Geometric gravity

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CONTENTS

Geometry

&

basic tools

I – Gravity & geometrical concepts

Newton's space-time : 2 things

space-time = **space** + **time**

No space-time ***geometry*** (invariant = dt)

But : space geometry | properties of time

Space geometry : (3-dimensional) euclidean space (→ homogeneity & isotropy)

Properties of time : uniformity

Minkowski's space-time

space-time geometry → time relativity $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

(pseudo-)euclidean → - space homogeneity & isotropy
- time uniformity

preserved for each observer (even if relative notions)

Inertial motions : determined by space-time properties only (no « force »)

Flatness → inertial motions = rectilinear and uniform (symmetries)

Observed motions (planets, …) → not inertial motions (if space-time flatness to be preserved !!!)

Newton's theory $$m\frac{d^2x^i}{dt^2} = f^i$$ (general)

$$\frac{d^2x^i}{dt^2} = \partial_i U$$ (gravitation)

Gravitation has nothing to do with space-time properties

Space-time properties have nothing to do with matter, energy, other fields, …

Question : to interpret observed motions, **is that possible to rule out forces ?**

back to **inertia** & rule out prior space-time symmetries

change space-time geometry, ie **rule out flatness**

Gravitation : the (space-time) metric

$$ds_N^2 = -(1-2U)\,dt^2 + dx^2 + dy^2 + dz^2$$

(let's refer it as Newton's metric) do the job, if one consider :

1 - geodesics (extremal length curves , or auto-parallel curves, see later…) are relevant definitions of inertial motions since :

- determined from space-time geometry properties only ;
- reduce to Newton's inertia principle in the « flatness » (see later…) case.

2 - Newton/Lagrange $\delta\int\left(\tfrac{1}{2}v^2 - U\right)dt = 0 \;\Leftrightarrow\; \delta\int ds_N = 0$ (geod. curves)

ie : the functions $x(t)$, $y(t)$, $z(t)$ obtained from the 2 variational principles are the same if $o(v^2)$ & $o(U)$ terms are discarded

Newton's theory successes → Newton's metric should be the lowest order solution of any admissible (geometric) gravity theory

➔ deal with **geometry** ? 2 geometrical concepts : **metric & connection**

0 – Tensors in brief

In a *N*-dim space (space-time) an *m*-order tensor (tensorial field) has N^m components

Considering a coordinates transform $x^a \to x'^a$, the components transform as :

Scalar (0th order tensor, one component) : invariant

Contravariant vector (« ordinary » vector, 1rst order contravariant tensor) :

$$A'^\alpha = \left(\sum_\beta \right) \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \qquad \left[\text{direct jacobian, as} \quad dx'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} dx^\beta \right]$$

Covariant vector (linear form, 1rst order cavariant tensor) :

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \qquad \left[\text{inverse jacobian, as} \quad \frac{\partial F}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial F}{\partial x^\beta} \right]$$

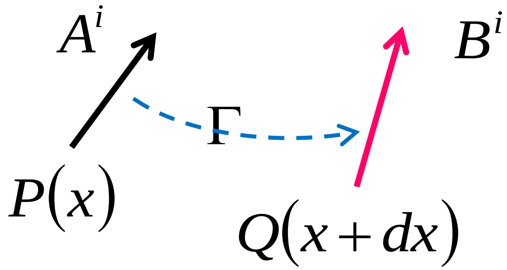
General tensor : $T'^{\mu\ldots}_{\alpha\ldots} = \frac{\partial x'^\mu}{\partial x^p} \ldots \frac{\partial x^i}{\partial x'^\alpha} \ldots T^{p\ldots}_{i\ldots}$

Tensorial densities : $D'^{\mu\ldots}_{\alpha\ldots} = \left| \frac{\partial x'}{\partial x} \right|^w \frac{\partial x'^\mu}{\partial x^p} \ldots \frac{\partial x^i}{\partial x'^\alpha} \ldots D^{p\ldots}_{i\ldots} \qquad (w = \text{weight})$

1 - Connections

define **local notion of parallelism**

Care : NOT tensors !!!



vector at P → [• displacement dx • connection Γ] → vector at Q

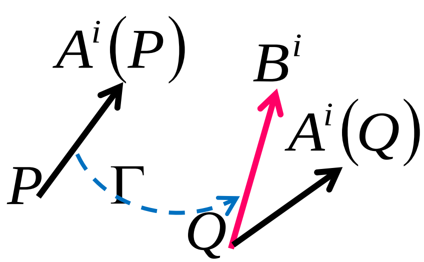
By definition , B is the vector **parallel** to A at point Q, w.r.t. the connection Γ

$$B^i = A^i + \delta A^i \quad \text{with} \quad \delta A^i = -\Gamma^i_{kj}\, A^j\, dx^k$$

change of comp. **by parallel transport**

Torsion-free connection ←→ $\Gamma^k_{ij} = \Gamma^k_{ji}$ → 40 indep. comp. In 4-dim space-times

Let us enforce that if A is the element at P of a vector field, B **is not** the element $A(Q)$ of this vector field at Q …



…but **compare** $A(Q)$ and B → notion of **covariant derivative**

$$\nabla_k A^i = \partial_k A^i + \Gamma^i_{kj}\, A^j \qquad (\leftrightarrow \quad \nabla A^i = dA^i + \Gamma^i_{kj}\, A^j dx^k \quad)$$

$$\partial_k\left(A^i B_i\right) = A^i \partial_k B_i + B_i\, \partial_k A^i = A^i\left(\partial_k B_i - \Gamma^j_{ik} B_j\right) + B_i\, \nabla_k A^i$$

Covariant derivates of scalars, cov. vectors, … (ok Leibniz rule)

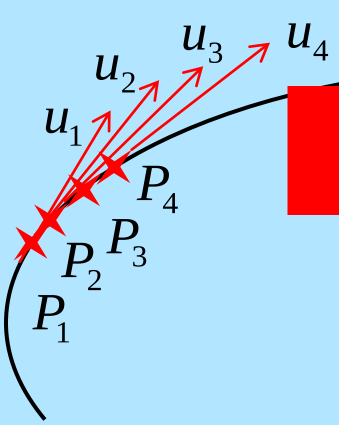
[Try (unsuccessfully !!!) to built] global parallelism → notion of curvature

Riemann-Christoffel (RC) curvature tensor :

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{\sigma k} \Gamma^\sigma_{jl} - \Gamma^i_{\sigma l} \Gamma^\sigma_{jk}$$

Space-time is flat (w.r.t. Γ) ←→ RC tensor = 0 (flatness def.)

Auto-parallel curves



$$dx \propto u$$

vector u_1 at P_1 → from P_1 to P_2 w.r.t. Γ → vector u_2 at P_2

vector u_2 at P_2 → from P_2 to P_3 w.r.t. Γ → vector u_3 at P_3

…and so on …

→ geodesics (defined as « auto-parallel curves » –APC –)

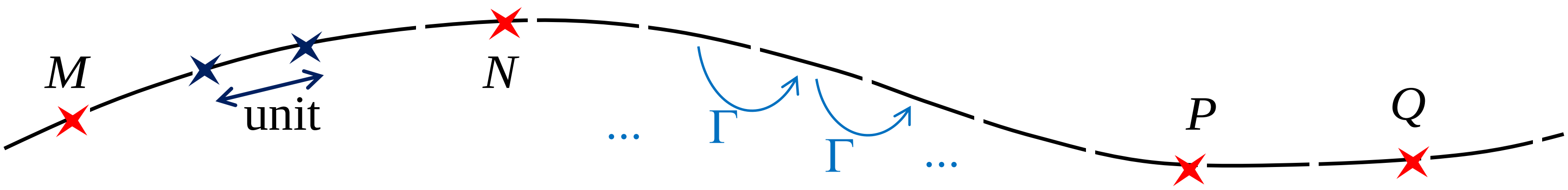
If parametrization p chosen so that between 2 « successive » points : $dx^k = u^k dp$

→ affine parameter APC equation : $u^i \nabla_i u^k = 0$ or $\frac{d^2 x^k}{dp^2} + \Gamma^k_{ij} \frac{dx^i}{dp} \frac{dx^j}{dp} = 0$

(if tangent vector U just prop to u : $U^i \nabla_i U^k \propto U^k$)

Remark that Γ induces a natural notion of « length » along each geodesic (APC) by :

1 - defining an « unitary length » by an infinitesimal segment on it
2 - carrying it along the APC
3 - counting the number of transported « unitary length » in each segment …



...but no way to compare the « lengths » of the bipoints [MN] & [PQ] if the four points do not belong to the same APC !!!

