Anomaly-Induced Effective Action of Gravity and Stability of Classical Solutions or New Perspectives for Quantum Gravity

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Summary

Why should we quantize gravity.

Semiclassical approach and higher derivatives.

Who's Afraid Of Ghosts?

Do they pose a danger?

S. W. Hawking, "**Who's Afraid Of (higher Derivative) Ghosts?"**, IN *BATALIN, I.A. (ED.) ET AL.: QUANTUM FIELD THEORY AND QUANTUM STATISTICS, VOL. 2*, 129-139

Gravitational waves.

Stability of classical solutions.

Conclusion.

Results and future perspectives of this work.

General Relativity

(GR) is a complete theory of classical gravitational phenomena. It proved valid in the wide range of energies and distances.

There are covariant equations for the matter (fields and particles, fluids etc) and Einstein equations for the gravitational field $g_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \, T_{\mu\nu} - \Lambda \, g_{\mu\nu} \, .$$

The most important solutions of GR have specific symmetries.

1) Spherically-symmetric solution. Stars ... Black holes.
2) Isotropic and homogeneous metric. **Universe.**

Both cases are characterized by singularities, when curvature and density of matter become infinite.

GR is not valid at all scales.

Dimensional arguments.

The expected scale of the quantum gravity effects is associated to the Planck units of length, time and mass.

The idea of Planck units is based on the existence of the 3 fundamental constants:

$$c = 3 \cdot 10^{10}\, cm/s\,,$$

$$\hbar = 1.054 \cdot 10^{-27}\, erg \cdot sec\,;$$

$$G = 6.67 \cdot 10^{-8}\, cm^3/sec^2\, g\,.$$

One can use them uniquely to construct the dimensions of

length $\quad l_P = G^{1/2}\, \hbar^{1/2}\, c^{-3/2} \,\approx\, 1.4 \cdot 10^{-33}\, cm;$

time $\quad t_P = G^{1/2}\, \hbar^{1/2}\, c^{-5/2} \,\approx\, 0.7 \cdot 10^{-43}\, sec;$

mass $\quad M_P = G^{-1/2}\, \hbar^{1/2}\, c^{1/2} \,\approx\, 0.2 \cdot 10^{-5} g \,\approx\, 10^{19}\, GeV.$

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Three choice for Quantum Gravity (QG)

One may suppose that the fundamental units indicate to the presence of fundamental physics at the Planck scale.

We can classify the possible approaches to three distinct groups, namely we can:

• **Quantize both gravity and matter fields. This is, definitely, the most fundamental possible approach**

•• **Quantize only matter fields on classical curved background (semiclassical approach).**
QFT and Curved space-time are well-established notions, which passed many experimental/observational tests.

••• **Quantize something else. E.g., in case of (super)string theory both matter and gravity are induced.**

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●● The renormalizable QFT in curved space requires introducing a generalized action of gravity (external field).
In this situation the form of the "vacuum action" is as follows:

$$S_{vac} = S_{EH} + S_{HD}$$

where $$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\{R + 2\Lambda\}\,.$$

is the Einstein-Hilbert action with the cosmological constant.

S_{HD} includes higher derivative terms. The most useful form is

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\}\,,$$

where $$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3\,R^2$$

is the square of the Weyl tensor in $d = 4$,

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2$$

is integrand of the Gauss-Bonnet term (topological term in d=4).
Introduction: *I.L. Buchbinder, S.D. Odintsov, I.Sh., Effective Action in Quantum Gravity. (1992 - IOPP).*

Observation about higher derivatives.

A consistent theory of **quantum matter fields on classical curved background** can be achieved only if we include

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\}\,,$$

along with Einstein-Hilbert and cosmological constant terms.

In quantum gravity such a higher derivative (HD) term means massive ghost, the gravitational spin-two particle with negative kinetic energy. This leads to the problem with unitarity, at least at the tree level.

In the semiclassical theory gravity is external and unitarity of the gravitational S**-matrix does not matter.**

- The consistency means existence of physically reasonable solutions and their stability under small perturbations.

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The semiclassical approach to gravity is usually associated with the equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = < T_{\mu\nu} > \qquad (1)$$

and implies that the gravity itself is not quantized.

The renormalizable theory of matter fields on curved space-time background requires that the action of gravity should be extended compared to the one of General Relativity (GR).

- R. Utiyama and B. S. DeWitt, **Renormalization of a classical gravitational field interacting with quantized matter fields**, J. Math. Phys. **3**, 608 (1962).
- N.D. Birell and P.C.W. Davies, **Quantum Fields in Curved Space**, (Cambridge University Press, Cambridge, 1982).
- I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, **Effective Action in Quantum Gravity** (IOP Publishing, Bristol, 1992).
- I. L. Shapiro, **Effective Action of Vacuum: Semiclassical Approach**, Class. Quant. Grav. **25**, 103001 (2008) [arXiv:0801.0216 [gr-qc]].

The full action includes Einstein-Hilbert term, which is the origin of the *r.h.s.* of (1) with the cosmological constant term

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(\,R + 2\Lambda\,) \qquad (2)$$

and also the higher derivative terms

$$S_{HD} \;=\; \int d^4x\sqrt{-g}\,\left\{a_1 C^2 + a_2 E + a_3\Box R + a_4 R^2\right\}, \qquad (3)$$

where $C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + (1/3)\,R^2$ is the square of the Weyl tensor and $E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$ is the integrand of the Gauss-Bonnet topological term. All terms of the action of vacuum

$$S_{vac} \;=\; S_{EH} + S_{HD}\,, \qquad (4)$$

In fact, the use of the terms (3) may lead to serious problems, because these terms are known to produce unphysical ghost terms for the linearized gravitational field on the flat background.

- K. S. Stelle, **Classical Gravity With Higher Derivatives**, Gen. Rel. Grav. **9**, 353 (1978).

One big problem is related to quantum corrections to (4).
The one-loop effective action of massless conformal fields is essentially controlled by conformal anomaly.

