1)/12$ numbers (n the dimension of the manifold) encoded in what is called the *Riemann tensor*.

B. Minkowski spacetime: a four dimensional manifold with a metric

An EVENT is an occurrence that is well localized both in space and time (it requires that it spreads little and occurs fastly, and is, of course, an idealization). The SPACETIME is the set of all EVENTS. Events require four coordinates to be labeled: three for the location and one for the time. According to the SPECIAL THEORY OF RELATIVITY, events in the spacetime can be adequately labeled with the Cartesian coordinates (x, y, z) and the time t measured by an inertial observer. The theory asserts that as a manifold, spacetime is $\mathbb{R}^4 = \{(ct, x, y, z)\}$, and comes equipped with the line element given below in equation (23): this defines MINKOWSKI SPACETIME.

Let's explore its geometry: massive particles cannot reach the the speed of light c whereas massless particles travel at exactly the speed of light. As a consequence, for a small piece of WORLDLINE of a particle (its trajectory in spacetime) holds the inequality

$$\underbrace{-c^2dt^2 + dx^2 + dy^2 + dz^2}_{ds^2} \leq 0 \tag{22}$$

with equality only for massless particles like a photon. Equation (23) defines the line element of Minkowski spacetime:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{23}$$

The metric in (t, x, y, z) coordinates is therefore $g_{ab} = \text{diag}(-c^2, 1, 1, 1)$. Note that it is not positive definite but has SIGNATURE $(-, +, +, +)$. The quantity $d\tau^2 \equiv -ds^2/c^2$ gives the square of the PROPER TIME interval (that is, measured by a clock carried along with the particle). Thus, for a particle with worldline $(t(u), x(u), y(u), z(u))$ (here u is an arbitrary parameter), the proper time τ elapsed between the events corresponding to $u = u_1$ and $u = u_2$ is

$$\tau_2 - \tau_1 = \frac{1}{c}\int_{u_1}^{u_2} \sqrt{-\left(-c^2 \tfrac{dt}{du}^2 + \tfrac{dx}{du}^2 + \tfrac{dy}{du}^2 + \tfrac{dz}{du}^2\right)}\, du \tag{24}$$

In particular, if we parametrize the worldline with proper time, $u = \tau$, then $d\tau/du = 1$ and the tangent vector $v^a = dx^a/d\tau$ has *negative* SQUARE NORM iqual to $-c^2$:

$$g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = -c^2 \tfrac{dt}{d\tau}^2 + \tfrac{dx}{d\tau}^2 + \tfrac{dy}{d\tau}^2 + \tfrac{dz}{d\tau}^2 = -c^2 \tag{25}$$

A SPACETIME DIAGRAM of a worldline is nothing different than what you get from an elementary Physics graph of the function of motion of a particle (Figure 2), if you switch the positions of the (in the case of motion in one dimension) t

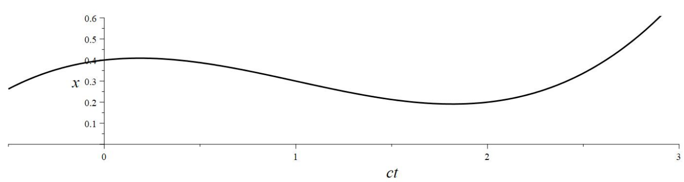

FIG. 2. A standard graph of a function of motion x vs t

and x axis, as in Figure 3. In view of (23), the slope of the tangents to worldlines are restricted (in one space dimension) to

$$-\frac{1}{c} \leq \frac{dt}{dx} \leq \frac{1}{c} \tag{26}$$

equality holding only for massless particles. The above condition defines the LIGHT CONE at a point of the spacetime. The FUTURE LIGHT HALF CONE corresponds to the half cone where tangent vector of worldlines point (in Minkowski space, the direction where t increases): the tangent to a worldline at a point p fits within the future light cone at p. Figure (3) illustrates the worldline of a massive particle and the future light cone at three of its events.

Since *proper time of segments of worldlines* are completely analogous to *length of segments of curves* on surfaces (compare equations (3) or (8) and (24)), it is for the same reason that the lengths of different curves on a surface joining A and B will in general be different, that the proper time elapsed along two different worldlines (of particles with mass) meeting at the events A and B will be different. A well known example is that of the so called "twin paradox": one twin remains at the origin of an inertial system, the other travels away and comes back after some time to find that

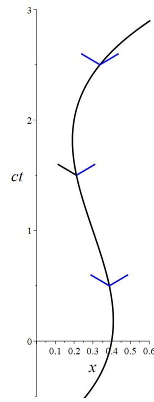

FIG. 3. The worldline in Figure 2 and the future light cones at three of its points.

the marks on their clocks (which were synchronized when he left) mark a different time. The situation is that in Figure 4, where I have worked out the case where the traveler twin has a worldline $(t, \ell \cos(\frac{\pi t}{2T}))$ –red in the figure– whereas his brother has a worldline $(t, 0)$ –blue in the figure–. Any situation with $\frac{\pi \ell}{2T} < c$ is admissible (why?). In the case of the figure the units are $[t]$ =years, $[\ell]$ =light years, and I chose $T = 5$ years and $\ell = 1$ light year. The proper time between encounters, measured by the blue twin is trivially $\Delta\tau_{blue} = 10$ years (just do the calculation (24)). For the red twin the integral (24) can be written in terms of elliptic functions. A numerical integration gives $\Delta\tau_{red} \simeq 9.75$ years.

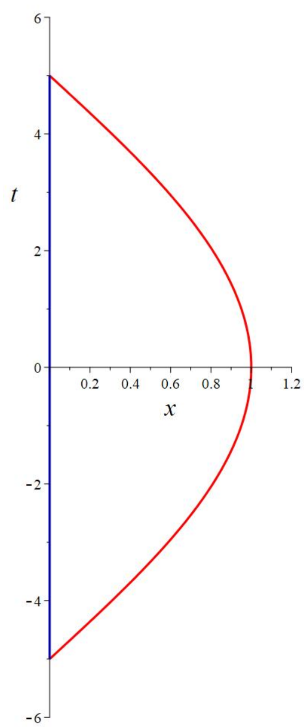

FIG. 4. The twin case: the "length" $\Delta\tau$ of the blue twin worldline between encounters is 10 years, that of his brother (in red) s 9.75 years. Units are years in the t axis and light years in the x axis.

In the real world, spacetime is four dimensional and, in Special Relativity condition (26) reads

$$dx^2 + dy^2 + dz^2 \leq c^2 dt^2 \tag{27}$$

Figure 5 is a depiction of worldline in Minkowski spacetime and the line cones at some points (@@@ DESCRIBIR MÁS)

As in ordinary geometry, we are free to choose any set of four coordinates that smoothly labels the events of the spacetime. In Minkowski spacetime, for example, we may choose spherical coordinates adapted to the worldline ($t =$

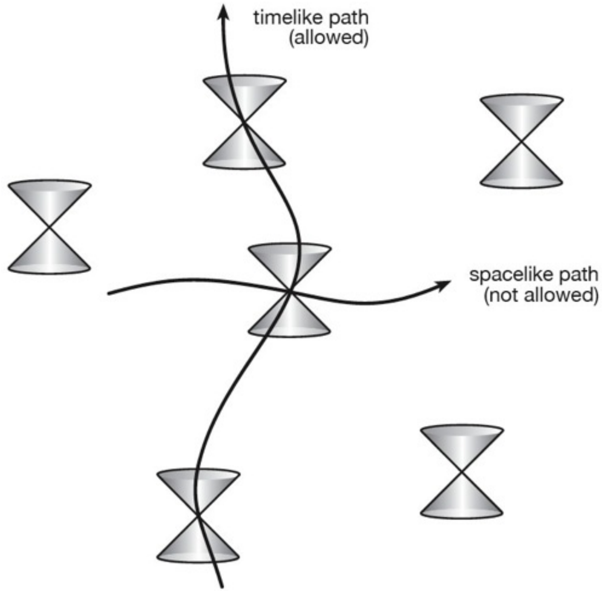

FIG. 5. A worldline and some light cones in Minkowski spacetime (taken from Sean Carroll blog `https://www.preposterousuniverse.com/` .