2 - Metrics (tensors)

define **local « distances »** …

…or, rather, the notion of **interval**, that generalises the usual euclidean distance notion

$$ds^2 = g_{\alpha\beta}\left(x^\lambda\right) dx^\alpha dx^\beta$$

→ allows to compare the « lengthes » of bipoints [MN] & [PQ], whatever these points

A metric → **extremal length curves** $\delta \int ds = 0$ where $ds = \sqrt{\left|ds^2\right|}$

$$\frac{d^2 x^k}{dp^2} + \Gamma(g)^k_{ij} \frac{dx^i}{dp} \frac{dx^j}{dp} = 0 \quad \text{with} \quad \underbrace{\Gamma(g)^k_{ij} \equiv \frac{1}{2} g^{km} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right)}_{\text{Christoffel (or metric) connection}}$$

g^{ij} defined by $g_{ij} g^{jk} = \delta_i^k$

Christoffel (or metric) connection

CARE ! : one can define a metric & a connection **without prior link** between them !

$\Gamma^k_{ij} \neq \Gamma(g)^k_{ij}$ a priori

Auto-parallel & extremal length curves are generally not the same !!!

metric → correspondance between covariance & contravariance

$$A^k, B_k \rightarrow A_i \equiv g_{ik} A^k \;,\; B^i \equiv g^{ik} B_k \;,\; ....$$

Contracted curvature tensors

RC curvature tensor :
may be contracted

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{\sigma k} \Gamma^\sigma_{jl} - \Gamma^i_{\sigma l} \Gamma^\sigma_{jk}$$

Ricci curvature tensor : $R_{ij} \equiv R^k{}_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^l_{kl} \Gamma^k_{ij} - \Gamma^l_{ik} \Gamma^k_{jl}$

- **metric not required !!!**
- **not symmetric** in general, even for torsion-free connection !!!

Scalar curvature : $R \equiv g^{ij} R_{ij}$ (metric required !!!)

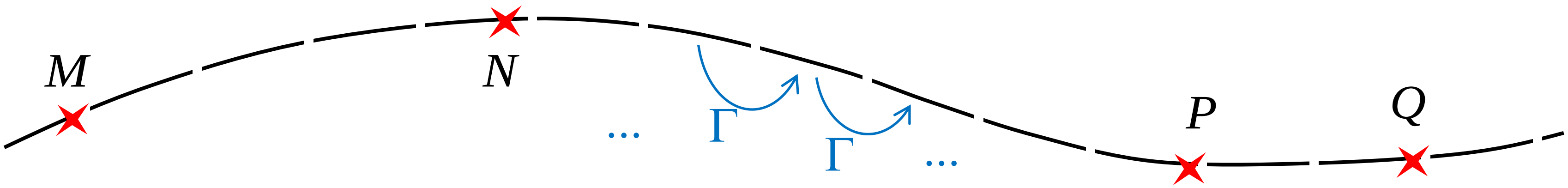
One may define Einstein's tensor : $E_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$ … **not symmetric** in general

Invariant integrals : $\int (\text{ any scalar }) \sqrt{|g|}\, d^4x$

…since g is a (weighted -2) scalar density
(as all determinant of any covariant 2d order tensor)

Metric-length vs Γ-length induced on auto-parallel curves

Let four points (M close to N, P close to Q) on a same (Γ)auto-parallel curve



Condition for $\left(\frac{l_\Gamma(MN)}{l_\Gamma(PQ)}\right)^2 = \frac{ds^2_{MN}}{ds^2_{PQ}}$ **???**

Answer (just considering torsion-free connections) : $\Gamma^k_{ij} = \Gamma(g)^k_{ij}$

Extremal length curves = auto-parallel curves (geodesics)

Ricci identity $\nabla_k g_{ij} = 0$

Ricci curvature is symmetric ! $R_{ij} = R_{ji}$

Einstein's tensor is (symmetric &) divergence-free $\nabla_i E^{ij} = 0$

Lorentzian space-times. Geodesic coordinates

Lorentzian space-time : for any given point *P*, it is possible to choose a coordinate system such that, at that point

$$g_{ab}(P)=m_{ab} \quad \text{where} \quad \underbrace{(m_{ab})=diag(-1,+1,+1,+1)}_{\text{Minkowski metric in cart coord}}$$

The coord transformation $\quad x'^i=\bar{x}^i+\frac{1}{2}\Gamma^i_{\alpha\beta}(P)\,\bar{x}^\alpha\bar{x}^\beta \quad \text{with} \quad \bar{x}^i=x^i-x^i(P)$

results in $\Gamma'=0$ at point *P* in the new coord *x*' , without changing metric components at that point (works for any connection)

→(math) simplifies some mathematical demonstrations

→(phys) free fall frame (geodesic frame) if Γ is the connection entering inertial motion equation (depending on the lagrangian of the theory)

$$\text{Rmk : if} \quad \Gamma=\Gamma(g_{ij}), \quad \Gamma'=0 \;\leftrightarrow\; \partial' g'_{ij}=0 \qquad (\text{but } \partial'\partial' g'_{ij}\neq 0 \text{, ie } \partial'\Gamma'\neq 0\,)$$

II – What can a geometric gravity lagrangian look like ?

Newton's metric ← Newton's universal gravity theory

Space-time geometry as the direct by-product of the theory → **theory = ???**

Lagrangian : a « gravitational part » + a « non- gravitational part »

- just gravitational field dependent
- determines vacuum field theory

$g_{ab}, \partial g_{ab}, \partial\partial g_{ab}, \ldots$

$\Gamma, \partial\Gamma, \ldots$

$\phi, \partial\phi, \ldots$

If a scalar field instead of Newton's *G*

- matter fields dependent → Ψ
- gravitational field dependent (matter-gravity couplings)

Matter couples with :
- metric, connection ?
- scalar field or not ?

Free Fall Universality (FFU) or not ?

Matter's equation of motion

Scalar density → action = integral invariant (L scalar)

Represents collectively all non-grav fields
(ℓ to be varied wrt each (dynamical) field)

$$\ell = \sqrt{-g}\,L_G\left(g_{ab},\partial g_{ab};\Gamma,\partial\Gamma;\phi,\partial\phi\right) + 2\sqrt{-g}\,L_{NG}\left(\Psi;g_{ab},\partial g_{ab};\Gamma;[\phi]\right)$$

> 1rst order formalism (Palatini) :
no prior link between Γ & the metric
→ 2 geometric dynamical fields

> 2d order formalism :
Γ = (a priori) metric connection
→ metric only as geom. dyn. field

Just metric if
2d order formalism

1rst order :
→ matter couples with what ?

Don't presume constant Newton's coupling
→ scalar-tensor gravity

Not present if matter is assumed to couple with geometry only…
…if univ of free fall required…
…in this representation
(conformal transformations,…)

1 - Vary the non-gravitational fields

Ψ_1 , Ψ_2 , … dynamical fields → vary → Lagrange equations

→ for each Ψ_n

$$\frac{\partial\left(\sqrt{-g}\,L_{NG}\right)}{\partial\Psi_n} - \partial_c \frac{\partial\left(\sqrt{-g}\,L_{NG}\right)}{\partial\left(\partial_c \Psi_n\right)} \quad (+\ldots) = 0$$

Each Lagrange equation (ie for each *n*) depends on all the Ψ_k (coupled equations) ….

…. unless non-grav fields enter **separately** the lagrangian :

$$L_{NG}(\Psi_1, \Psi_2, \ldots; g_{ab}, \ldots) = \overset{(1)}{L}_{NG}(\Psi_1; g_{ab}, \ldots) + \overset{(2)}{L}_{NG}(\Psi_2; g_{ab}, \ldots) + \ldots$$

→ decoupled equations

$$\frac{\partial\left(\sqrt{-g}\,\overset{(n)}{L}_{NG}\right)}{\partial\Psi_n} - \partial_c \frac{\partial\left(\sqrt{-g}\,\overset{(n)}{L}_{NG}\right)}{\partial\left(\partial_c \Psi_n\right)} \quad (+\ldots) = 0$$

(but each Ψ_k couples with the metric/connection)

2 - Vary the gravitational fields & define stress tensors

1rst order formalism : vary w.r.t. g_{ab} & Γ ⟶ two « families » of equations

δg^{ab} → « Einstein's » (even if not GR …) equation

$\delta \Gamma^{c}_{ab}$ → Link Γ & metric

2cd order formalism : vary w.r.t. g_{ab} ⟶ « Einstein's » (even if not GR) equation

In both cases, **variation w.r.t. metric** leads to

$$\delta \ell = \delta\left(\sqrt{-g}\,L_G\right) + 2\delta\left(\sqrt{-g}\,L_{NG}\right)$$