- M.J. Duff, **Observations On Conformal Anomalies**, Nucl. Phys. **B125** (1977) 334;
- S. Deser, M.J. Duff and C. Isham, **Nonlocal Conformal Anomalies**, Nucl. Phys. **B111** (1976) 45.

The anomaly-induced effective action of vacuum includes an arbitrary conformal functional.
This functional is a trivial constant for the homogeneous and isotropic metric, where the anomaly-induced quantum corrections produce Starobinsky inflation.

- A.A. Starobinski, Phys.Lett. **91B** (1980) 99; **Nonsingular Model of the Universe with the Quantum-Gravitational De Sitter Stage and its Observational Consequences**, Proceedings of the second seminar "Quantum Gravity", pp. 58-72 (Moscow, 1982); JETP Lett. **30** (1979) 719; **34** (1981) 460; Let.Astr.Journ. (in Russian), **9** (1983) 579.

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In the case of black holes, taking into account the conformal anomaly enables one to calculate Hawking radiation.

- S. M. Christensen and S. A. Fulling, **Trace Anomalies and the Hawking Effect**, Phys. Rev. D **15**, 2088 (1977).

Using the anomaly-induced action one can even classify the vacuum states in the vicinity of black hole.

- R. Balbinot, A. Fabbri and I. L. Shapiro, **Anomaly induced effective actions and Hawking radiation**, Phys. Rev. Lett. **83**, 1494 (1999) [hep-th/9904074].

The equations for gravitational waves calculated by using direct methods and anomaly-induced action produce equivalent results.

- A. A. Starobinsky, **Relict Gravitation Radiation Spectrum and Initial State of the Universe. (In Russian)**, JETP Lett. **30**, 682 (1979) [Pisma Zh. Eksp. Teor. Fiz. **30**, 719 (1979)];
- J. C. Fabris, A. M. Pelinson and I. L. Shapiro, **On the gravitational waves on the background of anomaly-induced inflation**, Nucl. Phys. B **597**, 539 (2001) [Erratum-ibid. B **602**, 644 (2001)] [arXiv:hep-th/0009197].

From the Quantum Field Theory viewpoint the anomaly-induced effective action of vacuum represents a well-defined quantum contribution which can be used to verify the compatibility of gravity and quantum effects.

The purpose of the present work is to verify whether these quantum terms are compatible with the well-known classical cosmological solutions for the different epochs of the history of the Universe.

The covariant form of anomaly-induced effective action of gravity is, in our point of view, the most complete available form of the quantum corrections to the gravitational action in four space-time dimensions.

- R. J. Riegert, **A Nonlocal Action for the Trace Anomaly**, Phys. Lett. B **134**, 56 (1984).
- E. S. Fradkin and A. A. Tseytlin, **Conformal Anomaly in Weyl Theory and Anomaly Free Superconformal Theories**, Phys. Lett. B **134**, 187 (1984).

The application to cosmology has been considered in the following works and led to the well-known Starobinsky model of inflation.

- S. G. Mamaev and V. M. Mostepanenko, **Isotropic Cosmological Models Determined By Vacuum Quantum Effects**, Sov. Phys. JETP **51** (1980) 9 [Zh. Eksp. Teor. Fiz. **78** (1980) 20].
- A. Vilenkin, **Classical and quantum cosmology of the Starobinsky inflationary model**, Phys. Rev. **D32** (1985) 2511.

The anomalous trace of the energy momentum tensor is given by the expression

$$< T^{\mu}_{\mu} > = -\left(wC^2 + bE + c\Box R\right), \tag{5}$$

- N.D. Birell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

Where the coefficients w, b and c depend on the number of active quantum fields of different spins,

$$w = \frac{1}{(4\pi)^2}\left(\frac{N_0}{120} + \frac{N_{1/2}}{20} + \frac{N_1}{10}\right), \tag{6}$$

$$b = -\frac{1}{(4\pi)^2}\left(\frac{N_0}{360} + \frac{11\,N_{1/2}}{360} + \frac{31\,N_1}{180}\right), \tag{7}$$

$$c = \frac{1}{(4\pi)^2}\left(\frac{N_0}{180} + \frac{N_{1/2}}{30} - \frac{N_1}{10}\right). \tag{8}$$

The anomaly-induced effective action $\bar{\Gamma}_{ind}$ represents an addition to the classical action of gravity, and can be found by solving the equation

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = < T^{\mu}_{\mu} > = (\omega C^2 + bE + c\Box R)\,. \qquad (9)$$

The covariant and generally non-local solution can be easily found in the form

$$\begin{aligned} \bar{\Gamma} &= S_c[g_{\mu\nu}] - \frac{3c+2b}{36} \int d^4x \sqrt{-g(x)}\, R^2(x) \qquad (10) \\ &+ \int d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)}\, (E - \frac{2}{3}\Box R)_x\, G(x,y) \left[\frac{w}{4} C^2 - \frac{b}{8} (E - \frac{2}{3}\Box R) \right]_y , \end{aligned}$$

where $G(x,y)$ is a Green function for the operator

$$\Delta_4 = \Box^2 + 2\, R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R\Box + \frac{1}{3} (\nabla^\mu R) \nabla_\mu \,.$$

Rewrite (10) in the local form by introducing two auxiliary fields ϕ and ψ

- I. L. Shapiro and A. G. Zheksenaev, *Gauge dependence in higher derivative quantum gravity and the conformal anomaly problem*, Phys. Lett. B **324**, 286 (1994).
- P. O. Mazur and E. Mottola, *Weyl cohomology and the effective action for conformal anomalies*, Phys. Rev. D **64**, 104022 (2001).

$$\begin{aligned}\bar{\Gamma}_{ind} &= S_c[g_{\mu\nu}] - \frac{3c+2b}{36}\int d^4x\sqrt{-g(x)}\,R^2(x) + \int d^4x\sqrt{-g(x)}\,\Big\{\frac{1}{2}\varphi\Delta_4\varphi - \\ &- \frac{1}{2}\psi\Delta_4\psi + \varphi\Big[\frac{\sqrt{-b}}{2}\,(E - \frac{2}{3}\,\Box R) - \frac{w}{2\sqrt{-b}}\,C^2\Big] + \frac{w}{2\sqrt{-b}}\,\psi\,C^2\Big\}. \qquad (11)\end{aligned}$$

The expression (11) is classically equivalent to (10), because if one uses the equations for the auxiliary fields φ and ψ, the nonlocal action (10) is restored.