$\tau, x = x_o, y = y_o, z = z_o$):

$$\begin{aligned} t &= t \\ x &= x_o + r \sin\theta \cos\phi, \\ y &= y_o + r \sin\theta \sin\phi, \\ z &= z_o + r \cos\theta \end{aligned} \tag{28}$$

Minkowski metrics assumes the form (Exercise 4)

$$ds^2 = -c^2 dt^2 + \underbrace{dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)}_{dx^2+dy^2+dz^2} \tag{29}$$

Further introducing the "advanced coordinates"

$$\begin{aligned} v &= ct + r \\ r &= r \\ \theta &= \theta \\ \phi &= \phi \end{aligned} \tag{30}$$

gives

$$ds^2 = \underbrace{-dv^2 + 2dv\, dr}_{-c^2dt^2+dr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \tag{31}$$

Below we write Minkowski's metric in coordinates $x^a = (ct, x, y, z)$, $\tilde{x}^a = (ct, r, \theta, \phi)$ and $\hat{x}^a = (v, r, \theta, \phi)$ (Exercise 6):

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \tilde{g}_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \qquad \hat{g}_{ab} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{32}$$

II. THE AXIOMS OF GENERAL RELATIVITY (GR)

According to GR, spacetime can be modeled as a four dimensional manifold M (in general, different from $\mathbb{R}^4$) and comes with a metric g_{ab} encoding all the information we need. As in Minkowski spacetime, the metric is not positive definite but has signature $(-+++)$ (that is, the metric matrix can be diagonalized with entries $-1, 1, 1, 1$). The worldline $x^a(u), a = 1, 2, 3, 4$ of a massive particle has tangent vectors with negative square norm:

$$g_{ab}(x)\dot{x}^a\dot{x}^b < 0, \quad \dot{x}^a = \frac{dx^a}{du} \tag{33}$$

and the proper time $\tau_2 - \tau_1$ measured along a piece of worldline from the event with coordinates $x^a(u_1)$ to the event of coordinates $x^a(u_2)$ is (compare to (41), note the sign change inside the square root),

$$\tau_2 - \tau_1 = \tfrac{1}{c} \int_{u_1}^{u_2} \sqrt{-g_{ab}(x)\dot{x}^a\dot{x}^b}\, du \tag{34}$$

Again, if we replace the upper limit by a variable, $u_2 \to u$, we find that $d\tau/du = \sqrt{-g_{ab}(x)\dot{x}^a\dot{x}^b}/c$, thus, if we parametrize worldlines with proper time, their tangent vectors $v^a = dx^a/d\tau$ satisfy (in view of $d\tau/du = 1$)

$$g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = -c^2 \tag{35}$$

The above equation generalizes the Minkowskian case (25).

A. Einstein's equations of General Relativity

In GR gravity is not regarded as a force: the mass and energy distributed in the Universe is accounted for by the *stress-energy-momentum field* T_{ab}, and this tensor field partially determines the metric g_{ab} ("mass and energy bend the spacetime") by means of Einstein field equations. This is outside the scope of this course and is briefly developed for completeness in the box below:

1. For a manifold of arbitrary dimension, the metric inverse, denoted simply using upper indices, is defined by the equation:

$$g^{ab}g_{bc} = \delta^a_c \tag{36}$$

2. The *Christoffel symbol* is obtained from the metric, its inverse and its first derivatives:

$$\Gamma^c_{ab} = \tfrac{1}{2} g^{cd} \, (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \, , \quad \left(\partial_a \equiv \frac{\partial}{\partial x^a} \right) \tag{37}$$

3. The *Riemman tensor field* is constructed from second derivatives of the metric

$$R^d{}_{cba} = -\partial_a \Gamma^d{}_{bc} + \partial_b \Gamma^d{}_{ac} + \Gamma^e_{ca}\Gamma^d_{be} - \Gamma^e_{cb}\Gamma^d_{ae} \tag{38}$$

In spite of having 4 indices, the number of independent components of the Riemann tensor is –due to algebraic constraints– not n^4, but $n^2(n^2-1)/12$, where n is the dimension of the manifold ($n = 4$ for the spacetime). For a surface in $\mathbb{R}^3$, $n = 2$ and $n^2(n^2-1)/12 = 1$: the only independent component of the Riemann tensor is the Gauss curvature that we calculated for the sphere in (20). For the spacetime $n = 4$ and the Riemann tensor has 20 independent components.

4. The *Ricci tensor R_{ab} field*, *Einstein tensor G_{ab}* and *Ricci scalar R* are

$$R_{ca} = R^d{}_{cda}, \quad R = R_{ca} g^{ca}, \quad G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} \tag{39}$$

Finally, *Einstein's equations* give a relation between the stress-energy-momentum tensor and spacetime metric through its Einstein's tensor:

$$G_{ab} = \frac{8\pi G}{c^4} \, T_{ab} \tag{40}$$

(right side is matter, left side is geometry). Note that G_{ab} is obtained from the metric and its first and second partial derivatives with respect to the coordinates and that it is *not* linear in the metric: the Einstein tensor of a sum of metrics is not the sum of their Einstein's tensors.

B. The geodesic equation

In the study of surfaces or other higher dimensional *Riemannian manifolds* (those equipped with a positive definite metric field), geodesics are the shortest path connecting a given pair of points. This length is given in equation (**??**)

$$s = \int_{u_1}^{u_2} \sqrt{g_{ab}(x)\dot{x}^a\dot{x}^b} \, du \tag{41}$$

where $x^a(u_1)$ and $x^a(u_2)$ are the coordinates of the end points. The problem of finding the stationary points of s under small variations $\delta x^a(u)$ of the trajectories with fixed end points ($\delta x^a(u_1) = 0 = \delta x^a(u_2)$) is *exactly the same* as that of finding the stationary points of the action of a mechanical problem with generalized coordinates x^a and Lagrangian

$$\hat{\mathcal{L}}(x^a, \dot{x}^a) = \sqrt{g_{ab}(x)\dot{x}^a\dot{x}^b} \tag{42}$$

As is well known, this problem leads to the Euler-Lagrange equations for the Lagrangian (42). The length s is parametrization invariant, and it turns out that if restrict to parametrizations such that the tangent vector has unit length (that is $g_{ab}(x)\dot{x}^a\dot{x}^b = 1$), we can replace $\hat{\mathcal{L}}$ by the much simpler Lagrangian:

$$\mathcal{L}(x^a, \dot{x}^a) = g_{ab}(x)\dot{x}^a\dot{x}^b \tag{43}$$

which is formally obtained from the line element by substituting $dx^a \to \dot{x}^a$ (see (11)).

As an example, consider a sphere of radius R, from (5) we find that

$$\mathcal{L}(\theta, \phi, \dot{\theta}, \dot{\phi}) = R^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{44}$$

The Euler-Lagrange equation for this Lagrangian are (exercise)

$$\begin{aligned} &\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \\ &\ddot{\phi} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0 \end{aligned} \tag{45}$$

Note that some solutions are the meridians $\theta(u) = \dot{\theta}_o u + \theta_o, \phi_u = \phi_o$ and the Equator $\theta(u) = \pi/2, \phi(u) = \dot{\phi}_o u + \phi_o$, and that no other parallel $\theta = \theta_o$ is geodesic. Note also that (44) is the Lagrangian of a free particle on the sphere. This extrapolates easily to the case of particle restricted to a surface $S \subset \mathbb{R}^3$: in view of the definition of the surface metric, equation (7), we find that $\dot{\vec{r}} \cdot \vec{r} = h_i j(q)\dot{q}^i\dot{q}^j$, thus the Lagrangian of a unit mass particle restricted to S (and subject to no external force) is (43) : free particles restricted to a surface follow geodesics!

Problem 2: i) Prove that the Equator and the meridians are geodesics of the sphere and that no parallel different from the Equator is a geodesic. ii) *Without doing any additional calculation* prove that geodesics connecting two points A and B of the sphere are segments of *great circles*: these are the curves obtained by intersecting the sphere with the plane defined by A, B and the sphere center. Since this the shortest path from A to B, this is the trajectory a plane would follow (as it minimizes distance) when there are no other relevant factors (such as weather issues or high mountains or political restrictions).

Problem 3: Prove that the geodesic equations, obtained by applying the Euler-Lagrange equations to the Lagrangian (43), lead to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \tag{46}$$

where Γ^a_{bc} is the Christoffel symbol for the metric, defined in equation (37).

In the case of a spacetime M with metric g_{ab}), the metric is, as for Minkowski spacetime, not positive definite. As a consequence, there are three kinds of curves of interest:
i) Timelike: those for which $g_{ab}\dot{x}^a\dot{x}^b < 0$. These are the worldlines of massive particles.
ii) Null or Lightlike: those for which $g_{ab}\dot{x}^a\dot{x}^b = 0$. These are the worldlines of massless particles and of light rays in the geometric optics approximation.
iii) Spacelike: those for which $g_{ab}\dot{x}^a\dot{x}^b = 0$. No particle can follow these curves.

Here comes one more axiom of GR:

Particles in free fall (that is, not acted upon by any force) follow timelike GEODESICS of the metric.

As an aside, these worldlines *maximize* proper time between fixed endpoint vents (compare with (41), note the sign change within the square root):

$$\tau = \int_{u_1}^{u_2} \sqrt{-g_{ab}(x)\dot{x}^a\dot{x}^b}\, du \tag{47}$$

III. NON ROTATING BLACK HOLES

A black hole was "hidden" in the first solution found for the Einsten's equations (Schwarzschild's solution, 1916). Understanding the incredible prediction in this solution took, however, many years. Schwarzschild's black hole is static, spherically symmetric, non rotating. In general, black holes rotate. Rotating black holes are described by Kerr's solution of the Einstein field equations, found only in 1963. Although non rotating black holes are highly ideal, most of the relevant physical effects can be explained using Schwarzschild's solution, which has the advantage of being much simpler than Kerr's. This is the subject of this section.