OK (see later for some specific cases)

depends on metric & its derivatives

$$\delta\left(\sqrt{-g}\,L_{NG}\right) = \left[\frac{\partial\left(\sqrt{-g}\,L_{NG}\right)}{\partial g^{ab}} - \partial_c \frac{\partial\left(\sqrt{-g}\,L_{NG}\right)}{\partial\left(\partial_c g^{ab}\right)} + \dots\right]\delta g^{ab}$$

up to div terms

$$\equiv \frac{\delta\left(\sqrt{-g}\,L_{NG}\right)}{\delta g^{ab}}$$

The **global** stress tensor is defined by

$$T_{ab}=-\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\,L_{NG}\right)}{\delta g^{ab}}$$

$$T^{ab}=g^{ac}g^{bd}T_{cd}$$

If non-grav fields **<u>separately</u>** enter the non-grav lagrangian :

$$L_{NG}\left(\Psi_1,\Psi_2,\ldots\right)=\overset{(1)}{L}_{NG}\left(\Psi_1\right)+\overset{(2)}{L}_{NG}\left(\Psi_2\right)+\ldots$$

δg^{ab}

$$T_{ab}=\overset{(1)}{T}_{ab}\left(\Psi_1\right)+\overset{(2)}{T}_{ab}\left(\Psi_2\right)+\ldots$$

Remark that

$$\delta\left(\sqrt{-g}\,L_{NG}\right)=\frac{\delta\left(\sqrt{-g}\,L_{NG}\right)}{\delta g^{ab}}\delta g^{ab}=\frac{\delta\left(\sqrt{-g}\,L_{NG}\right)}{\delta g_{ab}}\delta g_{ab}$$

$$-\frac{\sqrt{-g}}{2}T_{ab} \qquad -g^{ac}g^{bd}\delta g_{cd}$$

$$T^{ab}=+\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\,L_{NG}\right)}{\delta g_{ab}}$$

3 - From (the scalar nature of) action terms to « conservation laws »

$$\delta \int \ell d^4x = 0 \qquad \underbrace{\forall \ \delta g_{ab}}_{} \qquad \rightarrow \qquad \text{"Einstein" field eq}$$

$$\ell = \sqrt{-g}L$$

Postulate !!!

On the other hand, if the global lagrangian separates as

$$\int \ell d^4x = \int \ell_1 d^4x + \int \ell_2 d^4x + \ldots = A_1 + A_2 + \ldots$$

where each $\boldsymbol{\ell}_k$ is a (weighted -1) scalar density, then ***each*** A_k ***SHOULD*** be **invariant** under any **infinitesimal gauge transform**

$$x'^a = x^a + \xi^a \quad \rightarrow \quad \text{« effective » variation of each field}$$

$$\& \qquad \underbrace{\forall \xi \qquad \delta \int \ell_k d^4x = 0}_{}$$

NOT a postulate !!!

$g'_{ab}(P)=\frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}g_{cd}(P)$ $x'(P)=x(P)+\xi(P)$ $x(P)$	$\phi'(P)=\phi(P)$ (idem) (idem)	and so on	…

Metric : $$g'_{ab}(x)+\xi^c\partial_c g_{ab}+\ldots=\left(\delta_a^c-\partial_a\xi^c\right)\left(\delta_b^d-\partial_b\xi^d\right)g_{cd}(x)$$

ie, at 1rst order in ξ $$\underbrace{g'_{ab}(x)-g_{ab}(x)}_{\text{« effective » }\delta g_{ab}}=-g_{ac}\partial_b\xi^c-g_{cb}\partial_a\xi^c-\xi^c\partial_c g_{ab}$$

Scalar : $$\phi'_{ab}(x')=\phi(x)\xrightarrow{\text{1rst order}}\underbrace{\phi'(x)-\phi(x)}_{\text{« effective » }\delta\phi}=-\xi^c\partial_c\phi$$

……

These effective variations **necessarily result in** $\delta\int \ell_k d^4x=0 \quad \forall\xi$

IF able to put it in a form like $\int(\ldots)_a\xi^a d^4x=0$ → one identity per ℓ_k

$$(\ldots)_a=0$$

$$\ell_k\left(\Psi_1,\ldots;g_{ab};\Gamma^c_{ab};\phi\right) \;\rightarrow\; \int\left(\frac{\delta\ell_k}{\delta\Psi_1}\delta\Psi_1+\ldots+\frac{\delta\ell_k}{\delta g_{ab}}\delta g_{ab}+\frac{\delta\ell_k}{\delta\Gamma^c_{ab}}\delta\Gamma^c_{ab}+\frac{\delta\ell_k}{\delta\phi}\delta\phi\right)d^4x=0$$

Consider now cases where :

just $\frac{\delta\ell_k}{\delta\Psi_k}\delta\Psi_k$ remains

(1) ℓ_k just depends on Ψ_k (no $\Psi_k - \Psi_l$ coupling, whatever l)

& **besides**

(2) Ψ_k just enters ℓ_k → results in 2 points :

$$\frac{\delta\ell_k}{\delta\Psi_k}=0$$

$$\frac{\delta\ell_k}{\delta g_{ab}}=\frac{\sqrt{-g}}{2}\overset{(k)}{T}{}^{ab}$$

& **besides**

(3) if now Γ does not explicitly enter ℓ_k for any reason (2cd order formalism, or ….)

$$\int\left[\frac{\sqrt{-g}}{2}\overset{(k)}{T}{}^{ab}\left(-g_{ac}\partial_b\xi^c-g_{cb}\partial_a\xi^c-\xi^c\partial_c g_{ab}\right)-\frac{\delta\ell_k}{\delta\phi}\xi^c\partial_c\phi\right]d^4x=0$$

Perform partial integrations & find

$$\int\left[\sqrt{-g}\nabla_a\overset{(k)}{T}{}^a_{\ c}-\frac{\delta\ell_k}{\delta\phi}\partial_c\phi\right]\xi^c d^4x=0$$

(+ usual requirements)

True whatever ξ → $\sqrt{-g}\nabla_a \overset{(k)}{T}{}^a_{\ c} = \frac{\delta \ell_k}{\delta \phi}\partial_c \phi$

IF

- * no scalar in grav field
- * ϕ does not couple with matter
- * …. (no ϕ term)

$$\nabla_a \overset{(k)}{T}{}^a_{\ c} = 0$$

Some remarks :

Rmk 1 : one identity per *k* (if all decoupled)

Rmk 2 : if Ψ_1 & Ψ_2 are coupled, ie

Ψ_3, Ψ_4, …are not here

Ψ_1 & Ψ_2 are not here

$$L_{NG}(\Psi_1,\Psi_2,\Psi_3,\ldots) = \overset{(1-2)}{L}{}_{NG}(\Psi_1,\Psi_2) + \overset{(3)}{L}{}_{NG}(\Psi_3) + \ldots\ldots$$

$$\frac{\partial\left(\sqrt{-g}\,\overset{(1-2)}{L}{}_{NG}\right)}{\partial\Psi_1} - \partial_c \frac{\partial\left(\sqrt{-g}\,\overset{(1-2)}{L}{}_{NG}\right)}{\partial(\partial_c\Psi_1)} = 0 \quad \& \quad \frac{\partial\left(\sqrt{-g}\,\overset{(1-2)}{L}{}_{NG}\right)}{\partial\Psi_2} - \partial_c \frac{\partial\left(\sqrt{-g}\,\overset{(1-2)}{L}{}_{NG}\right)}{\partial(\partial_c\Psi_2)} = 0$$

$$\& \quad \frac{\partial\left(\sqrt{-g}\,\overset{(3)}{L}{}_{NG}\right)}{\partial\Psi_3} - \partial_c \frac{\partial\left(\sqrt{-g}\,\overset{(0)}{L}{}_{NG}\right)}{\partial(\partial_c\Psi_3)} = 0 \quad \& \ldots.$$

…while the $A_{(1\text{-}2)}$ invariance results in

$$\int \left(\underbrace{\frac{\delta \ell_{(1-2)}}{\delta \Psi_1}}_{=0} \delta\Psi_1 + \underbrace{\frac{\delta \ell_{(1-2)}}{\delta \Psi_2}}_{=0} \delta\Psi_2 + \frac{\delta \ell_{(1-2)}}{\delta g_{ab}} \delta g_{ab} + \frac{\delta \ell_{(1-2)}}{\delta \Gamma^c_{ab}} \delta\Gamma^c_{ab} + \frac{\delta \ell_{(1-2)}}{\delta \phi} \delta\phi \right) d^4x = 0$$

ie, if Γ does not enter $\ell_{(1\text{-}2)}$

$$2 g_{bc} \nabla_a \underbrace{\left(\frac{\delta \ell_{(1-2)}}{\delta g_{ab}} \right)}_{} = \frac{\delta \ell_{(1-2)}}{\delta \phi} \partial_c \phi \qquad (= 0 \text{ if no scalar})$$

sometimes refered as $\overset{(1)}{T}_{ab}(\Psi_1) + \overset{(2)}{T}_{ab}(\Psi_2)$ but …

… how defining $\overset{(1)}{T}_{ab}(\Psi_1)$ & $\overset{(2)}{T}_{ab}(\Psi_2)$? ? ?