Consider now the background cosmological solution for the theory with the action including quantum corrections,

$$S_{total} = - M_P^2 \int d^4x \sqrt{-g}\, R + \bar{\Gamma}\,, \qquad (12)$$

where $M_P^2 = 1/16\pi G$ is the square of the Planck mass, and the quantum correction $\bar{\Gamma}$ is taken in the form (11).

Looking for the isotropic and homogeneous solution, the starting point is to choose the metric in the form $g_{\mu\nu} = a^2(\eta)\, \bar{g}_{\mu\nu}$, where η is conformal time. It proves useful to introduce the notation $\sigma = \ln a$.

The theory includes the equations for the three fields, namely for φ, ψ, and σ. For the sake of simplicity we will consider conformally flat background and therefore set $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$.

Equations for φ and ψ have especially simple form

$$\sqrt{-g}\,\left[\Delta_4\,\varphi + \frac{\sqrt{-b}}{2}\,(E - \frac{2}{3}\,\Box R)\, - \,\frac{w}{2\sqrt{-b}}\,C^2\,\right] = 0\,,$$
$$\sqrt{-g}\,\left[\Delta_4\,\psi - \frac{w}{2\sqrt{-b}}\,C^2\,\right] = 0\,.$$

By using the transformation laws for the quantities in the last expression, one can obtain

$$\sqrt{-g}\,C^2 = \sqrt{-\bar{g}}\,\bar{C}^2\,, \qquad \sqrt{-g}\,\Delta_4 = \sqrt{-\bar{g}}\,\bar{\Delta}_4\,, \tag{13}$$

$$\sqrt{-g}(E - \frac{2}{3}\Box R) = \sqrt{-\bar{g}}(\bar{E} - \frac{2}{3}\Box\bar{R} + 4\bar{\Delta}_4\sigma)\,. \tag{14}$$

Taking into account our choice for the fiducial metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ all the terms in the *r.h.s.* of the last equation are equal to zero and we arrive at the following equations

$$\Box^2 \varphi + 8\pi\sqrt{-b}\,\Box^2\sigma = 0\,, \qquad \Box^2 \psi = 0\,. \tag{15}$$

The solutions of (15) can be presented in the form

$$\varphi = -8\pi\sqrt{-b}\,\sigma + \varphi_0\,, \qquad \psi = \psi_0\,. \tag{16}$$

where $\Box$ is the flat-space D'Alembertian and φ_0, ψ_0 are general solutions of the homogeneous equations $\Box^2 \varphi_0 = 0\,,\ \Box^2 \psi_0 = 0$.

There is an obvious arbitrariness related to the choice of the initial conditions for the auxiliary fields φ, ψ. However, replacing Eq. (16) back into the action and taking variation with respect to σ we arrive at the unique equation for σ.

$$\frac{\ddddot{a}}{a} + \frac{3\,\dot{a}\dddot{a}}{a^2} + \frac{\ddot{a}^2}{a^2} - \left(5 + \frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3} - 2k\left(1 + \frac{2b}{c}\right)\frac{\ddot{a}}{a^3}$$
$$-\frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{2\Lambda}{3}\right) = -\frac{\rho_m^0}{c\,a^3}\,. \qquad (17)$$

There are several relevant observations we have to make about the solutions of Eq. (17) in different physical situations.

In the theory without matter, when $\rho_m = 0$, there are two exact solutions, namely

$$a(t) \;=\; a_0 \cdot \exp(Ht) \tag{18}$$

where

$$H \;=\; \frac{M_P}{\sqrt{-32\pi b}} \left(1 \pm \sqrt{1 - \frac{64\pi b}{3} \frac{\Lambda}{M_P^2}} \right)^{1/2} . \tag{19}$$

As far as the cosmological constant is quite small compared to the square of the Planck mass, $\Lambda \ll M_P^2$, we meet two very different values of H (here $\Lambda > 0$)

$$H_c \;\approx\; \sqrt{\frac{\Lambda}{3}} \qquad \text{and} \qquad H_S \;\approx\; \frac{M_P}{\sqrt{16\pi b}} . \tag{20}$$

It is easy to see that the first solution with H_c is the one of the theory without quantum corrections, while the second value H_S corresponds to the inflationary solution of Starobinsky.

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To better understand the situation, let us replace the corresponding FRW solution, e.g., $a(t) \sim t^{2/3}$, into the equation (17).

Is easy to see that the classical part, composed by Einstein and matter terms, do behave like $1/t^2$, while the quantum corrections, which are given by higher derivative terms, behave like $1/t^4$.

This means that in the unstable phase the quantum terms do decay rapidly, such that the classical solution $a(t) \sim t^{2/3}$ is an excellent approximation to the solution of Eq. (17) in the corresponding epoch. The same is true for the radiation-dominated $a(t) \sim t^{1/2}$ and cosmological constant-dominated epochs too.

Derivation of the gravitational waves

Let us derive the equation for the tensor modes of metric perturbations.