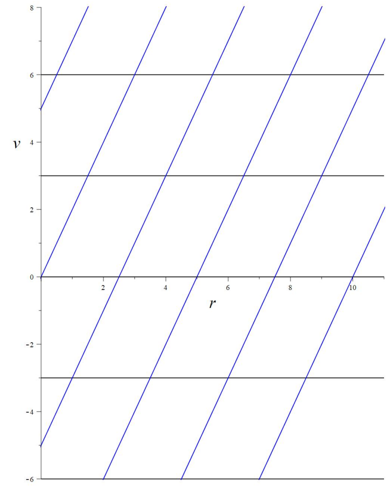

FIG. 6. Radial null geodesics in Minkowski spacetime, the black ones are ingoing whereas the blue ones are outgoing. Some future light cones are indicated.

A. Schwarzschild static black hole

1. Radial null geodesics (light rays) in Minkowski spacetime

Consider Minkowski spacetime in (v, r, θ, ϕ) coordinates, equation (31)

$$ds^2 = -dv^2 + 2dv\ dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{48}$$

Radial light rays follow null lines (that is, $ds^2 = 0$) with $d\theta = 0 = d\phi$:

$$0 = ds^2 = -dv^2 + 2dvdr = dv(-dv + 2dr) \tag{49}$$

There are two kinds of radial light rays:

- radial *ingoing* null geodesics (RINGs), for which $dv = 0$

$$\frac{dv}{dr} = 0, \quad v = v_o + 2r \quad \text{parametrically: } (v(s), r(s), \theta(s), \phi(s)) = (v_o, -s + r_o, \theta_o, \phi_o) \tag{50}$$

  (s grows towards the future, then r gets smaller)

- radial outgoing null geodesics (RONGs), for which $dv/dr = 2$

$$\frac{dv}{dr} = 2, \quad v = v_o \quad \text{parametrically: } (v(s), r(s), \theta(s), \phi(s)) = (2s + v_o, s + r_o, \theta_o, \phi_o) \tag{51}$$

  (s grows towards the future, then r gets larger)

Note that *every* RONG in Minkowski spacetime can reach regions of arbitrarily large values of r. The region of spacetime towards RONGs are directed in the far future, that is

$$(v, r, \theta, \phi) = (2s + v_o, s + r_o, \theta_o, \phi_o) \quad \text{for very large } s \quad (\text{equivalently } v \to \infty, r \to \infty, \text{keeping } v - 2r \text{ finite}), \tag{52}$$

is called *future null infinity*, and denoted $\mathscr{I}^+$. Mathematically, $\mathscr{I}^+$ can be added as the boundary of a manifold that what we call *conformal spacetime*, but we don't care about this technicality here).

2. Schwarzschild solution

Minkowski spacetime is a trivial solution of Einstein's vacuum (that is $T_{ab} = 0$) equations. Schwarzschild found the first non trivial vacuum solution of the Einstein equations just a few months after Einstein's GR was published. In suitable coordinates (not the originally used by Schwarzschild), the line element looks like a deformation of Minkowski's (as given in (57)):

$$- f(r)dv^2 + 2dv\, dr + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad f(r) = 1 - \frac{r_S}{r} \tag{53}$$

This metric models *the vacuum exterior region of a static spherically symmetric distribution of total mass M*. The mass M is related to the SCHWARZSCHILD RADIUS by the equation

$$r_S = \frac{2MG}{c^2} \tag{54}$$

Given the magnitude of Newton's constant G and the speed of light c, the Schwarzschild radius is ridiculously small, for example

$$\begin{aligned} r_S(\text{Sun}) &= 3 \text{ kilometers} \\ r_S(\text{Earth}) &= 9 \text{ millimeters} \end{aligned} \tag{55}$$

A piece of matter needs to be comprised to extremely high densities to fit within a sphere of radius less than its Schwarzschild radius.

Note that for $r \gg r_s = 2MG/c^2$, Schwarzschild metric tends to agree with Minkowski metric (57). We express this by saying that Schwarzschild metric is "asymptotically Minkowskian". In view of the smallness of r_S, we don't actually need to take r to be too large. For example, if we model the metric of the solar system (outside the Sun) with Schwarzschild's solution (which is ok outside the Sun, where there is essentially vacuum, neglecting the Sun's rotation and its departure from perfect sphericity), since the average distance Venus-Sun is about $10^8 km$, then for $r > r_{Venus}$ we have

$$ds^2 = ds^2_{Mink} - \frac{r_S}{r} dv^2, \quad \frac{r_S}{r} < \frac{3km}{10^8 km} = 3 \times 10^{-8} \tag{56}$$

So assuming that the spacetime metric in our labs is Minkowskian is an excellent approximation.

3. Radial null geodesics (light rays) in Schwarzschild spacetime

Consider Schwarzschid spacetime in (v, r, θ, ϕ) coordinates, equation (53):

$$ds^2 = -f(r)dv^2 + 2dv\, dr + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad f(r) = 1 - \frac{r_S}{r} \tag{57}$$

Let's find the outgoing and ingoing null rays, as we did in the Minkowski case. Again, radial light rays follow null lines (that is, $ds^2 = 0$) with $d\theta = 0 = d\phi$:

$$0 = ds^2 = -f(r)dv^2 + 2dvdr = dv(-fdv + 2dr) \tag{58}$$

and this equation gives us the two kinds of rays:

- radial ingoing null geodesics (RINGs), for which $dv = 0$ and r:

$$\frac{dv}{dr} = 0, \quad v = v_o \quad \text{parametrically: } (v(s), r(s), \theta(s), \phi(s)) = (v_o, -s + r_o, \theta_o, \phi_o) \tag{59}$$

- radial outgoing null geodesics (RONGs), for which $(1 - r_S/r)dv = 2dr$:

$$\frac{dv}{dr} = \frac{2}{f(r)}. \tag{60}$$

In this case we find three different cases:

$$\begin{cases} v = 2r + 2r_S \ln\left(\frac{r}{r_S} - 1\right) + \text{const.} & \text{, geodesics that stay in the region } r_S < r < \infty \\ v = 2r + 2r_S \ln\left(1 - \frac{r}{r_S}\right) + \text{const.} & \text{, geodesics that stay in the region } 0 < r < r_S < \\ r = r_S, -\infty < v < \infty & \text{, geodesics that stay in the hypersurface defined by } r = r_S \end{cases} \tag{61}$$

Note in Figures 7 and 8 that every RONG in Minkowski spacetime can reach regions of arbitrarily large values of r. The region of spacetime reached by RONGs in the far future, that is

$$(v, r, \theta, \phi) = (2s, s + r_o, \theta_o, \phi_o) \quad \text{for very large } s \quad (\text{equivalently } v \to \infty, r \to \infty, \text{keeping } v - 2r = \text{finite}), \tag{62}$$

is called *future null infinity*, and denoted $\mathscr{I}^+$

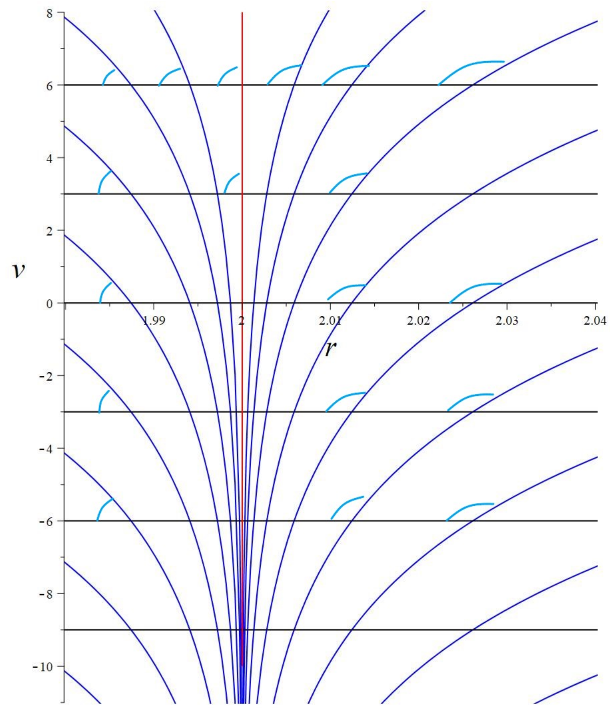

FIG. 7. Detail of ingoing (black) and "outgoing" (blue, and the red one) radial null geodesics for a Schwarzschild spacetime near the Schwarzschild radius (in the example at $r_S = 2$). The tangents at their intersections trace the future null cone at every point: future timelike or null curves must cross the event with a tangent inside this half cone. In particular, timelike/null curves can only cross the horizon from right to left and, although it is possible to send signals and travel from the $r > r_S$ to arbitrarily large r regions (future null infinity) and also to the Black Hole region $0 < r < r_S$, it is not possible to get out of the Black Hole region.