Rmk 3 : what if the scalar curvature lagrangian is considered ? (2cd order formalism)

$$A_{Ricci\ curv.} = \int R \sqrt{-g}\, d^4x$$

invariance w.r.t. $x'^a = x^a + \xi^a$

$$\nabla_a E^{ab} = 0$$

ie (back to) Einstein's tensor zero divergence (ie twice contracted Bianchi)

Rmk 4 (terminology) : $\nabla_a \overset{(k)}{T}{}^a_c = 0$ is usually refered as a **« conservation law »**

but … what is conserved ?... **Is even <u>something conserved ???</u>**

4 – (Digression) dynamical vs non-dynamical

One may imagine theories including **non-dynamical fields** as part of the gravitationnal field (a theory = a lagrangian) :

- bimetric gravity theories with background minkowski metric, or another metric not affected by the matter content (Rosen's theory, Visser's massive gravity, …)
- scalar-tensor like theories with an « external » (non-dynamical) scalar field (Renormalization Group GR)

Non-dynamical fields → do not generate field equations attached to these fields

→ theories a priori less constrained than their fully dynamical version

However, the mere existence of **geometrical identities** (Bianchi …) results in **redondance in the field equations** resulting from **fully dynamical theories**

➔ **part of the (dynamical) equations carry no information**

→ the difference between the fully and not fully dynamical versions of a theory may be thiner, or at least **more subtle than what could be first expected** (see examples later)

5 – A special case of matter : the perfect fluid

Perfect fluid : stress tensor of the form $T^{ab} = (\varepsilon + P) u^a u^b + P g^{ab}$

with : ε & P : scalar fields

u^a : normalized vector field $\quad g_{ab} u^a u^b = -1$

$T_{ab} u^a u^b = \varepsilon$

interpretation !!!

ε = fluid's proper energy

u = fluid's quadri-velocity

« conservation » laws : $\nabla_a \overset{(k)}{T}{}^a_{\ c} = 0$

CARE : properties if Γ=Γ(*g*) !!!

energy conservation

$$\partial_\alpha \left(\sqrt{-g}\, \rho u^\alpha \right) = 0$$

where $\rho = \exp\left\{ \int \frac{d\varepsilon}{\varepsilon + P(\varepsilon)} \right\}$

Euler's equation

$$(\varepsilon + P) u^\beta \nabla_\beta u^\alpha = -\left(g^{\alpha\beta} + u^\alpha u^\beta \right) \partial_\beta P$$

P = fluid's pressure

$P = P(\varepsilon)$: state equation (of « barotropic fluids »)

Dust : $P = 0 \quad \rightarrow \quad \partial_\alpha\left(\sqrt{-g}\,\varepsilon u^\alpha\right) = 0 \quad \& \quad u^\beta \nabla_\beta u^\alpha = 0 \quad (\text{geodesics})$

Radiation (photon gas) or **ultra-relativistic particles** :

$$P = \varepsilon/3 \quad \rightarrow \quad \partial_\alpha\left(\sqrt{-g}\,\varepsilon^{3/4} u^\alpha\right) = 0 \quad \& \quad u^\beta \nabla_\beta u^\alpha = -\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\partial_\beta \ln \varepsilon^{1/4}$$

Care : u = quadri-velocity of the photon gas, not related to velocity of light !

Vacuum/cosmological constant : $P = -\varepsilon \quad \rightarrow \quad T^{\alpha\beta} = -\varepsilon g^{\alpha\beta} \quad \rightarrow \quad \partial\varepsilon = 0$

Stiff matter : $P = +\varepsilon \quad (\text{sound vel} = c) \quad \rightarrow \quad \ldots\ldots$

Linear barotropic fluids :

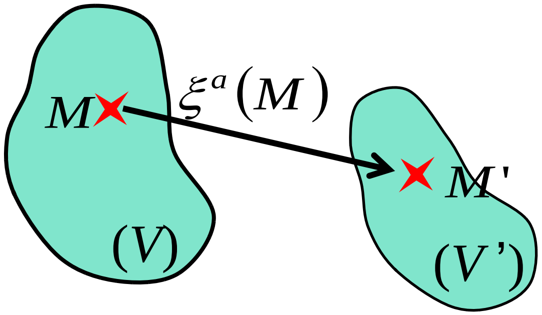
B. fl. with linear eq of state : $P = \lambda\varepsilon$ (λ = 0 , 1/3 , –1 , +1 , ...)

$$\partial_\alpha\left(\sqrt{-g}\,\varepsilon^{1/(1+\lambda)} u^\alpha\right) = 0 \quad \& \quad u^\beta \nabla_\beta u^\alpha = -\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\partial_\beta \ln \varepsilon^{\lambda/(1+\lambda)}$$

III – Symmetries, conservation laws, laws of conservation

Consider : - an integration domain (*V*)

- an infinitesimal mapping ξ : *M* → *M*’ : $x^a(M') = x^a(M) + \xi^a[x(M)]$
- (*V*’) the image of (*V*) by ξ



and the 2 integrals :

$$\begin{cases} A = \int_{(V)} l(....)d^4x \\ \& \\ A' = \int_{(V')} l(....)d^4x \end{cases}$$

* **One has shown that (sect. II-3) :**

ALWAYS !!!

- if *l* is **a part** (a priori) **of the lagrangian** of the theory
- if A = scalar & (*V*) = the **full space-time** thence *V*’ = *V* & $\forall \xi^a\ ,\ A' = A$

$x' = x + \xi$
as a diffeomorphism

theory covariance !

if the integrand of *A*’-*A* takes the form $(...)_a \xi^a$

A **vector** identity

stress-energy tensor(s) « conservation »

* **Consider now another situation,** where :

sometimes …

- the **full lagrangian** is into consideration
- $\exists \xi^a$ such that $\forall (V)$ one has : $A' = A$

Compute A'-A as an integral over (V)

Get one **scalar** relation (symmetry, Noether theorem, one par ξ)

Integrand $= 0$

Symmetries do not always exist !!!

Noether theorem : **one way** to deal with symmetries

Another way : Killing vectors

These 2 ways are not strictly equivalent !!!

refers to lagrangian symmetries

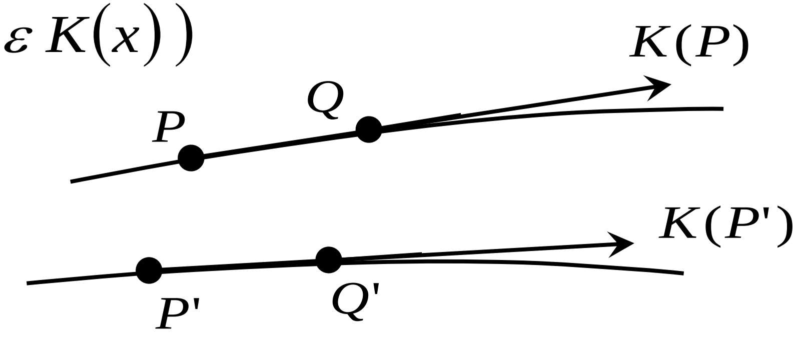
refers to space-time symmetries (isometries)

Let us have a closer look on Killing vector fields (rather than Noether) ….

1 – Isometries & Killing vectors

Let us consider a vector field *K* & an infinitesimal number ε

and the mapping $P(x) \rightarrow Q(x + \varepsilon K(x))$



Consider 2 close points *P* & *P*'
and their transformed *Q* & *Q*'

This mapping is an isometry iff : $\forall\ P\ \&\ P' \quad ds^2_{QQ'} = ds^2_{PP'}$

and *K* is the corresponding Killing vector

$$\forall\ x\ \&\ dx: \underbrace{g_{ab}(x+\varepsilon K(x))\big(dx^a + \varepsilon K^a(x+dx) - \varepsilon K^a(x)\big)(\ldots\text{idem}\quad a \leftrightarrow b\ldots)}_{ds^2_{QQ'}} = \underbrace{g_{ab}(x)\,dx^a dx^b}_{ds^2_{PP'}}$$

Develop & just keep 1rst order ε terms

$$\nabla_a K_b + \nabla_b K_a = 0$$

2 – Killing vs conservations laws

$\nabla_a T^{ab} = 0$ is usually refered as a conservation law ….but ….

….while $\partial_a W^{a\ldots} = 0$ **IS** a conservation law in the (integral) physical meaning

$$\partial_0 W^{0\ldots} + \partial_k W^{k\ldots} = 0 \longrightarrow \frac{d}{dt}\int_{(Vol)} W^{0\ldots} dxdydz = -\oint_{(\partial Vol)} W^{k\ldots} dS_k$$

$\nabla_a T^{ab} = 0$ does (generally) not resort such an interpretation !!!