It proves useful to present the action (12) of a more useful way. After performing some integrations by parts, it can be cast into the form

$$S = \int d^4x\, L\,, \tag{21}$$

with

$$L = \sum_{s=0}^{5} f_s\, L_s \tag{22}$$

$$= \sqrt{-g}\left[f_0\, R + f_1\, R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} + f_2\, R^{\alpha\beta} R_{\alpha\beta} + f_3\, R^2 + \right.$$

$$+ \left. f_4\, \varphi \Box R + f_5\, \varphi \Delta \varphi \right], \tag{23}$$

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Where the f-terms are defined as

$$
\begin{aligned}
f_0 &= -\frac{{M_P}^2}{16\pi}\,; \\
f_1 &= a_1 + a_2 - \frac{b+\omega}{2\sqrt{-b}}\,\varphi + \frac{\omega}{2\sqrt{-b}}\,\psi; \\
f_2 &= -2a_1 - 4a_2 + \frac{\omega+2b}{\sqrt{-b}}\,\varphi - \frac{\omega}{\sqrt{-b}}\,\psi; \\
f_3 &= \frac{a_1}{3} + a_2 - \frac{3c+2b}{36} - \frac{3b+\omega}{6\sqrt{-b}}\,\varphi + \frac{\omega}{6\sqrt{-b}}\,\psi; \\
f_4 &= -\frac{4\pi\sqrt{-b}}{3}; \\
f_5 &= \frac{1}{2}\,,
\end{aligned}
\tag{24}
$$

where the coefficients $a_{1,2}$ are the same defined in (4).

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Perturbation equations

Using the conditions (with $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$),

$$\partial_i h^{ij} = 0 \text{ and } h_{kk} = 0 \,, \tag{25}$$

together with the synchronous coordinate condition $h_{\mu 0} = 0$, we introduce metric perturbation in the equation (22) as follows

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \,, \quad h_{\mu\nu} = \delta\, g_{\mu\nu} \,. \tag{26}$$

Here $g^0_{\mu\nu} = \{1, -\delta_{ij}\, a^2(t)\}$ are the background cosmological solutions.

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Bilinear parts of the partial Lagrangians from Eq. (22):

$$
\begin{aligned}
L_0 &= a^3 f_0\Big[h^2\Big(\frac{3}{2}\dot{H}+3H^2\Big)+h\ddot{h}+4Hh\dot{h}+\frac{3}{4}\dot{h}^2-\frac{h}{4}\frac{\nabla^2 h}{a^2}\Big]+\mathcal{O}(h^3),\\
L_1 &= a^3 f_1\Big[\dot{h}^2\Big(2H^2-2\dot{H}\Big)-h\ddot{h}\Big(4H^2+4\dot{H}\Big)-h^2\Big(3\dot{H}^2+6\dot{H}H^2+6H^4\Big)-\\
&- h\dot{h}\Big(8H\dot{H}+16H^3\Big)+\ddot{h}^2+4H\dot{h}\ddot{h}+\Big(\frac{\nabla^2 h}{a^2}\Big)^2+2\dot{h}\frac{\nabla^2\dot{h}}{a^2}+\\
&+ \Big(H^2 h-2H\dot{h}\Big)\frac{\nabla^2 h}{a^2}\Big]+\mathcal{O}(h^3),\\
L_2 &= a^3 f_2\Big[-h\dot{h}\Big(12\dot{H}H+24H^3\Big)-\frac{\dot{h}^2}{2}\Big(5\dot{H}+\frac{18}{4}H^2\Big)-\\
&- h^2\Big(3\dot{H}^2+9\dot{H}H^2+9H^4\Big)-h\ddot{h}\Big(4\dot{H}+6H^2\Big)+\frac{\ddot{h}^2}{4}+\frac{3}{2}H\dot{h}\ddot{h}+\\
&+ \frac{1}{4}\Big(\frac{\nabla^2 h}{a^2}\Big)^2-\frac{1}{2}\Big(\ddot{h}+3H\dot{h}-\dot{H}h-3H^2 h\Big)\frac{\nabla^2 h}{a^2}\Big]+\mathcal{O}(h^3),\\
L_3 &= -6a^3 f_3\Big(\dot{H}+2H^2\Big)\Big[h^2\Big(\frac{3}{2}\dot{H}+3H^2\Big)+2h\ddot{h}+\\
&+ 8Hh\dot{h}+\frac{3}{2}\dot{h}^2-\frac{h}{2}\frac{\nabla^2 h}{a^2}\Big]+\mathcal{O}(h^3),
\end{aligned}
$$

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$$
\begin{aligned}
L_4 &= a^3 f_4 \left[\frac{3}{2}\left(\dot{H}+2H^2\right)\left(\ddot{\varphi}+3H\dot{\varphi}\right)h^2 + \left(\ddot{\varphi}+3H\dot{\varphi}\right)h\ddot{h} + \left(18H^2\dot{\varphi}+4H\ddot{\varphi}\right)h\dot{h} + \right.\\
&+ \left. \frac{3}{4}\left(\ddot{\varphi}+3H\dot{\varphi}\right)\dot{h}^2 - \frac{1}{4}\left(\ddot{\varphi}+3H\dot{\varphi}\right)\frac{1}{a^2}h\nabla^2 h \right] + \mathcal{O}(h^3),\\
L_5 &= a^3 f_5 \left\{ \left[\ddot{\varphi}h^2 - \frac{3}{2}H\dot{\varphi}h^2 - \dot{\varphi}\dot{h}h \right]\ddot{\varphi} + \right.\\
&+ \left. \dot{\varphi}^2 \left[-\frac{7}{4}H^2h^2 - \frac{1}{4}\dot{H}h^2 - \frac{7}{3}H\dot{h}h - \frac{1}{3}\ddot{h}h - \frac{1}{6}\frac{h}{a^2}\nabla^2 h \right] \right\} + \mathcal{O}(h^3).
\end{aligned}
\tag{27}
$$

One can perform a comparison of these equations with the ones known from the literature. A very similar expansion was obtained by Gasperini in

- M. Gasperini, **Tensor perturbations in high curvature string backgrounds**, Phys. Rev. D **56**, 4815 (1997)

to explore metric perturbations in the pre-Big-Bang inflationary scenario. Quite different from our case, when we intend to consider the semiclassical gravity with the action induced by anomaly.