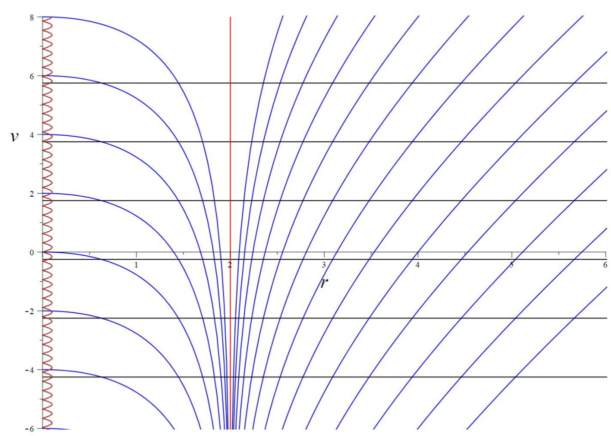

FIG. 8. A zoom out vision of RINGs and RONGs (in blue). The black hole horizon is in red, the wavy line signals the singularity at $r = 0$, which is a boundary of spacetime, and so no traversable. Note how null geodesics approach the shape of Minkowskian geodesics for $r > r_S$ and how the future cones in the black hole region $r < r_S$ forces worldines into the singularity at $r = 0$.

4. The coordinate singularity in Schwarzschild's original derivation

When Schwarzschild set out to find a solution for the spherically symmetric static vacuum Einstein's equations he (correctly) proposed the ansatz

$$ds^2 = -g(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{63}$$

which looks like Minkowski line element in spherical coordinates (equation (29)). It happens that, if we want to put (53) into the diagonal form (63) the transformation (30), $t = (v - r)/c$, which works in the Minkowski case, doesn't do the job (leaves a crossed term $dt\,dr$). We need instead define t by

$$c\,t = v - r - r_S \ln\left|\frac{r}{r_S} - 1\right| = \begin{cases} v - r - r_S \ln\left(\frac{r}{r_S} - 1\right) & , r > r_S \\ \text{undefined} & , r = r_S \\ v - r - r_S \ln\left(1 - \frac{r}{r_S}\right) & , r < r_S \end{cases} \tag{64}$$

The way t is defined makes the set (t, r, θ, ϕ) appropriate coordinates in the open set $r > r_S$ and also in the set set $0 < r < r_S$, but this coordinates do not cover the entire manifold (is not 1-1 in the entire manifold, is not defined at $r = r_S$) and cannot be used, for example, to study geodesics that cross the horizon r_S. It's not a surprise that the transformation $(v, r, \theta, \phi) \to (t, r, \theta, \phi)$ puts the metric (53) into a form that is singular for $r = r_S$:

$$ds^2 = -c^2(1 - r_S/r)dt^2 + \frac{dr^2}{1 - r_S/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{65}$$

which is indeed of the desired form (63), and the one that Schwarzschild found. The singularity of (65) at $r = r_S$, that is, of the element g_{rr} of the metric in these coordinates (see (12)) is a consequence of the transformation rule for the metric under a coordinate change, equation (14), and the fact that the Jacobian matrix of the coordinate transformation $(v, r, \theta, \phi) \to (t, r, \theta, \phi)$ is singular at $r = r_S$. Simple stated: the coordinate system (v, r, θ, ϕ) is *global* (covers the entire manifold), whereas the coordinates (t, r, θ, ϕ) are not. This singularity caused some confusion before the advanced coordinates (v, r, θ, ϕ) and the form (53) of the metric was available. It was then understood that t, not being defined at $r = r_S$ and diverging as this surface is approached (see (64)), was not suitable to form part of a global coordinate system.

5. *The future boundary of Schwarzschild solution*

The chronology of events leading to the understanding of the Schwarzschild solution started with the puzzling singularity at $r = r_S$ in (65), resolved when it was found that it was a coordinate singularity. One is compelled to ask if there is a coordinate change that resolves the surviving singularity at $r = 0$ in (53). That the answer is in the negative can be shown by a simple and smart argument: although the transformation under coordinate changes of *tensor fields* such as the metric involve the Jacobian of the coordinate transformation $\frac{\partial \tilde{x}^a}{\partial x^b}$ and its inverse (see, for example, (14)), and these may introduce of lift singularities, for a scalar field $\Phi(x)$ we simply re-express it as $\Phi(x(\tilde{x}))$. Now consider the following scalar field, made out of the Riemann tensor and the metric and its inverse (these fields where defined in (36)-(38)):

$$\Phi := R^a{}_{bcd} R^e_{fgh} g_{ae} g^{bf} g^{cg} g^{dh} \tag{66}$$

we find, for the Schwarzschild solution,

$$\Phi(v, r, \theta, \phi) = \frac{M^2}{r^6} \tag{67}$$

so, no matter which coordinate change $(r, v, \theta, \phi) = F(x^1, x^2, x^3, x^4)$ we propose, when the x^a tend to a point for which $r = 0$, $\Phi \to \infty$.
Since Φ is a scalar field made out of the Riemann tensor and –as we briefly explained above– this tensor encodes all the information about *curvature*, we say that there is a *curvature singularity* at $r = 0$. There is no way to extend the spacetime beyond $r = 0$.

Now that we know that $r = 0$ is *not* a coordinate artifact but a boundary of the spacetime manifold we may wonder what kind of boundary is this. To our perplexity, this boundary is not "a place in space" but, instead, "the end of time" for a family of observers, in some sense similar to the Big Band a "beginning of time" in cosmological solutions, except that in this case time ends only for those unfortunate inside the black hole, that is, those who enter the $r < r_S$ region (more on this below).

6. *The black hole in the Schwarzschild solution*

Schwarzschild spacetime has two very distinct regions: the Domain of Outer Communications (DOC), which is the subset defined by $r > r_S$ and the Black Hole Region (BH): the set where $0 < r < r_S$. The boundary between these two regions, the hypersurface (3-dimensional manifold) defined by $r = r_S$ and (v, θ, ϕ) free, is called the Event Horizon. The terminology will become clear as we go through the following facts:

1. The solution is asymptotically Minkowskian: for large r the metrics (53) and (48) are indistinguishable. As in the case of Minkowski spacetime, we call future null infinity $\mathscr{I}^+$ the region where $v \to \infty$, $r \to \infty$ keeping $v - 2r$ finite. It turns out that a radial null geodesic in the family of so called "outer going" (the blue ones in Figures 7 and 8) are *truly* outer going (and can reach $\mathscr{I}^+$) *only if they originate in the DOC*. Those at the horizon stay at the horizon, whereas those in the BH region crush into the $r = 0$ boundary: they never enter the DOC. This is illustrated n Figures 7 and 8. Since *any* timelike or null future worldline must fit within the future light cones (some of them are depicted in light blue in Figures 7 and 8), we conclude that events in the BH region are causally disconnected from those in the DOC: there is no way to send radio waves or spaceships from the BH to the DOC.

2. Every future directed timelike or null curve through a point in the BH region ends up at the curvature singularity at $r = 0$. We can even give an upper bound for the amount of proper time a massive particle can survive inside the BH region. This calculation is easier working in Schwarzschild coordinates (65), which can be used since we are dealing with worldines entirely restricted to the BH region, not crossing the horizon. For a particle with mass, equation (65) gives (keep in mind that $f < 0$ in this region!!)

$$c^2 d\tau^2 = -ds^2 = c^2(1 - r_S/r)dt^2 - \frac{dr^2}{1 - r_S/r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) < -\frac{dr^2}{1 - r_S/r} = \frac{dr^2}{r_S/r - 1} \tag{68}$$

therefore (Exercise) the proper time τ to get from $r_o < r_S$ to the $r = 0$ boundary satisfies

$$\tau < \frac{1}{c}\int_0^{r_o} \frac{dr}{\sqrt{\frac{r_S}{r} - 1}} = \frac{R\pi}{2c} \simeq 10^{-5}\frac{M}{M_{Sun}}\text{seconds} \tag{69}$$

where in the last step we used the fact that the Schwarzschild radius for a solar mass in $\sim 3km$. Note that, if you are inside a BH, the best you can do to survive the longest is to let you go: free fall (geodesic motion) maximizes proper time. WE have just proved our claim above: $r = 0$ is not a place in the Universe but a final time for particles inside the BH (don't take to seriously the names of coordinates: moving in the r direction while keeping (t, θ, ϕ) fixed is a timelike curve inside the black hole).

3. The horizon acts as a one-way membrane: it allows particles and light rays from the DOC into the BH but not the other way. It is called *event horizon* because is the past boundary of those events that are causally disconnected from "the far away events" at $\mathscr{I}^+$.

4. In the BH region there are TRAPPED SURFACES: closed and compact surfaces such that light emitted towards its exterior as well as that emitted towards the interior have *both* decreasing wave fronts, contrary to what happens in Minkowski spacetime, where the first have increasing wave fronts whereas the second have decreasing wave fronts. This can be inferred by inspecting Figures 7 and 8, in which it is clear that both RINGs and "RONGs" in the BH region move towards regions of *decreasing* radius, whereas in the DOC they behave "normally", as in Minkowski spacetime, Figure 6.