BUT : if there is a Killing vector

$$\underbrace{\nabla_a \left(T^{ab} K_b\right)}_{=\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,T^{ab}K_b\right)} = \underbrace{K_b \nabla_a T^{ab}}_{=0} + \underbrace{T^{ab}\nabla_a K_b}_{=\frac{1}{2}T^{ab}\underbrace{(\nabla_a K_b + \nabla_b K_a)}_{=0}} = 0$$

$\partial_a\left(\sqrt{-g}\,T^{ab}K_b\right) = 0$ ie a "true" conserv. law !!!

involves the gravitational field

Stationary field : existence of a time-like Killing vector → ***energy*** conservation

Geometric gravity :

General Relativity

& sisters

IV – General relativity & ways to alternatives

The GR lagrangian reads $\ell = \sqrt{-g}R + 2\sqrt{-g}L_{NG}\left(\Psi; g_{ab}, \partial g_{ab}; \Gamma\right)$

Thence its variation $\delta\ell = R_{ab}\delta\left(\sqrt{-g}g^{ab}\right) + \sqrt{-g}g^{ab}\delta R_{ab} + 2\delta\left(\sqrt{-g}L_{NG}\right)$

First geometric term

$$\delta\left(\sqrt{-g}g^{ab}\right) = \sqrt{-g}\left(\delta g^{ab} - \frac{1}{2}g^{ab}g_{cd}\delta g^{cd}\right) \rightarrow R_{ab}\delta\left(\sqrt{-g}g^{ab}\right) = \sqrt{-g}\left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\delta g^{ab}$$

Second geometric term : let us use connection's (Γ !!!) geodesic coordinates

$$\delta R_{ab} = \partial_c \delta\Gamma^c_{ab} - \partial_b \delta\Gamma^c_{ac} \rightarrow \sqrt{-g}g^{ab}\delta R_{ab} = \sqrt{-g}g^{ab}\partial_c\delta\Gamma^c_{ab} - \sqrt{-g}g^{ab}\partial_b\delta\Gamma^c_{ac}$$

CARE : <u>NOT</u> metric's connection geodesics coordinates !!! → $\partial g_{ab} \neq 0$ **!!!!!!**

Nevertheless, one can note that (using Γ = 0) :

$$\sqrt{-g}g^{ab}\delta R_{ab} = \partial_c\left(\sqrt{-g}g^{ab}\delta\Gamma^c_{ab}\right) - \partial_b\left(\sqrt{-g}g^{ab}\delta\Gamma^c_{ac}\right) - \delta\Gamma^c_{ab}\nabla_c\left(\sqrt{-g}g^{ab}\right) + \delta\Gamma^c_{ac}\nabla_b\left(\sqrt{-g}g^{ab}\right)$$

tensorial equality : ok in one coord system → **ok in all coord**

…and since exact divergence terms do not contribute

$$\delta\ell = \sqrt{-g}\,E_{ab}\delta g^{ab} - \left\{ \nabla_c\left(\sqrt{-g}\,g^{ab}\right) - \delta_c^b \nabla_e\left(\sqrt{-g}\,g^{ae}\right) \right\} \delta\Gamma^c_{ab} + 2\delta\left(\sqrt{-g}\,L_{NG}\right)$$

with $E_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$

Now, remark that : $\nabla_c\left(\sqrt{-g}\,g^{ab}\right) = 0 \quad \Leftrightarrow$ connection = Christoffel's

…and conclude that **first & second order formalisms → same field equations**

…as soon as the **connection does not enter the non-gravitational lagrangian**

Remark this last conclusion is indebted to :

- lagrangian depends linearly on Ricci → ~~***f*****(*****R*****) theories, pure geom alternatives**~~
- no field (dynamical or not) in front of Ricci → ~~**scalar-tensor theories**~~
- (once more …) **matter does not couple with the connection**

(usual) GR field equations :

$$E_{ab} = \overset{(1)}{T}_{ab} + \overset{(2)}{T}_{ab} + \dots$$

$$\nabla_a \overset{(1)}{T}{}^{ab} = 0 \quad \& \quad \nabla_a \overset{(2)}{T}{}^{ab} = 0 \quad \& \quad \dots$$

compatibility ensured thanks to Bianchi !!!

$$\nabla_a \left[\overset{(1)}{T}{}^{ab} + \overset{(2)}{T}{}^{ab} + \dots \right] = 0$$

This suggests some « minimal » alternative ways to gravity :

- lagrangian non-linearly Ricci dependent
  → ***f*(*R*) theories, squared Ricci, ...**

- metric(s) + …
  → **scalar-tensor theories (& vector-tensor, bimetric, …)**

first & second order versions

- GR lagrangian vs **first order** formalism **& matter/Γ coupling**

V – f(R) theories

Lagrangian having the form $\ell = \sqrt{-g}\, f(R) + 2\sqrt{-g}\, L_{NG}(\Psi; g_{ab}, \partial g_{ab}; \Gamma)$

Variation $\delta\ell = f\delta\left(\sqrt{-g}\right) + \sqrt{-g}\, f'\delta\left(g^{ab}R_{ab}\right) + 2\delta\left(\sqrt{-g}\, L_{NG}\right)$

$$\delta\ell = \sqrt{-g}\left(f' R_{ab} - \frac{1}{2} f g_{ab}\right)\delta g^{ab} + f' \sqrt{-g}\, g^{ab}\delta R_{ab} + 2\delta\left(\sqrt{-g}\, L_{NG}\right)$$

If $f(R) = R \rightarrow$ back to GR ($f(R) = R + \Lambda \rightarrow$ back to ΛGR)

The following of the story changes **depending on 1rst/2cd order** approach if *f* not linear/affine (event if matter lagrangian not explicitly Γ-dependent)

Indeed, let's use Γ's geodesic coordinates and get

$$f'\sqrt{-g}\, g^{ab}\delta R_{ab} = -\delta\Gamma^{c}_{ab}\nabla_c\left(f'\sqrt{-g}\, g^{ab}\right) + \delta\Gamma^{c}_{ac}\nabla_b\left(f'\sqrt{-g}\, g^{ab}\right) + \text{div terms}$$

that may also be written, using the **conformal metric** $\bar{g}_{ab} = f' g_{ab}$

$$f'\sqrt{-g}\, g^{ab}\delta R_{ab} = -\delta\Gamma^{c}_{ab}\nabla_c\left(\sqrt{-\bar{g}}\, \bar{g}^{ab}\right) + \delta\Gamma^{c}_{ac}\nabla_b\left(\sqrt{-\bar{g}}\, \bar{g}^{ab}\right) + \text{div terms}$$

$$\delta\ell = \sqrt{-g}\left(f' R_{ab} - \frac{1}{2} f g_{ab}\right)\delta g^{ab} - \left\{\underbrace{\nabla_c\left(f'\sqrt{-g}\,g^{ab}\right)}_{=\nabla_c\left(\sqrt{-\bar{g}}\,\bar{g}^{ab}\right)} - \delta_c^b \nabla_e\left(f'\sqrt{-g}\,g^{ae}\right)\right\}\delta\Gamma_{ab}^c + 2\delta\left(\sqrt{-g}\,L_{NG}\right)$$

First order theory (if matter lagrangian not Γ-dependent)

$$\nabla_c\left(\sqrt{-\bar{g}}\,\bar{g}^{ab}\right) - \delta_c^b \nabla_e\left(\sqrt{-\bar{g}}\,\bar{g}^{ae}\right) = 0 \;\rightarrow\; \nabla_c\left(\sqrt{-\bar{g}}\,\bar{g}^{ab}\right) = 0$$

Γ= Christoffel's connection of **conformal metric** (not the metric g)

$$\Gamma_{ab}^c = \frac{1}{2}\bar{g}^{ce}\left(\partial_a \bar{g}_{be} + \partial_b \bar{g}_{ae} - \partial_e \bar{g}_{ab}\right) = \frac{1}{2} g^{ce}\left(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab}\right) + \frac{1}{2}\left(\delta_b^c \partial_a + \delta_a^c \partial_b - g^{ce} g_{ab} \partial_e\right)\ln f'$$

…with the metric-field equation $f'(R) R_{ab} - \frac{1}{2} f(R) g_{ab} = T_{ab}$ (second order PDE)

…& all the usual matter equations (including all stress-energy divergences = 0)

Second order theory

δΓ from the metric's variation in geod frame → $\delta\Gamma_{ab}^c = \frac{1}{2} g^{ce}\left(\partial_a \delta g_{be} + \partial_b \delta g_{ae} - \partial_e \delta g_{ab}\right)$

Do some algebraic manipulations, discard div terms …, and finally get the field equation

$$f' R_{ab} - \frac{1}{2} f g_{ab} - \nabla_a \partial_b f' + g_{ab} g^{ce} \nabla_c \partial_e f' = T_{ab}$$

…& all the usual matter equations (including all stress-energy divergences = 0)

First order theory ← lagrangian depends on **1rst order deriv** of dyn grav fields

second order fields equations

$$\nabla_c\left(f'(R)\sqrt{-g}\,g^{ab}\right)=0$$

$$f'(R)R_{ab}-\frac{1}{2}f(R)g_{ab}=T_{ab}$$

Second order theory ← lagrangian depends on **2cd order deriv** of dyn grav fields

fourth order fields equations

$$f'R_{ab}-\frac{1}{2}fg_{ab}-\nabla_a\partial_b f'+g_{ab}g^{ce}\nabla_c\partial_e f'=T_{ab}$$

fourth order terms

$\partial\partial\partial\partial\, g_{ab}$

…unless $f' = \text{cst}$ ↔ GR with/without cosmological constant
(very exceptional case !!!)