The equation for tensor mode can be cast into the form

$$
\begin{aligned}
& \left(2f_1 + \frac{f_2}{2}\right) \ddddot{h} + \left[3H\left(4f_1 + f_2\right) + 4\dot{f}_1 + \dot{f}_2\right] \dddot{h} + \left[3H^2\left(6f_1 + \frac{f_2}{2} - 4f_3\right)\right. \\
+ \quad & H\left(16\dot{f}_1 + \frac{9}{2}\dot{f}_2\right) + 6\dot{H}\left(f_1 - f_3\right) + 2\ddot{f}_1 + \frac{1}{2}(\ddot{f}_2 + f_0 + f_4\ddot{\varphi}) + \frac{3}{2}f_4 H\dot{\varphi} - \left.\frac{2}{3}f_5\dot{\varphi}^2\right] \ddot{h} \\
- \quad & \left(4f_1 + f_2\right)\frac{\nabla^2\ddot{h}}{a^2} + \left[\dot{H}\left(4\dot{f}_1 - 6\dot{f}_3\right) - 21H\dot{H}\left(\frac{1}{2}f_2 + 2f_3\right) - \ddot{H}\left(\frac{3}{2}f_2 + 6f_3\right)\right. \\
+ \quad & 3H^2\left(4\dot{f}_1 + \frac{1}{2}\dot{f}_2 - 4\dot{f}_3\right) - 9H^3(f_2 + 4f_3) + H\left(4\ddot{f}_1 + \frac{3}{2}\ddot{f}_2\right) + \frac{3}{2}f_4\dot{\varphi}\left(3H^2 + \dot{H}\right) \\
+ \quad & H(3f_4\ddot{\varphi} + \frac{3}{2}f_0 - 2f_5\dot{\varphi}^2) + \frac{1}{2}f_4 \dddot{\varphi} - \left.\frac{4}{3}f_5\dot{\varphi}\ddot{\varphi}\right]\dot{h} - \left[H\left(4f_1 + f_2\right) + 4\dot{f}_1 + \dot{f}_2\right]\frac{\nabla^2\dot{h}}{a^2} \\
+ \quad & \left[5f_4 H \dddot{\varphi} + f_4 \ddddot{\varphi} - (36\dot{H}H^2 + 18\dot{H}^2 + 24H\ddot{H} + 4\dddot{H})\left(f_1 + f_2 + 3f_3\right)\right. \\
- \quad & H\dot{H}\left(32\dot{f}_1 + 36\dot{f}_2 + 120\dot{f}_3\right) - 8\ddot{H}\left(\dot{f}_1 + \dot{f}_2 + 3\dot{f}_3\right) - H^2\left(4\ddot{f}_1 + 6\ddot{f}_2 + 24\ddot{f}_3\right) \\
- \quad & 4\dot{H}\left(\ddot{f}_1 + \ddot{f}_2 + 3\ddot{f}_3\right) - 9f_4\dot{\varphi}\left(H^3 + H\dot{H}\right) + f_4\ddot{\varphi}\left(3H^2 + 5\dot{H}\right) - H^3\left(8\dot{f}_1 + 12\dot{f}_2 + 48\dot{f}_3\right) \\
+ \quad & f_5\dot{\varphi}^2\left(\frac{1}{2}H^2 + \frac{1}{3}\dot{H}\right) + \frac{2}{3}f_5 H\dot{\varphi}\ddot{\varphi} - \frac{1}{6}f_5\ddot{\varphi}^2 + \left.\frac{1}{3}f_5\dot{\varphi}\dddot{\varphi}\right] h + f_0\left[2\dot{H} + 3H^2\right] h \\
+ \quad & \left[H^2\left(4f_1 + 4f_2 + 12f_3\right) + H\left(2\dot{f}_1 + \frac{1}{2}\dot{f}_2\right) + 2\dot{H}\left(f_1 + f_2 + 3f_3\right)\right. \\
- \quad & \frac{1}{2}(\ddot{f}_2 + f_4\ddot{\varphi} + f_0 + 3f_4 H\dot{\varphi}) - \left.\frac{1}{3}f_5\dot{\varphi}^2\right]\frac{\nabla^2 h}{a^2} + \left[2f_1 + \frac{1}{2}f_2\right]\frac{\nabla^4 h}{a^4} = 0\,. \qquad (28)
\end{aligned}
$$

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Finally, if we take $H = constant$ in (28), we find the result which perfectly fits the one of

- J. C. Fabris, A. M. Pelinson and I. L. Shapiro, **On the gravitational waves on the background of anomaly-induced inflation**, Nucl. Phys. B **597**, 539 (2001) [Erratum-ibid. B **602**, 644 (2001)] [arXiv:hep-th/0009197].

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Now we can start to deal with our main task and see whether Eq. (28) indicated that there is a stability of the cosmological solutions or not.

We can analyze this problem in two ways: numerically and analytically. We will start with an approximate analytical analysis.

Let us start by rewriting the terms in (28) by using the plane wave representation in flat space section,

$$\frac{\nabla^2 \ddot{h}}{a^2} = -n^2 \frac{\ddot{h}}{a(t)^2}\,, \qquad \frac{\nabla^2 \dot{h}}{a^2} = -n^2 \frac{\dot{h}}{a(t)^2}\,,$$
$$\frac{\nabla^2 h}{a^2} = -n^2 \frac{h}{a(t)^2}\,, \qquad \frac{\nabla^4 h}{a^2} = n^4 \frac{h}{a(t)^4}\,. \tag{29}$$

Then the equation for tensor perturbations can be presented as follows

$$b_4 \ \overset{....}{h} \ + b_3 \ \dddot{h} \ + b_2 \ \ddot{h} \ + b_1 \ \dot{h} \ + b_0 \ h = 0\,, \tag{30}$$