7. *Some historical milestones*

- 1783: John Michell (in a letter to *Philosophical Transactions of the Royal Society of London*) conjectured the existence of *black stars*. He combined Newton's gravity with the speed of light c (first measured by er Ole Rømer in 1676) and treated light as corpuscles: the escape velocity v_e from a spherical star of radius r_* can be obtained from

$$\frac{m}{2}{v_e}^2 - \frac{GMm}{r_*} = 0, \quad \Rightarrow \quad {v_e}^2 = \frac{2GM}{r_*} \tag{70}$$

so v_e must be *larger than* c to escape the surface of the star if

$$r_* < \frac{2GM}{c^2} \equiv r_S \qquad \text{(Schwarzschild radius!!!)} \tag{71}$$

If the inequality (71) holds, light falls back onto the star surface: the star looks black from any point at a distance

$$d > \frac{r_* r_S}{r_S - r_*} \tag{72}$$

- 1796: Laplace made the same prediction as Michell and published it in his book *Le Systeme du Monde*.

- 1801: Young proved the wave nature of light by means of a (sort of) two slit experiment.
- 1809: In view of Young's wave theory, Laplace removed black stars from the third edition of his book.
- 1915: Einstein's completed his General Relativity theory, which predicts influence of gravity on light propagation in spite of the wave nature of light (NOT by slowing the light wave speed, but bending the direction of propagation.)
- 1916: Schwarzschild presented the first exact solution to Einstein's equation. The solution was meant to represent the geometry for a vacuum static solution, for example, the exterior of a non-rotating star of mass M:

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) dt^2 + \frac{dr^2}{1 - \frac{2MG}{c^2 r}} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{73}$$

  The singularity at $r = 0$ was to be expected for a point-like mass, but the one at $r = r_S$ was unexpected. In any case, Schwarzschild's metric had no problem to model, as intended, the *exterior field* of ordinary objects because the radius of ordinary objects is much greater than their Schwarzschild radius (see equation (55)) and then *the exterior vacuum space* of the modeled astrophysical object –which is the vacuum region represented by (73)– extends in a region where $r \gg r_s$ and (73) is free of singularities. This was a pragmatical, temporary way out of the puzzle of why there is a singularity at $r = r_S$ in (73).

- 1924: Eddington found the coordinate transformation (64) that solves the coordinate singularity in Schawrzschild's solution.
- 1939: Chandrasekhar predicted the collapse of electron-degenerate non rotating bodies if their mass is above 1.44 solar masses.
- 1958: Finkelstein correctly interpreted the $r = r_S$ hypersurface as an event horizon: "The Schwarzschild surface $r = r_S$ is not a singularity but acts as a perfect unidirectional membrane: causal influences can cross it but only in one direction."
- 1963: Roy Kerr find the metric for a rotating BH.
- 1967: John Wheeler coin the expression "black hole" at a lecture.

Problem 4: Explore the many phenomenological differences between John Michell's "black stars" and the GR black holes. In Michell's model: is there a DOC and a BH region? Is there a horizon? Are there trapped spheres?

8. *Schwarzschild timelike (and null) geodesics*

To find the geodesics of Schwarzschild we have to solve the Euler-Lagrange equations for the Lagrangian (43)

$$\mathcal{L}(v, r, \theta, \phi, \dot{v}, \dot{r}, \dot{\theta}, \dot{\phi}) = -f(r)\dot{v}^2 + 2\dot{v}\dot{r} + r^2(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) \tag{74}$$

We look (without lack of generality) solutions of the Euler-Lagrange equation for which $\theta(\tau) = \pi/2$. Since v and ϕ are cyclic coordinates we have the conserved quantities (setting $\theta = \pi/2$, signs and normalizations for the constants of motion ε and ℓ are conventional):

$$\begin{aligned} -2\varepsilon &= \frac{\partial \mathcal{L}}{\partial \dot{v}} = -2\dot{v}f + 2\dot{r} \quad \Rightarrow \quad \dot{v} = \frac{\varepsilon + \dot{r}}{f} \\ 2\ell &= \frac{\partial \mathcal{L}}{\partial \dot{v}} = 2r^2\dot{\phi}^2 \quad \Rightarrow \quad \dot{\phi} = \frac{\ell}{r^2} \end{aligned} \tag{75}$$

Since the Lagrangian does not depend explicitely on τ, $\dot{x}^a \frac{\partial \mathcal{L}}{\partial \dot{x}^a} - \mathcal{L} = \mathcal{L}$ is itself conserved (and of course, equal to $-c^2$ for timelike geodesics and to zero for null geodesics). Putting everything together we arrive at

$$\mathcal{L} = -f\left(\frac{\varepsilon + 2\dot{r}}{2f}\right)^2 + 2\dot{r}\left(\frac{\varepsilon + 2\dot{r}}{2f}\right) + \frac{\ell^2}{r^2} = -\kappa \equiv \begin{cases} -c^2 & \text{, if timelike} \\ 0 & \text{, if null} \end{cases} \tag{76}$$

which can be simplified to

$$\tfrac{1}{2}\dot{r}^2 + \frac{f(r)}{2}\left(\frac{\ell^2}{r^2} + kappa\right) = \frac{\varepsilon^2}{2} \tag{77}$$

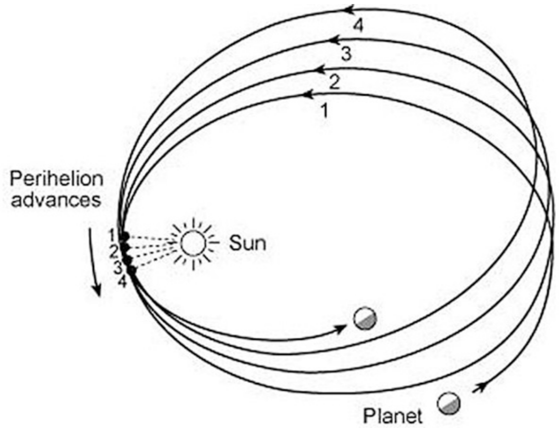

FIG. 9. Orbits under the effective potential (78) have a different period that that required for $\phi \to \phi + 2\pi$ and so they are not closed. When the effect is small we can still regard a planetary orbit as if the planet were mounted on a Keplerian ellipse which is slowly rotated (this is called "perihelion precession", and it is about 43"/century for Mercury).

which is formally equivalent to setting the kinetic ($\dot{r}^2/2$) plus potential ($V_{eff}(r)$) energy of a unit mass in one dimensional motion equal to $(\varepsilon^2 - kappa)/2$:

$$\tfrac{1}{2}\dot{r}^2 + \underbrace{\left(\frac{\ell^2}{2r^2} - \frac{MG}{r} - \frac{MG\ell^2}{c^2r^3}\right)}_{V_{eff}(r)} = \frac{\varepsilon^2 - \kappa}{2} \tag{78}$$

This is exactly what happens when we reduce Kepler's central object problem using cyclic coordinates, except that the effective potential has a nontrivial term $\sim r^{-3}$. In (78) we recognize: i) the centrifugal term (in blue), ii) Newton's term (black) and a General Relativity effect (red) which is a tiny correction to the centrifugal term if $r \gg r_S$:

$$\frac{\ell^2}{2r^2} \to \frac{\ell^2}{2r^2}\left(1 + \frac{r_S}{r}\right) \tag{79}$$

It is indeed the matching of (78) with the Newtonian limit what indicates that the, *a priori* a constant of integration r_S equals $2GM/c^2$. The relativistic term spoils a very special property of the Newtonian potential: only Newtonian ($\sim 1/r$) and harmonic ($\sim r^2$) central potentials allow closed trajectories. For any other central potentials admitting bounded solutions ($r_{min} < r < r_{max}$), in general (we assume motion happens in the $\theta = \pi/2$ plane, as usual) the period for $r : r_{min} \to r_{max} \to r_{min}$ divided by the period for $\phi : 0 \to 2\pi$ is not rational, and thus closed orbits are not allowed. In the solar system, the Schwarzschild radius of the Sun is so small when compared to the Sun-planet distances that the relativistic term can be treated as a perturbation. In this way, elliptical orbits can be regarded as a "slowly rotating ellipse" (not indeed a closed orbit). The tiny rotation of the perihelion of Mercury (43"/century) is accounted for by this term. The behavior of the effective potential depends crucially on the value of ℓ: i) if $\ell^2 > 12(GM/c)^2$ there are two critical points $r_1 < r_2$, which are respectively a local maximum and a local minimum. These correspond to circular orbits (unstable the first, stable the second); ii) if $\ell^2 \leq 12(GM/c)^2$ the potential is monotonic, increasing with r. This is illustrated in Figure 10.

Problem 5: Consider a bounded motion with energy just above the local minimum of the effective potential in (78). Treat it as a perturbation of the stable circular orbit. Calculate the precession of the perihelium. (Note: this is a difficult problem, the proposal is that you follow how the problem is worked out in GR textbooks such as Wald's or Carroll's)

B. The Black Hole experience

You may have probably heard that if someone falls into a BH, people outside never actually get to *see* him falling. If he is sending messages at fixed proper time intervals, the messages arrive at increasingly longer periods of (proper) time of the receivers which stay far away from the BH horizon. In this section we prove all these assertions.