Should drastically change the properties of solutions (w.r.t. GR) …
…but maybe not so much as could be expected … (scalar-tensor equivalence)

As many $f(R)$ theories as functions f …

…

… **you can choose f in such a way the Robertson-Walker scale factor fits any given cosmology** (accelerated or not !!!)

→ does it make sense invoking $f(R)$ gravity to « explain » the accelerated expansion ???

→ at minima : the required $f(R)$ should be tested by other means !!!

VI – Other purely metric alternatives vs Ostrogradsky theorem

Purely geom gravity : a priori not limited to $f(R)$!!!

$$\ell = \sqrt{-g}\,F\left(R, \partial R, R_{ab}R^{ab}, R_{abcd}R^{abcd}, Weyl, \ldots\right) + 2\sqrt{-g}\,L_{NG}\left(\Psi; g_{ab}, \partial g_{ab}\right)$$

(consider 2d order formalism) → 4th order PDEs (except $F = R + cst$)

Digression : the Ostrogradskian instability

First order lagrangian $L_F(q, q')$

$$\underbrace{\frac{d}{dt}\frac{\partial L_F}{\partial q'} = \frac{\partial L_F}{\partial q}}_{\text{Euler-Lagrange}} \quad \& \quad \underbrace{E_F = -L_F + q'\frac{\partial L_F}{\partial q'}}_{\text{« energy » (Noether)}}$$

Second order lagrangian $L_S(q, q', q'')$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L_S}{\partial q'} - \frac{d}{dt}\frac{\partial L_S}{\partial q''}\right) = \frac{\partial L_S}{\partial q}}_{\text{Euler-Lagrange}} \quad \& \quad \underbrace{E_S = -L_S + q'\left(\frac{\partial L_S}{\partial q'} - \frac{d}{dt}\frac{\partial L_S}{\partial q''}\right) + q''\frac{\partial L_S}{\partial q''}}_{\text{« energy »}}$$

Lagrange equation → evolution → dynamical system form $\frac{d\vec{X}}{dt} = \vec{f}\left(\vec{X}\right)$

First order case $\frac{d}{dt}\frac{\partial L_F}{\partial q'} = \frac{\partial L_F}{\partial q} \rightarrow P = \left(\frac{\partial L_F}{\partial q'}\right)_q$ —nondegeneracy→ P depends on q'

Thence

- $q' = A(q,P)$
- E-L = 2cd order

$$\frac{dP}{dt} = \left(\frac{\partial L_F}{\partial q}\right)_{q'} = f_P(q,P) \quad \& \quad \frac{dq}{dt} = A(q,P)$$

Hamilton's proposal (comply with some tradition …)

Define $H_F(q,P) = PA(q,P) - L_F(q, A(q,P))$

$$\frac{dP}{dt} = -\left(\frac{\partial H_F}{\partial q}\right)_P \quad \& \quad \frac{dq}{dt} = \left(\frac{\partial H_F}{\partial P}\right)_q$$

Energy (numerically = Hamiltonian) $E_F = -L_F(q, A(q,P)) + PA(q,P)$

Complex (non-linear !) *P*-dependance

(may be bound…)

Second order case

$$\frac{d}{dt}\left(\frac{\partial L_S}{\partial q'} - \frac{d}{dt}\frac{\partial L_S}{\partial q''}\right) = \frac{\partial L_S}{\partial q} \quad \rightarrow \quad P_1 = \left(\frac{\partial L_S}{\partial q'}\right)_{q,q''} - \frac{d}{dt}\left(\frac{\partial L_S}{\partial q''}\right)_{q,q'}$$

$$P_2 \equiv \left(\frac{\partial L_S}{\partial q''}\right)_{q,q'} \xrightarrow{\text{nondegeneracy}} P_2 \text{ depends on } q''$$

Thence

- $q'' = B(q, q', P_2)$
- E-L = 4rth order

$$\frac{dq}{dt} = q' \quad \& \quad \frac{dq'}{dt} = B(q, q', P_2)$$

$$\& \quad \frac{dP_1}{dt} = \left(\frac{\partial L_S}{\partial q}\right)_{q',q''} \quad \& \quad \frac{dP_2}{dt} = \left(\frac{\partial L_S}{\partial q'}\right)_{q,q''} - P_1$$

as fct of $(q, q', B(q, q', P_2))$

…or any other form, like Hamilton's, but…

… is that really mandatory to comply with some tradition for understanding physics ?

$$E_S = -L_S(q, q', B(q, q', P_2)) + q' P_1 + P_2 B(q, q', P_2)$$

Complex in q, q', P_2, but … **LINEAR in P_1 !!!** **energy is unbound** (up & below !)

➔ **Interactions** with other systems **leads to** (ostrogradskian) **instability !!!**

Some examples

Example 1 : consider a system described by (harmonic oscillator)

$$L(q,q')=\frac{1}{2}mq'^2-\frac{1}{2}m\omega^2q^2 \;\;\rightarrow\;\; E=\frac{1}{2}mq'^2_0+\frac{1}{2}m\omega^2q_0^2 \geq E_{\min} \quad \text{(lower bound)}$$

→ If several lower energy bounded systems are **interacting** with it, no part of the global system may arbitrarily increase its energy because of global energy conservation

Example 2 : consider a system described by (« second order harmonic oscillator »)

$$L(q,q')=\frac{1}{2}\beta mq''^2+\frac{1}{2}mq'^2-\frac{1}{2}m\omega^2q^2 \;\;\rightarrow\;\; E=f(\text{init. cond.})$$

…& find the **energy is unbound** (from up & below !) **whatever the sign of β !!!**

Example 3 : consider a system described by the lagrangian

$$L=\frac{1}{2}\alpha mq'^3-\frac{1}{2}m\omega^2q^2 \;\;\rightarrow\;\; E=\alpha mq'^3_0+\frac{1}{2}m\omega^2q_0^2 \quad (\text{unbound since } q'^3 > \text{or} < 0)$$

→ thence may exchange energy with no limit …

→ **First order is not a garanty against instability !!!**

…but… do you know a physical system resorting such a lagrangian ???

Question : **a way** for a 2cd order lagrangian **to escape** the ostrogradskian **instability ?**

The key point for unstability $P_2 \equiv \left(\frac{\partial L_S}{\partial q''}\right)_{q,q'} \rightarrow q'' = B(q, q', P_2)$

→ the only hope to escape Ostrogradskian instability is the case where the second order lagrangian is such that the so-defined P_2 cannot be inverted with respect to q'', and as far as we know,

that is the only way to escape this instability !!!

end of the digression , and let us be back to alternative metric gravity (& GR)

The GR lagrangian reads $\ell = \sqrt{-g}R + (\text{matter})$

If just metric as a field, *R* depends on metrics, 1rst & 2cd derivatives (terminology !)

from $\partial\Gamma$ terms

But, « false » 2cd order lagrangian ! Indeed

$$\partial\Gamma \xrightarrow{\ell=\sqrt{-g}R} \sqrt{-g}\,g^{**}\partial_*\Gamma^*_{**} = \partial_*\left(\sqrt{-g}\,g^{**}\Gamma^*_{**}\right) - \Gamma^*_{**}\partial_*\left(\sqrt{-g}\,g^{**}\right)$$

div terms → no effect

(squared) first order

→ **GR not concerned** by Ostrogradskian instability at all …

…but **the problem stands for alternatives** ($f(R)$, squared Ricci, …) **!!!!!!**

$$f(R) \xrightarrow{\ell=\sqrt{-g}\,f(R)} \sqrt{-g}\,f\left(g^{**}\partial_*\Gamma^*_{**} - .... + g^{**}\Gamma^*_{**}\Gamma^*_{**} + ...\right) = ..$$

Other geom theories .... even worse !!!