Where we used the notations

$$
\begin{aligned}
b_4 &= 2f_1 + \frac{f_2}{2}\ , && (31)\\
b_3 &= 3H\Big(4f_1 + f_2\Big) + 4\dot{f}_1 + \dot{f}_2\ , && (32)\\
b_2 &= \Big(4f_1 + f_2\Big)\frac{n^2}{a^2} + 3H^2\Big(6f_1 + \frac{f_2}{2} - 4f_3\Big) + H\Big(16\dot{f}_1 + \frac{9}{2}\dot{f}_2\Big) + 6\dot{H}\Big(f_1 - f_3\Big)\\
&+ 2\ddot{f}_1 + \frac{1}{2}\left(\ddot{f}_2 + f_0 + f_4\ddot{\varphi}\right) + \frac{3}{2}f_4 H\dot{\varphi} - \frac{2}{3}f_5\dot{\varphi}^2\ , && (33)\\
b_1 &= \Big[H\Big(4f_1 + f_2\Big) + 4\dot{f}_1 + \dot{f}_2\Big]\frac{n^2}{a^2} + \dot{H}\Big(4\dot{f}_1 - 6\dot{f}_3\Big) - 21H\dot{H}\Big(\frac{1}{2}f_2 + 2f_3\Big)\\
&- \ddot{H}\Big(\frac{3}{2}f_2 + 6f_3\Big) + 3H^2\Big(4\dot{f}_1 + \frac{1}{2}\dot{f}_2 - 4\dot{f}_3\Big) - 9H^3\left(f_2 + 4f_3\right) + H\Big(4\ddot{f}_1 + \frac{3}{2}\ddot{f}_2\Big)\\
&+ \frac{3}{2}f_4\dot{\varphi}\Big(3H^2 + \dot{H}\Big) + H\left(3f_4\ddot{\varphi} + \frac{3}{2}f_0 - 2f_5\dot{\varphi}^2\right) + \frac{1}{2}f_4\ \dddot{\varphi}\ - \frac{4}{3}f_5\dot{\varphi}\ddot{\varphi}\ , && (34)\\
b_0 &= 5f_4 H\ \dddot{\varphi}\ + f_4\ \ddddot{\varphi}\ - \left(36\dot{H}H^2 + 18\dot{H}^2 + 24H\ddot{H} + 4\ \dddot{H}\ \right)\Big(f_1 + f_2 + 3f_3\Big)\\
&- H\dot{H}\Big(32\dot{f}_1 + 36\dot{f}_2 + 120\dot{f}_3\Big) - 8\ddot{H}\Big(\dot{f}_1 + \dot{f}_2 + 3\dot{f}_3\Big) - H^2\Big(4\ddot{f}_1 + 6\ddot{f}_2 + 24\ddot{f}_3\Big)\\
&- 4\dot{H}\Big(\ddot{f}_1 + \ddot{f}_2 + 3\ddot{f}_3\Big) - 9f_4\dot{\varphi}\Big(H^3 + H\dot{H}\Big) + f_4\ddot{\varphi}\Big(3H^2 + 5\dot{H}\Big)\\
&- H^3\Big(8\dot{f}_1 + 12\dot{f}_2 + 48\dot{f}_3\Big) + f_5\dot{\varphi}^2\Big(\frac{1}{2}H^2 + \frac{1}{3}\dot{H}\Big) + \frac{2}{3}f_5 H\dot{\varphi}\ddot{\varphi} - \frac{1}{6}f_5\ddot{\varphi}^2 + \frac{1}{3}f_5\dot{\varphi}\ \dddot{\varphi}\\
&+ f_0\Big[2\dot{H} + 3H^2\Big] - \Big[H^2\Big(4f_1 + 4f_2 + 12f_3\Big) + H\Big(2\dot{f}_1 + \frac{1}{2}\dot{f}_2\Big) + 2\dot{H}\Big(f_1 + f_2 + 3f_3\Big)\\
&+ \frac{1}{2}\left(\ddot{f}_2 + f_4\ddot{\varphi} + f_0 + 3f_4 H\dot{\varphi}\right) - \frac{1}{3}f_5\dot{\varphi}^2\Big]\frac{n^2}{a^2} + \Big[2f_1 + \frac{1}{2}f_2\Big]\frac{n^4}{a^4}\ . && (35)
\end{aligned}
$$

We can easily reduce this fourth-order equation to a system of four first-order equations. Making the new change of variables we introduce

$$h_0 = h\,, \quad h_1 = \dot{h}_0 = \dot{h}\,, \quad h_2 = \dot{h}_2 = \ddot{h}\,, \quad h_3 = \dot{h}_2 = \dddot{h}\ . \quad (36)$$

Rewriting the differential equation, we arrive at

$$\begin{aligned} \dot{h}_3 &= -\frac{1}{b_4}\Big(b_3\,h_3 + b_2\,h_2 + b_1\,h_1 + b_0\,h_0\Big)\,, \\ \dot{h}_2 &= h_3\,, \\ \dot{h}_1 &= h_2\,, \\ \dot{h}_0 &= h_1\,. \end{aligned}$$

We can rewrite the linear system of four equations given above in a matrix form to compute its eigenvalues and eigenvectors.

$$\dot{h}_k = A^l_k \, h_l \,, \tag{37}$$

where $k = 0, 1, 2, 3$ and the matrix $A = A^l_k$ has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d_0 & d_1 & d_2 & d_3 \end{pmatrix},$$

Here we called $d_k = -b_k / b_4$.
So, the next task is to find the eigenvalues of A and hence we consider

$$det \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ d_0 & d_1 & d_2 & (d_3 - \lambda) \end{pmatrix} = 0 \,. \tag{38}$$

The algebraic equation

$$\lambda^4 - d_3 \, \lambda^3 - d_2 \, \lambda^2 - d_1 \, \lambda^1 - d_0 = 0 \,. \tag{39}$$