Let's pose the problem: Kamikaze and his Friend meet at $\theta = 0, \phi = 0, r = r_i \gg r_S$ at some v. Kamikaze tells Friend he will let free fall radially to the BH, and assures him that he will stay in touch by sending a WhatsApp message every

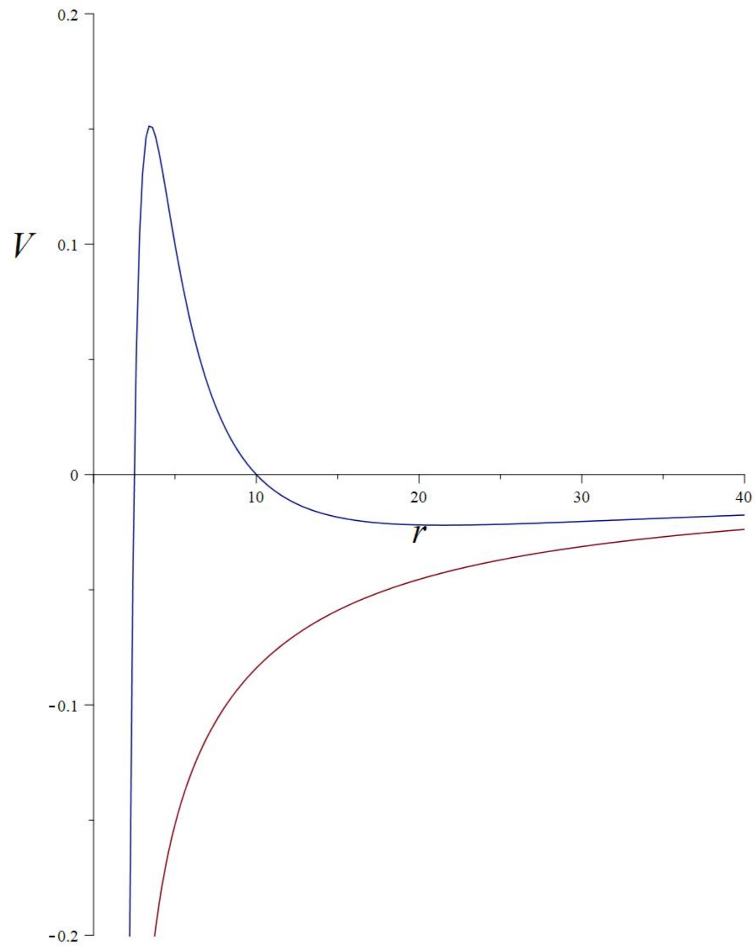

FIG. 10. The effective potential for the radial motion, equation (78) has a qualitatively different behavior for small and large values of the particle angular momentum. The examples above correspond to $r_S = 2$ units in the graph and to two different values of ℓ one with $\ell^2 = 4(GM/c)^2$ (in red), the other with $\ell^2 = 25(GM/c)^2$ (in blue). The local maximum and minimum of the second one are located at $r \simeq 3.4$ and $r \simeq 26.8$ respectively.

$\Delta\tau_K$ of its proper time τ_K. They synchronize their clocks so that both indicate $\tau = 0$ when their depart (the departure event has coordinates $(v_i, r_i \gg r_S, \theta = \pi/2, \phi = 0)$. Friend traces the $(v, r = r_i, \theta = 0, \phi = 0), v > v_i$ worldline, which is nearly free fall if $r_i \gg r_S$ (check this assertion). Kamikaze does send messages every fixed $\Delta\tau_K$ intervals. However, Friends notices that Kamikaze's messages get to him over increasingly longer intervals $\Delta\tau_F$.

We would like to find the relation between the $\Delta\tau_F$ and $\Delta\tau_K$ as Kamikaze approaches the BH horizon. We proceed as follows:

a) Calculate Kamikaze's worldline $(v_K(\tau_K), r_K(\tau_K))$ (recall that $\theta_K = \pi/2$ and $\phi_K = 0$).

b) Calculate the *trajectories* $v(r)$ of the RONGs emitted by Kamikaze (its message) starting at $(v_K(\tau_K), r_K(\tau_K))$ (these are the WhatsApp signals).

c) Determine the intersections of the RONG with Friends worldline, that is, the detection events. For the v coordinate of the detection of the signal sent at $(v_K(\tau_K), r_K(\tau_K)$ I will use the notation $V_F(\tau_K)$, so that the detection event has coordinates $(v \equiv V_F(\tau_K), r = r_i, \theta = \pi/2, \phi = 0)$.

Since $r_i \gg r_s$, at $r = r_i$ is $1 - r_S/r \simeq 1$ and Friends stays at $r = r_i$ (that is, $dr = d\theta = d\phi = 0$ along Friend's worldline), Friend's proper time satisfies $d\tau_F^2 = -ds^2/c^2 \simeq -dv^2/c^2$ In our notation, the quotient of the proper time intervals at which Friends receives the messages, divided by the rate of emission proper time intervals is (assuming $\Delta\tau_K$ small)

$$\frac{\Delta\tau_F}{\Delta\tau_K} \simeq \frac{1}{c}\frac{dV_F(\tau_K)}{d\tau_K} \tag{80}$$

The solution then goes as follows (you are supposed to fill in the blanks left in the following calculations, I got help from MAPLE)

a) Calculation of Kamikaze's worldline $(v_K(\tau_K), r_K(\tau_K))$:
Since Kamikaze "lets himself fall radially", his worldline is a timelike geodesic starting at $v = v_i, r = r_i \gg r_S$ and constant $\theta = \pi/2$ and $\phi = 0$, In particular, $\dot{\phi} = 0 = \dot{\theta} = 0$, then according to (78) with $\ell = 0$ and $\kappa = c^2$ (which correponds to *timelike* geodesics)

$$\frac{1}{2}\dot{r}_K^2 - \frac{2MG}{r_K} = \frac{\varepsilon^2 - c^2}{2c^2} = 0 \tag{81}$$

Because of the initial conditions $r_K(0) = r_i \gg r_S$, $\dot{r}_K(0) = 0$, the constant on the right is zero, then $\varepsilon = c$. Also, along the geodesic $\dot{r} \leq 0$, so that

$$\dot{r}_K^2 = \frac{2GM}{r_K}, \quad \text{and} \quad \dot{r}_K = -c\sqrt{\frac{r_S}{r_K}} \tag{82}$$

which can be integrated and gives

$$r_K(\tau_K) = \left(r_i^{3/2} - \tfrac{3}{2}c\sqrt{r_S}\,\tau_K\right)^{2/3}. \tag{83}$$

Kamikaze will cross the horizon at τ_K^0 and get into the singularity at τ_K^1 where

$$\tau_K^0 = \frac{2}{3}\frac{r_i^{3/2} - r_S^{3/2}}{\sqrt{r_S}\,c}, \quad \tau_K^1 = \frac{2}{3}\frac{r_i^{3/2}}{\sqrt{r_S}\,c} \tag{84}$$

Note that $\tau_K^1 - \tau_K^0 = \frac{2}{3}\frac{R}{c}$, in consistency with (69).

We now use $\varepsilon = c$ and $\dot{r}_K = -c\sqrt{\frac{r_S}{r_K}}$ in the first equation (75). This gives

$$\frac{\dot{v}_K}{c} = \frac{1 - \sqrt{\frac{r_S}{r_K}}}{1 - \frac{r_S}{r_K}} = \frac{1}{1 + \sqrt{\frac{r_S}{r_K}}}. \tag{85}$$

If we replace above $r_K(\tau)$ from (83) we get and explicit form for $\dot{v}_K$ that we can integrate in terms of elementary functions to get an explicit form for $v_K(\tau_K)$. The result is complicated (and actually not needed), I merely include it for completeness:

$$v_K(\tau_K) = \frac{4}{3}r_S\left(\frac{r_K(\tau_K)}{r_S}\right)^{3/4} - \frac{4}{3}r_S\ln\left(\left(\frac{r_K(\tau_K)}{r_S}\right)^{3/4} + \sqrt{2}\right) + c\,\tau_K + \text{const} \tag{86}$$

The additive can be obtained by setting $v_k(\tau_K = 0) = v_i$.

b) We are interested in finding the function $v_F(\tau_K)$ for the RONG with initial condition $(v_K(\tau_K), r_K(\tau_K))$ (that is, sent by Kamikaze) and final condition corresponding to Friend's worldline: $(v = V_F(\tau_K), r = r_i)$ (note that this equation *defines* the function $V_F(\tau_K)$).