It turns out that :

- $f(R)$ theories are Ostrogradsky-stable …
- …**and that's all !!!** (in the pure (one) metric family !)

This is related to the fact R involves the second derivative of just one metric's « effective component »… (another interpretation later)

…while this is not the case considering

$$g^{ab}\partial_a R \partial_b R\,,\ R_{ab}R^{ab}\,,\ R_{abcd}R^{abcd}\,,\ C_{abcd}C^{abcd}\,,\ ...$$

VII – Scalar-tensor theories

$$\ell = \sqrt{-g}\left[F(\phi)R + G(\phi)g^{ab}\partial_a\phi\partial_b\phi + \lambda(\phi)\right] + 2\sqrt{-g}\,L_{NG}(\Psi; g_{ab}, \partial g_{ab}; \Gamma; \phi)$$

Definition of ϕ → one could choose $F(\phi) = \phi$, or …

➔ Let us decide later and continue with 3 functions F, G & λ

Matter's eq of motion ← $L_{NG}(\Psi; g_{ab}, \partial g_{ab}; \Gamma; \phi)$

Matter couples with : geometry ($g_{ab}, \partial g_{ab}; \Gamma$) scalar field ($\phi$)

If just matter-geometry coupling required (FFU) → $L_{NG}(\Psi; g_{ab}, \partial g_{ab}; \Gamma)$

If just matter-metric coupling required (for 1rst order) → $L_{NG}(\Psi; g_{ab}, \partial g_{ab})$

Let us consider the lagrangian

$$\ell = \sqrt{-g}\left[F(\phi)R + G(\phi)g^{ab}\partial_a\phi\partial_b\phi + \lambda(\phi)\right] + 2\sqrt{-g}\,L_{NG}(\Psi; g_{ab}, \partial g_{ab})$$

Geometric part : **scalar curvature coupled with the scalar field**

1rst or 2cd order ? → 2 different theories !

Let us ***choose*** the **2cd order formalism** (ie comply with a tradition, but being fully aware of that !!!)

1 – Field equations

Vary ϕ → one get, up to a divergence term

$$\delta\ell = \sqrt{-g}\left[F'R + G'(\partial\phi)^2 - 2\sqrt{-g}^{\,-1}\partial_b\left(G\sqrt{-g}\,g^{ab}\partial_a\phi\right) + \lambda'\right]\delta\phi$$

Thence the field equation $\quad 2Gg^{ab}\nabla_a\partial_b\phi + G'(\partial\phi)^2 - F'R - \lambda' = 0$

Now vary the metric

$$\delta\ell = F\delta\left(\sqrt{-g}R\right) + G\partial_a\phi\partial_b\phi\underbrace{\delta\left(\sqrt{-g}\,g^{ab}\right)}_{\text{ok} \ldots} + \lambda\underbrace{\delta\sqrt{-g}}_{\text{ok} \ldots} + 2\underbrace{\delta\left(\sqrt{-g}L_{NG}\right)}_{= -\frac{1}{2}\sqrt{-g}\,T_{ab}\delta g^{ab}}$$

$$\delta\ell = F\delta\left(\sqrt{-g}R_{ab}g^{ab}\right) + \sqrt{-g}\left[G\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}G(\partial\phi)^2 - \frac{\lambda}{2}g_{ab} - T_{ab}\right]\delta g^{ab}$$

$$= R_{ab}\delta\left(\sqrt{-g}g^{ab}\right) + \sqrt{-g}g^{ab}\delta R_{ab} = \sqrt{-g}E_{ab}\delta g^{ab} + \sqrt{-g}g^{ab}\delta R_{ab}$$

$$\delta\ell = F\sqrt{-g}g^{ab}\delta R_{ab} + \sqrt{-g}\left[FE_{ab} + G\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}G(\partial\phi)^2 - \frac{\lambda}{2}g_{ab} - T_{ab}\right]\delta g^{ab}$$

Do some algebra for Ricci variation ……and get the field equations

$$FE_{ab} - \nabla_a\partial_b F + g_{ab}g^{ce}\nabla_c\partial_e F + G\left[\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}(\partial\phi)^2\right] - \frac{\lambda}{2}g_{ab} = T_{ab}$$

with $\nabla_a\partial_b F = F'\nabla_a\partial_b\phi + F''\partial_a\phi\partial_b\phi$

Contraction → $-FR + 3g^{ce}\nabla_c\partial_e F - G(\partial\phi)^2 - 2\lambda = T$

eliminate R

Scalar equation $\left(3F'^2 - 2FG\right)g^{ab}\nabla_a\partial_b\phi + \left(3F'F'' - (FG)'\right)(\partial\phi)^2 - 2F'\lambda + F\lambda' = F'T$

Let us now benefit the definition of ϕ **freedom**, and choose $F(\phi) = \phi$

$$\phi E_{ab} - \nabla_a \partial_b \phi + g_{ab} g^{ce} \nabla_c \partial_e \phi + G\left[\partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} (\partial\phi)^2\right] - \frac{\lambda}{2} g_{ab} = T_{ab}$$

$$(3 - 2\phi G) g^{ab} \nabla_a \partial_b \phi - (\phi G)' (\partial\phi)^2 - 2\lambda + \phi\lambda' = T$$

$F(\phi) = \phi$ **and** no ϕ in the matter lagrangian → ST theory in **Jordan's representation**

Let us read $G(\phi)$ under the (Brans-Dicke) form $G(\phi) = -\frac{\omega(\phi)}{\phi}$

and $\lambda(\phi) = -2\phi U(\phi)$ read the field equations as

$$E_{ab} + U g_{ab} = \frac{T_{ab}}{\phi} + \frac{\omega}{\phi^2}\left[\partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} (\partial\phi)^2\right] + \frac{1}{\phi}\left(\nabla_a \partial_b \phi - g_{ab} g^{ce} \nabla_c \partial_e \phi\right)$$

$$(3 + 2\omega) g^{ab} \nabla_a \partial_b \phi + \omega' (\partial\phi)^2 + 2\phi U - 2\phi^2 U' = T$$

2 – ST vs GR

Consider the Brans-Dicke (BD) case, with no potential (ω = cst & $\lambda = 0$)

$$\ell = \sqrt{-g}\left[\phi R - \frac{\omega}{\phi} g^{ab} \partial_a \phi \partial_b \phi\right] + 2\sqrt{-g} L_{NG}(\Psi; g_{ab}, \partial g_{ab})$$

Remark that for solutions such that $\phi = \text{cst}$ the lagrangian reduces to GR. But considering the scalar equation (BD case)

$$g^{ab}\nabla_a \partial_b \phi = \frac{T}{3+2\omega}$$

one sees this requires $\omega = \infty$ (at least if $T \neq 0$)

ie the BD lagrangian reduces to GR in the $\omega \to \infty$ limit. However, let us remark that the situation is not so obvious, as the scalar kinetic term got undetermined, as $\omega \times 0$.

In fact, the $\omega \to \infty$ limit leads back to GR, but **plus a massless scalar field** (that may be zero) up to the sources Ψ present from the start. This has **no trivial consequences !**

For instance, let us consider that :

- cosmological measurements result in the value H_0 of the Hubble parameter ;
- we want to constrain the age of a dust-filled Universe from H_0, considering the Universe is euclidean RW.

Our aim is to compare the constraints got in the framework of :

- GR
- BDT with $\omega >> 1$ ($\omega = \infty$ in the limit case)

If large ω BDT is equivalent to GR, the contraints should be the same.
If the constraints are different, this means that large ω BDT is not equivalent to GR.