After some algebra, we can reduce the above equation to the following form

$$z^2 + \xi_1\, z + \xi_2 \;=\; 0\,. \tag{40}$$

The most important quantity is

$$\Delta \;=\; \xi_1 + \frac{4}{27}\,\xi_2^3 \;=\; 4\left[\left(\frac{\xi_1}{2}\right)^2 + \left(\frac{\xi_2}{3}\right)^3\right]. \tag{41}$$

The value of Δ, obtained by using the Cardano formula. Eq. (41), will tell us the nature of these roots.

$$\begin{aligned}
\xi_1 &= \frac{-\alpha}{3} + \beta \quad \text{and} \quad \xi_2 = \left(\frac{2\alpha^3}{27} + \frac{3\gamma - \beta\,\gamma}{3}\right), \\
\alpha &= \frac{5}{2}\,p\,; \quad \gamma = \frac{1}{8}\left(q^2 - 4p^2 + 4\,p\,r\right) \quad \text{and} \quad \beta = 2p^2 - r\,, \\
p &= -\frac{39}{8}\,d_3^2 + d_2\,; \quad q = \frac{d_3^2}{8} - \frac{d_3 d_2}{2} + d_1 \quad \text{and} \quad r = -\frac{3d_3^4}{256} + \frac{d_2 d_3^2}{16} - \frac{d_2 d_1}{4} + d_0\,.
\end{aligned} \tag{42}$$

Let us remember that $b_k/b_4 \;=\; -d_k$.

Exponential expansion

For the three cases of our interest, namely for exponential expansion, radiation and matter epochs, we find,

- When we choose $a_1 < 0$ we found $\Delta < 0$. This is consistent because when we analyze equation (39) directly, we find all eigenvalues to be real and negative. So, we have the stability in this case, exactly as we could expect from comparison to the inflationary case.

  However, if we choose $a_1 > 0$, then we find $\Delta < 0$ too. But analyzing equation (39) directly by numerical method (this means deriving the roots numerically by use of Mathematica software), we find three negative and one positive eigenvalue. So, we can observe the instability in this case.

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Radiation

- When we choose $a_1 < 0$ we found $\Delta > 0$. This is consistent because when analyzing the Eq. (39) directly, we find two real eigenvalues, which are both negative and also two complex eigenvalues with negative real parts. So, we have stability in this case.

  But if we take $a_1 > 0$ it turns out that $\Delta < 0$. Analyzing Eq. (39) directly, we find two negative eigenvalues and two positive ones. So, we have instability in this case. Again, the stability of the classical solution is completely dependent on the classical term (3).

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Matter

- With $a_1 < 0$ we find $\Delta > 0$. This is consistent the direct numerical analysis of Eq. (39), because in this way we find two real negative eigenvalues and also two complex eigenvalues with negative real parts. So, we have the stability for $a_1 < 0$.

  However, if we choose $a_1 > 0$, we find $\Delta > 0$, indicating instability. By analyzing Eq. (39) directly one confirms this result, for we meet two real eigenvalues (one negative and other positive) and two complex ones, both with negative real parts.

Let us remark that the sign of a_1 defines whether the massless tensor mode in the classical theory is a graviton or a ghost.

- K. S. Stelle, **Classical Gravity With Higher Derivatives**, Gen. Rel. Grav. **9**, 353 (1978).

From this perspective our result means that the stability property of the theory with higher derivative classical term

$$S_{HD} = \int d^4x\sqrt{-g}\left\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\right\},$$

and quantum correction

$$\begin{aligned}\bar{\Gamma}_{ind} &= S_c[g_{\mu\nu}] - \frac{3c+2b}{36}\int d^4x\sqrt{-g(x)}\,R^2(x) + \int d^4x\sqrt{-g(x)}\left\{\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi\right.\\ &+ \left.\varphi\left[\frac{\sqrt{-b}}{2}\,(E - \frac{2}{3}\,\Box R) - \frac{w}{2\sqrt{-b}}\,C^2\right] + \frac{w}{2\sqrt{-b}}\,\psi\, C^2\right\}.\end{aligned}$$

is completely defined by classical part.

As a result of our consideration we can conclude that there is a stability for Eq. (28), if and only if a_1 is negative!

Numerical analysis

In order to ensure that our qualitative and analytic consideration of the stability is correct, let us present the analysis of the stability of the differential equation (28) by means of numerical methods, using the software *Mathematica*.

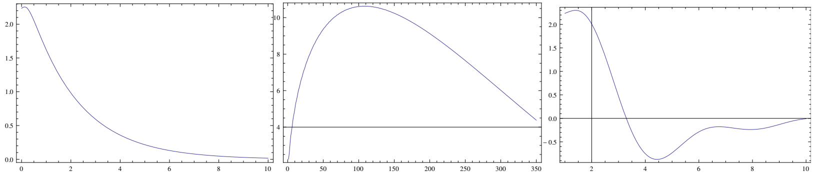

Figure: Graph of perturbation $h(t)$ in function of time analyzed in the cases for $a(t) = a_0 e^{H_0 t}$, $a(t) = a_0 t^{1/2}$ and $a(t) = a_0 t^{2/3}$ respectively, with the initial conditions $h_0 = \frac{1}{\sqrt{2n}}$, $\dot{h} = \sqrt{\frac{n}{2}}$,$\ddot{h} = \frac{n^{3/2}}{\sqrt{2}}$, $\dddot{h} = \frac{n^{5/2}}{\sqrt{2}}$, where we adopt $a_1 < 0$. Stable behavior.