It is interestint to rederive the RONGs using equations (75)-(78). From (78, $\dot{r} = |\varepsilon|$. Using this in the first equation (75) gives

$$\dot{v} = \frac{\varepsilon + |\varepsilon|}{f} = \begin{cases} \dot{v} = 0, \dot{r} = -\varepsilon & , \text{ if } \varepsilon < 0 \\ \dot{v} = \frac{2\varepsilon}{f},\ \dot{r} = \varepsilon & , \text{ if } \varepsilon > 0 \end{cases} \tag{87}$$

Clearly, the case $\varepsilon < 0$ corresponds to the RINGs, whereas $\varepsilon > 0$ corresponds to the RINGs. Moreover, $\dot{r} = \varepsilon$ indicates that r is an affine parameter so we can take $\varepsilon = \pm 1$. For RONGs, then, $\frac{dv}{dr} = \frac{dotv}{\dot{r}} = \frac{2}{f}$ and we are led back to (60)

and (61): $v = 2r^*(r) + const$, where we introduced $r^*(r) = r + r_S \ln\left(\frac{r}{r_S} - 1\right)$, known in the literature as the *tortoise radial coordinate* (we will use that it satisfies $dr^*/dr = 1/f(r)$).
The RONG starting at $(v_K(\tau_K), r_K(\tau_K))$, parametrized with r, is (the subscript S stands for *signal*)

$$v_S(r) = 2r^*(r) - 2r^*(r_K(\tau_K)) + v_K(\tau_K), \quad r_K < r \tag{88}$$

Friend's worldline is the curve $(v, r_i, \theta = \pi/2, \phi = 0), v_i < v < v_\dagger$, where $(v_\dagger, r_i, \pi/2, 0)$ is the event "Friend's death".)
The signal meets Friend's worldline at at $v = v_F(\tau_K) = v_S(r_i)$, that is

$$v_F(\tau_K) = 2r^*(r_i) - 2r^*(r_K(\tau_K)) + v_K(\tau_K) \tag{89}$$

From (86) follows that $v_K(\tau_K)$ is bounded in entire relevant interval $0 \leq \tau \leq \tau_K^1$, so the first and third terms in (89) are bounded. However, the second term, $-2r^*(r_K(\tau_K)) \simeq -r_S \ln\left(\frac{r_K}{r_S} - 1\right)$ tends to +infinity as $r_K \to {r_S}^+$ (equivalently, as $\tau_K \to {\tau_K^0}^+$. Taking the derivative of $v_F(\tau_K)$ with respect to τ_K gives, for $\tau_K \to \tau_K^0$ (that is, $r_K \to r_S$), and using (82) and $dr^*/dr = 1/f$,

$$\frac{\Delta\tau_F}{\Delta\tau_K} \simeq \frac{1}{c}\frac{dV_F(\tau_K)}{d\tau_K} \simeq -\frac{2}{f(r_K(\tau_K))}\frac{\dot{r}_K}{c} = \frac{2r_K(\tau_K)}{r_K(\tau_K) - r_S}\sqrt{\frac{r_S}{r_K(\tau_K)}} \simeq \frac{2r_S}{r_K(\tau_K) - r_S} \tag{90}$$

which effectively diverges as $r_K \to {r_S}^+$.
A first order Taylor expansion of $r_K(\tau)$ around τ_K^0 gives $r_K(\tau) = r_S - c(\tau_K - \tau_K^0) + ...$ therefore

$$\frac{d\tau_F}{d\tau_K} = \frac{1}{c}\frac{dV_F(\tau_K)}{d\tau_K} \simeq \frac{2r_S}{c(\tau_K^0 - \tau_K)}, \tag{91}$$

that is, the proper time τ_F at which Friends receives the signal sent by Kamikaze at his proper time τ_K behaves, as $\tau_k \to {\tau_K^0}^+$, as $\tau_F \simeq -\frac{2r_S}{c}\ln(1 - \tau_K/\tau_K^0)$, and tends to infinity as Kamikaze approaches the BH horizon.

In Figure 11 Kamikaze and Friends worldlines appear in black, the horizon sets the units ($r_S = 2$), the blue lines are the signals sent out by Kamikaze *at equal* $\Delta\tau_K$ (note a tiny last signal sent when already inside the BH, falling into the $r = 0$ singularity). The spread of $\Delta\tau_F$ of Friend's worldline is evident.

Problem 6: Complete all the steps not given explicitly in the calculations in this section.

Problem 7: The quotient of the frequencies of the light ray as measured by Friend over the emission frequency is

$$\frac{\omega_F}{\omega_K} = \frac{g_{ab}u_K^a k^b\mid_E}{g_{cd}u_F^a k^b\mid_D} \tag{92}$$

where $u^a = dx_K^a/d\tau_K$ is the four-velocity ok Kamikaze and $|_E$ and $|_D$ indicates evaluation at the emission and detection events.

Prove that

$$\frac{\omega_F}{\omega_K} = 1 - \sqrt{\frac{r_S}{r_K}} \tag{93}$$

and so that the light get infinitely redshifted as $r_K \to r_S$.

C. Collapse and BH formation

A few months after publishing his vacuum solution, Schwarzschild presented a solution for a static perfect fluid interior $r < r_{star}$ of constant density that matched the vacuum exterior (65) for $r > r_{star}$. He found that the solution was possible only if $r_{star} > \frac{9}{8}r_S$. As this limit was approached, the pressure at the star center diverges! It was then established that under more general equations of state the same happens: no static interior solution matches Schwarzschild'a exterior if the star mass is compressed beyond 9/8 of the Schwarzschild radius (71). This, of course, does not rule out the possibility of matching with a *non static* spherically symmetric interior: a spherically collapsing star.

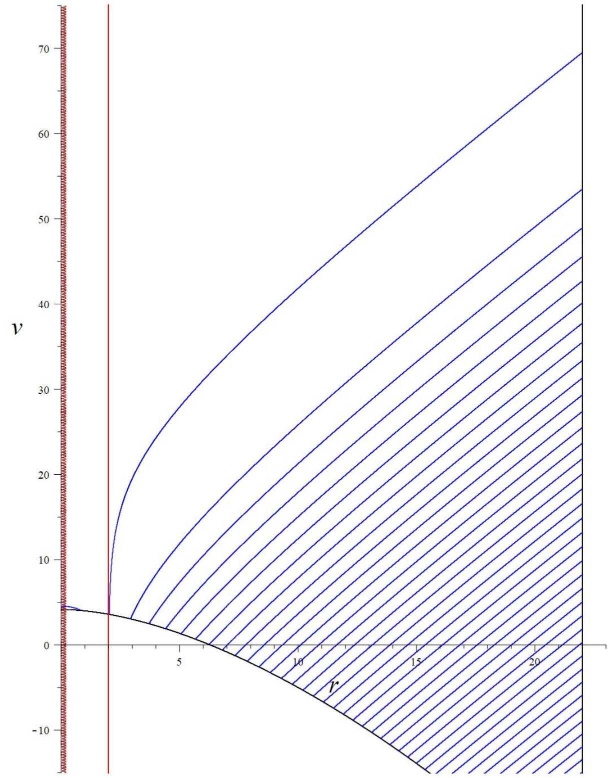

FIG. 11. Kamikaze and Friend worldlines (black) and the signals sent from the first to the second (blue) at fixed sender time intervals. Since $\Delta v \simeq c\Delta\tau$ at Friend's worldline, the spread of the intervals between consecutive signal arrivals is evident. The chosen units are such that $r_S = 2$.

Since the Schwarzschild metric is the *unique vacuum spherical symmetric solution of Einstein's equations* (a theorem due to Birkhoff), the *outside metric* of the star must be (53). A generic point at the *boundary* of the imploding star would follow –since we expect a zero pressure gradient at such a point– a radial timelike geodesic like, for example, that of Kamikaze in Figure 11. To be more precise, let $(r_{star}(\tau), v_{star}(\tau), \theta, \phi)$ (θ and ϕ constant) be the worldline of a star surface point, then the metric is (53) for $v > v_{star}(r_s tar)$, whereas in the region $v \leq v_{star}(r_s tar)$ it will be a solution of (40) for whatever matter field T_{ab} contains the star, matching (continuously and with a continuous normal derivative) (53) at the hypersurface $v = v_s tar(r_s tar)$. This is depicted in Figure 12. The "gray fog" in the figure is the region of spacetime whose metric we don't give details (as we have not proposed any star model yet). Once given, the metric could be solved for and the RONGs continued within the star. In any case, they are continuous. In particular, the horizon enters the star. The center of symmetry $r = 0$ in the matter filled region, however, is not a singularity but a regular worldline corresponding to a star particle. It is only outside the star that RONGs curve towards $r = 0$, which is located "at the future", and is also a curvature singularity.