In **GR**, the RW scale factor reads $a_{GR}(t) \propto t^{2/3} \rightarrow \dfrac{d\ln a}{dt} = 2/(3t)$

in such a way H_0 values **fixes the age of the Universe** $T_{GR} = \dfrac{2}{3H_0}$

In **BDT**, the RW scale factor reads $a_{\omega BD}(t) \propto t^{q}(t+C)^{2\frac{1+\omega}{4+3\omega}-q}$

where C is an integration constant, and $q = \dfrac{s\sqrt{3+2\omega}-\sqrt{3}}{3s\sqrt{3+2\omega}-\sqrt{3}} \quad (s=\pm)$

In the case where $C > 0$, the resulting age of the Universe reads

$$T_{\omega BD} = \frac{1}{H_0}\left[\frac{1+\omega}{4+3\omega} - CH_0 + \sqrt{\left(\frac{1+\omega}{4+3\omega} - CH_0\right)^2 + 2\,\frac{s\sqrt{3+2\omega}-\sqrt{3}}{3s\sqrt{3+2\omega}-\sqrt{3}}\,CH_0}\,\right]$$

This shows that for large ω, one has asymptotically

$$T_{\omega BD} = \frac{1}{3H_0}\left(1-b+\sqrt{1+b^2}\right) \quad \text{with } b = \text{positive(integr.) cst}$$

in such a way it is always possible to **adjust *b* (ie *C*) so that $T_{\omega BD}$ takes any value in the interval**

$$\left[\frac{1}{3H_0}, \frac{2}{3H_0}\right] \quad \text{ie} \quad \left[\frac{1}{2}T_{GR}, T_{GR}\right] \quad (\text{while } \phi = cst)$$

3 – Conformal transformations

A conformal transform (CT) is a **dependent** variable transform having the form

$$\bar{g}_{ab} = F\left(x^c\right) g_{ab} \qquad \text{with} \qquad F > 0$$

→ **not** a coordinate change !!!
→ it changes the geometry, but not the related causal structure

$$\bar{g}^{ab} = F^{-1} g^{ab} \quad \& \quad \sqrt{-\bar{g}} = F^2 \sqrt{-g}$$

$$\bar{\Gamma}^c_{ab} = \Gamma^c_{ab} + \Delta^c_{ab}(F) \quad \text{with} \quad \Delta^c_{ab}(F) = \frac{1}{2}\left(\delta^c_b \partial_a + \delta^c_a \partial_b - g_{ab} g^{ce} \partial_e\right) \ln F$$

$$\bar{R} = F^{-1}\left(R - 3 g^{ab} \nabla_a \partial_b \ln F - \frac{3}{2} g^{ab} \partial_a \ln F \partial_b \ln F\right)$$

or, reciprocally

$$g_{ab} = F^{-1} \bar{g}_{ab} \quad \& \quad g^{ab} = F \bar{g}^{ab} \quad \& \quad \sqrt{-g} = F^{-2} \sqrt{-\bar{g}}$$

$$R = F\left(\bar{R} + 3 \bar{g}^{ab} \bar{\nabla}_a \partial_b \ln F - \frac{3}{2} \bar{g}^{ab} \partial_a \ln F \partial_b \ln F\right)$$

The lagrangian $\ell = \sqrt{-g}\left[\phi R - \frac{\omega}{\phi} g^{ab}\partial_a\phi\partial_b\phi - 2\phi U\right] + 2\sqrt{-g}\,L_{NG}(\Psi; g_{ab}, \partial g_{ab})$

thus reads, if one chooses $F = \phi$ (discard div terms)

$$\ell = \sqrt{-\bar{g}}\left[\bar{R} - \left(\omega + \frac{3}{2}\right)(\bar{\partial}\ln\phi)^2 - 2\frac{U}{\phi}\right] + 2\phi^{-2}\sqrt{-\bar{g}}\,L_{NG}\left(\Psi; \phi^{-1}\bar{g}_{ab}, \partial\left(\phi^{-1}\bar{g}_{ab}\right)\right)$$

➔ this is GR lagrangian (terminology : TS theory in Einstein representation), but

(1) + a scalar field (not initially present in the matter content)

(2) (originally present) matter now couples with the scalar field → ~~EFU~~

Remark the scalar kinetic term's sign change if ω > or < –3/2

Ostrogradsky stability requires **ω > –3/2**

Einstein's representation :

- (math) sometimes leads to easier calculations (simpler field equations) ….
- (phys) ….be aware that geodesics do not represent free motions (in this « frame »)

VIII – *f*(*R*) vs scalar-tensor

The idea :

- *f*(*R*) gravity → metric + one (free) function *f*
- ST gravity → metric + scalar field + 2 (free) functions ω & *U*

→ one could suspect it **possible to interpret *f*(*R*) as a subset of ST** …(?)

…& even that there should exist **many solutions !!!** (depending on the way to define a scalar field with the help of *f*, …)

(second order) *f*(*R*) gravity → $f' R_{ab} - \frac{1}{2} f g_{ab} = T_{ab} + \nabla_a \partial_b f' - g_{ab} g^{ce} \nabla_c \partial_e f'$

The rhs of this form of the field equation strongly reminds ST with **ω=0 !!!**...

$$\phi\left(E_{ab} + U g_{ab}\right) = T_{ab} + \nabla_a \partial_b \phi - g_{ab} g^{ce} \nabla_c \partial_e \phi$$

…& suggests the scalar field definition $\phi = f'(R)$

$$R_{ab} - \frac{1}{2} R g_{ab} + \frac{1}{2\phi}(R\phi - f) g_{ab} = \frac{T_{ab}}{\phi} + \frac{1}{\phi}\left(\nabla_a \partial_b \phi - g_{ab} g^{ce} \nabla_c \partial_e \phi\right)$$

The identification with the (*ab*) ST equation suggests to the potential definition

$$U = \frac{1}{2\phi}\left(R\phi - f\right) \quad \text{with} \quad \phi = f'\left(R\right)$$

The equivalence with the corresponding ST theory requires recovering also the scalar field equation, It should come from the (*ab*) equation + U & ϕ definitions

Contract the (*ab*) equation $\quad 3g^{ab}\nabla_a\partial_b\phi + 4\phi U - \phi R = T$

& remark that $\quad 2\phi U = R\phi - f \;\rightarrow\; 2Ud\phi + 2\phi dU = Rd\phi + \left[\phi - f'\left(R\right)\right]dR = Rd\phi$

eliminate R

…and thus be back to $\quad 3g^{ab}\nabla_a\partial_b\phi + 2\phi U - 2\phi^2 U' = T$

Morality : $f(R)$ gravity has been put under ($\omega = 0$)-ST form

in that sense, $f(R)$ is a subset of ST gravity

Remark : → $f(R)$ escapes the Ostrogradskian instability (consider E. representation)

IX – Non-dynamical scalar-tensor

Let us be back to the ST lagrangian, but consider the **scalar field is non-dynamical** , ie $\phi(x)$ is given a priori, from « external » considerations

→ the lagrangian

$$\ell = \sqrt{-g}\left[\phi R - \frac{\omega}{\phi} g^{ab}\partial_a\phi\partial_b\phi + \lambda\right] + 2\sqrt{-g}\,L_{NG}$$

has to be varied wrt metric and ψ fields only.

$$\begin{cases} E_{ab} - \dfrac{\lambda}{2\phi} g_{ab} = \dfrac{T_{ab}}{\phi} + \dfrac{\omega}{\phi^2}\left[\partial_a\phi\partial_b\phi - \dfrac{1}{2} g_{ab}(\partial\phi)^2\right] + \dfrac{1}{\phi}\left[\nabla_a\partial_b\phi - g_{ab} g^{ce}\nabla_c\partial_e\phi\right] \\ \text{matter field equations} \quad \Rightarrow \quad \nabla_a T^{ab} = 0 \end{cases}$$

ie one equation less than in the full dynamical case ….

….but remember Bianchi identity → recover 1 equation ….

…. → the same number as in the full dynamical case ….but ….

….does it mean **back to the full dynamical case ???**

ie : are we able to recover the scalar field equation ???

→ Let us check carrefully, as a conscientious student ….

Take the field equation divergence, and get, using stress tensor conservation

$$E_{ab}\nabla^a\phi - \frac{1}{2}\nabla_b\lambda = \left[\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}(\partial\phi)^2\right]\nabla^a\left(\frac{\omega}{\phi}\right) + \frac{\omega}{\phi}\partial_b\phi\nabla^a\partial_a\phi + (\nabla_a\nabla_b - \nabla_b\nabla_a)\partial^a\phi$$

Use the covariant derivative commutation rule and get

$$\left[R + \frac{d\lambda}{d\phi} + (\partial\phi)^2\frac{d}{d\phi}\left(\frac{\omega}{\phi}\right) + 2\frac{\omega}{\phi}\nabla^a\partial_a\phi\right]\partial_b\phi = 0$$

Now compare with the dynamical equation obtained varying ϕ, in the (usual) dynamical scalar case

$$R + \frac{d\lambda}{d\phi} + (\partial\phi)^2\frac{d}{d\phi}\left(\frac{\omega}{\phi}\right) + 2\frac{\omega}{\phi}\nabla^a\partial_a\phi = 0$$

….& conclude that **non-dynamical ST is equivalent to the reunion of :**

- **dynamical ST**
- **constant scalar field, ie GR (with cosmological constant)**

The meaning ?

It exactly means that

both ω–dynamical **ST & GR solutions**

are solutions of ω–non dynamical ST !!!

One knows :

- solar system's dynamics is well described by GR gravity
- cosmological expansion may be described in the framework of $f(R)$, thence ST, gravity

(Provocative !) question :

If one **strongly believe** (act of faith !!!) in **ST gravity**, could the **coexistence** in a same Universe (ours !!!) of **GR-like &** (far from GR) **ST-like solutions** be regarded as an **argument against the dynamical nature of the scalar field ?**