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Numerical analysis

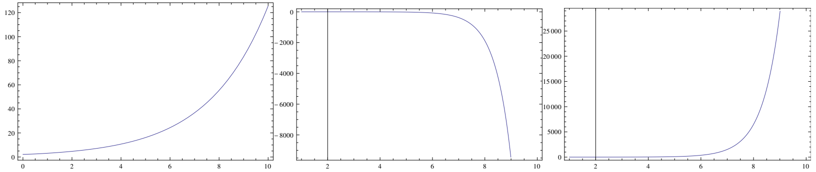

Figure: Graph of perturbation $h(t)$ in function of time analyzed in the cases for $a(t) = a_0 e^{H_0 t}$, $a(t) = a_0 t^{1/2}$ and $a(t) = a_0 t^{2/3}$ respectively, with the initial conditions $h_0 = \frac{1}{\sqrt{2n}}$, $\dot{h} = \sqrt{\frac{n}{2}}$,$\ddot{h} = \frac{n^{3/2}}{\sqrt{2}}$, $\dddot{h} = \frac{n^{5/2}}{\sqrt{2}}$, where we adopt $a_1 < 0$. Stable behavior.

Initial data problem and the spectrum of gravitational waves

Here we develop a linearization process to found the spectral index k for our theory. But first we will test and fix our numerical method using the traditional inflation.

The perturbations, which originate from the fluctuations of the zero point energy of the quantum fields, have the spectrum characteristic of a scalar quantum field in Minkowski space. This "vacuum state" is well known:

$$h(x,\eta) = h(\eta)\, e^{\pm i\mathbf{n}.\mathbf{x}} \quad , \quad h(\eta) \propto \frac{e^{\pm in\eta}}{\sqrt{2n}} \quad . \tag{43}$$

where we employed the conformal time, because the FRW metric becomes conformal to the Minkowski metric in flat space; $\mathbf{n}$ is the wavenumber vector.

How the initial amplitude depends on **n**. In our case, it becomes

$$h_0 \propto \frac{1}{\sqrt{2n}} \quad , \quad \dot{h}_0 \propto \sqrt{\frac{n}{2}} \quad , \quad \ddot{h}_0 \propto \frac{n^{3/2}}{\sqrt{2}} \quad , \quad \dddot{h}_0 \propto \frac{n^{5/2}}{\sqrt{2}} \quad . \tag{44}$$

In order to study the dynamics of $h(t, \mathbf{x})$, it is necessary to make a Fourier transform,

$$h_{\mathbf{n}}(t) = \frac{1}{(2\pi)^{3/2}} \int h(t, \mathbf{x})\, e^{i\mathbf{n}\cdot\mathbf{x}}\, d^3x. \tag{45}$$

We will need the total square of the amplitude, namely

$$h^2(t) = \int h_{\mathbf{n}}^2(t)\, d^3n\,. \tag{46}$$

The above equation can be rewritten in the form

$$h^2(t) = 4\pi \int h_{\mathbf{n}}^2(t)\, n^2\, d\, n = 4\pi \int h_{\mathbf{n}}^2(t)\, n^3\, d\, \ln\, n = 4\pi \int P_n^2(t)\, d\, \ln\, n\,, \tag{47}$$

where

$$P_n(t) = h_{\mathbf{n}}^2(t)\, n^3\,. \tag{48}$$

The last quantity is called the **“power spectrum”**.

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We can find the power spectrum and the spectral index for the gravitational wave. For this end one has to square the value of the gravitational perturbation h at a given time and for a given wave number n. We will vary this n for a fixed t and simultaneously solve our fourth-order differential equation (Eq. (28)) numerically. After this, we linearize the graph by plotting the relation

$$\ln n^3 h_{\mathbf{n}}^2(t) \times \ln n\,. \tag{49}$$

As a result we obtain the linear proportionality coefficient, which will be denoted as k and called spectral index. Then we have, $P_n^2(t) \propto n^k$, i.e., it is proportional to the spectral index. It is the power spectrum that will tell us how the amplitude of the perturbations depends on the wavelength.

Now we will compare our results with the inflationary scenario based on the Einstein's equations with a cosmological constant. We can consider that this cosmological constant arise from some inflaton potential.

$$h(\eta) = \sqrt{\eta}\, g_{\pm}(n) H^{(1)}_{\pm\frac{3}{2}}(n\eta) \quad , \tag{50}$$

where $g_{\pm}$ are integration constants which may depend on n, and $H^{(1)}(x)$ is the Hankel function of first kind. This dependence must be fixed by the initial spectrum. Taking the vacuum state as described above, we find that those constants do not depend of n. With the long wavelength limit approximation $n \to 0$ and considering the dominant mode in the above expression, we find

$$P_n \propto n^{3/2-3/2} = \text{constant} \quad . \tag{51}$$

The traditional inflationary scenario predicts a flat spectrum, with $k = 0$.

We can use these results to gauge our numerical procedure. Fixing the initial spectrum according to (44), integrating the equation for the gravitational wave for the traditional inflationary scenario,

$$\ddot{h} - \frac{\dot{a}\,\dot{h}}{a} + \left\{ \frac{n^2}{a^2} - 2\,\frac{\ddot{a}}{a} \right\} h = 0 \quad , \tag{52}$$

using $a(t) = e^{Ht}$, and the numerical procedure described above, we found, for the (50) case,

$$k \simeq 0.01\,. \tag{53}$$

this numerical result is vary close to the analytical one, which is zero. We have considered a variation of n between zero and one. Thus we find that to initial perturbations whose scales are of the order of the Planck length.

Now, applying the same procedure for our model of equation (28), we find, in the case of exponential expansion without quantum corrections (this means $a_1 = a_2 = 0$),

$$k = 0.00894167 \approx 0.01 \tag{54}$$

where we use the units with the Planck mass equal to 1 in the equation (28) and the Eqs. (24), for perform the numerical analysis.

Conclusions

- **For QG with higher derivative (HDQG) the propagator includes massive nonphysical mode(s) called ghosts.**

- **These massive ghosts are capable to produce terrible instabilities, but ... for this end there should be at least one such ghost excitation in the initial spectrum.**

Conclusions

- Starting from approximate classical solution we observe, in the cosmological metric case, that the ghost is not actually generated below Planck scale.

- **Ghosts are likely not posing a threat below the Planck scale!**

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Thanks!

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