A simple model like this was worked out in detail by Oppenheimer and Snyder in 1939. They matched the vacuum exterior region with an interior that is a piece of a cosmology. If you are interested in this model, you can work out the following problem:

Problem 8: The Oppenheimer and Snyder (OS) metric matches Schwarzschild vacuum exterior with an imploding spherically symmetric cosmological solution. This is best worked out in the so called Painlevè-Gustard coordinates (for details on the calculations below see Matthias Blau Lecture Notes on GR at `http://www.blau.itp.unibe.ch/GRLecturenotes.html`).

a) Painlevè-Gullstrand (PG) coordinates for Schwarzschild: starting from (65), introduce

$$T = ct + 2\sqrt{r_S r} + r_S \ln\left|\frac{\sqrt{r/r_S} - 1}{\sqrt{r/r_S} + 1}\right| \tag{94}$$

Note that

$$c\,dt = dT - \frac{\sqrt{r/r_S}}{r/r_S - 1} dr \tag{95}$$

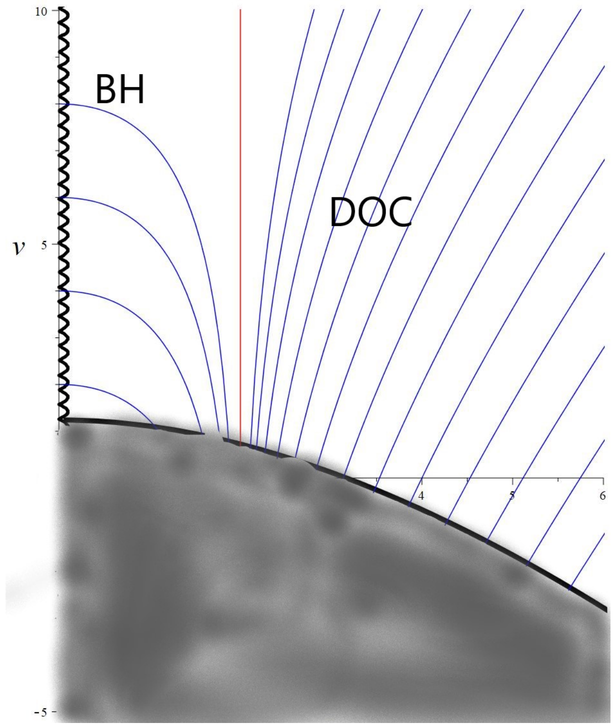

FIG. 12. The spacetime (v, r)-diagram (that is, suppressing (θ, ϕ)) of a spherical collapse and BH formation. The surface of the star is represented by a timelike geodesic (thick black, compare with Kamikaze's worldline in Fig. 11). The metric in the *interior* of the star is not vacuum (therefore not Schwarzschild's) but some spherical solution of the Einstein's equation (40) with non trivial T_{ab} modeling the star content. This corresponds to the gray area. Note that $r = 0$ is *not* a singularity in the star interior but the center of the spherical symmetry. The BH and DOC regions *outside the star are indicated.* To see how they extend within the star we would need to know the metric there in order to be able to integrate the null geodesics towards the interior of the star surface.

Prove that (65) gives

$$ds^2 = -dT^2 + \left(dr + \sqrt{r_S/r}\, dT\right)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{96}$$

Note that, as happens with Eddington coordinates, PG coordinates lift the coordinate singularity of (65) at $r = r_S$.

b) The interior metric

$$ds^2 = -dT^2 + (dr - rH(T)\, dT)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{97}$$

corresponds to a portion of a flat cosmological model ($H(T) = \dot{a}/a$ being the Hubble function!):

$$ds^2 = -dT^2 + a(T)^2(d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{98}$$

Verify that (97) follows from the well know cosmological metric (98) if we define

$$r \equiv a(T)\tilde{r} \tag{99}$$

We assume that $a(T) = kT^{2/3}$, which corresponds to dust (that is, pressureless) perfect fluid. This implies that

$$H = \frac{\dot{a}}{a} = \frac{2}{3T} \tag{100}$$

c) Prove that

$$r(T) = \left(\tfrac{9}{4} r_S\right)^{1/3} (-T)^{2/3}, T < 0, \tag{101}$$

parametrizes a timelike , radial, ingoing geodesic (θ and ϕ are held constant) of *both* metrics: (96) and (97)

d) The surface Σ of the star swaps in spacetime a 3-dimensional hypersurface Σ that is the union of the worldlines (101) of all its particles, that is, for every (θ, ϕ)

$$\Sigma = \{(T, r, \theta, \phi) \mid r = R(T) \text{ in } (101)\} \tag{102}$$

Prove that the induced metric on this hypersurface (obtained by replacing $r \to r(T)$ and $dr \to \frac{dr(T)}{dT} dT$) agrees on both sides (a necessary continuity requirement).

Note that the spacetime metric can be written in a more compact way as

$$ds^2 = -dT^2 + \left(dr + \sqrt{\frac{\mathcal{R}(T,r)}{r}}\, dT \right)^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{103}$$

where

$$\mathcal{R}(T,r) = \begin{cases} r_S & , r > R(T) \\ \frac{2r^3}{9T^2} & , r < R(T) \end{cases} \tag{104}$$

An alternative, more academic (meaning, less realistic) model, is that of Vaidya: a beam of massless particles coming from infinity accumulates in Minkowski space, concentrating and forming a Schwarzschild BH. The proposed metric to describe the full process is very simple:

$$ds^2 = -f(v,r)\, dv^2 + 2dv\, dr + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \quad f(v,r) = 1 - \frac{R(v)}{r}. \tag{105}$$

If we insert this metric into Einstein's equations (40), we find that the matter field coming out (represented by $T_{ab} = c^4/(8\pi G) G_{ab}$) is physically meaningful only if $dR/dv \geq 0$, so we will further assume this condition. If we do so, T_{ab} corresponds to an influx of massless particles falling radially *while* $dR/dv \neq 0$. An interesting situation is then when $m(v) = 0$ for $v \leq v_i$, grows monotonically in the interval (v_i, v_f), and stabilizes at $R(v) = r_S$ for all $v > v_f$, r_S a constant). The metric will then be the $v < v_i$ piece of Minkowski metric (48), interpolated by (105) and matched with the Schwarzschild metric (53) for $v > v_f$. One can show that trapped spheres will all lie within what is called the *apparent horizon* at $r = R(v)$. However, the event horizon $r_E(v)$ lies outside the apparent horizon (that is $r_E(v) \geq R(v)$ for all v). So there is aregion free of trapped surfaces which, however, is inside the BH.

Problem 9: Assume that, in (105), $R(v)$ satisfies $R'(v) > 0$ for all v and $\lim_{v\to\infty} R(v) = r_S$ (a constant). Prove that the horizon is the hypersurface $r = \mathfrak{r}(v)$ defined by the differential equation:

$$\frac{d\mathfrak{r}}{dv} = \frac{1}{2} - \frac{R(v)}{2\mathfrak{r}(v)} \tag{106}$$

subject to the boundary condition $\mathfrak{r}(v) \to r_S$ as $v \to \infty$.

An example is given in Figure 13, where $R(v) = 0$ is interpolated with $R(v) = r_S$ with a simple sine function, and equation (106) (and those of two other RONGs) are solved numerically. Note that the apparent horizon $r = R(v)$ lies within the event horizon.

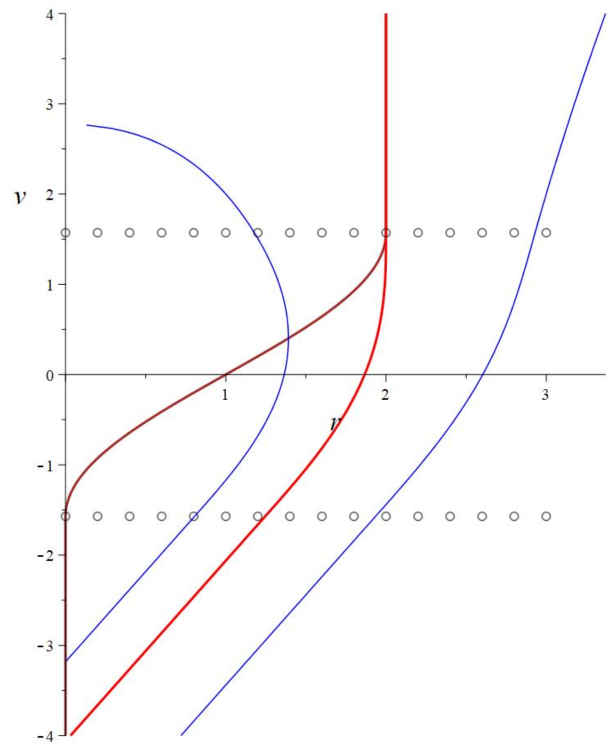

FIG. 13. Vaydia spacetime diagram. Below the lower pointed line the metric is that of Minkowski spacetime and above the upper pointed line is that of Schwarzschild. The Vaydia transition, where $R(v)$ increases from zero to two units occurs the Vaydia transition: an inflow of massless particles carry energy that builds up into a Schwarzschild BH. The *apparent horizon* (in brown) is the boundary of the trapped sphere region: all trapped spheres lie to its left. The BH region is to the left of the the event horizon (drawn in red) and enters the flat Minkowskian region: part of the BH lies in a flat portion of the spacetime! This is so because some initially outgoing radial geodesics (light rays) get bent in the future in such a way that they enter the $r < r_S$ piece of the Schwarschild piece of the metric (example: the left blue curve). The right blue geodesic, instead, gets into the Schwarzschilean portion at an event with $r > r_S$, so it can safely go to $\mathscr{I}^+$. In this example $R(v) = M(1 + \sin(v/M))$ for $-M\pi/2 \leq v \leq M\pi/2$, 0 for $v < M\pi/2$ and $2M$ for $v > M\pi/2$, and $M = 1$.