Anomaly-induced inflation and gravitational stability

Ilya L. Shapiro

Universidade Federal de Juiz de Fora, MG, Brazil

Partially supported by: CNPq, FAPEMIG, ICTP

Escola Prof. Plínio em Cosmologia - September 2016

Contents

- **Conformal anomaly and anomaly-induced effective action.**
- **Light massive fields case.**
- **Starobinsky model: unstable and stable versions.**
- **Transition from stable to unstable regimes.**
- **Generating a right-size R^2-term.**

Based on:

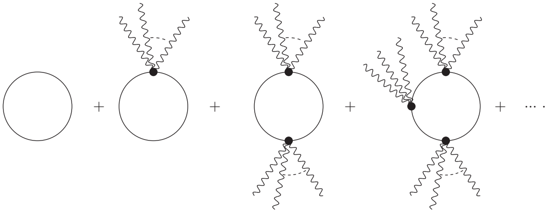
- *J. Fabris, A. Pelinson, I.Sh., Grav. Cosmol. (2000) , gr-qc/9810032; NPB (2001), hep-th/0009197];*
- *J. Fabris, A. Pelinson, I.Sh, F.Takakura, NPB (2003) hep-ph/0208184, NPB (PS) (2004) hep-ph/0311308;*
- *Tibério de Paula Netto, A. Pelinson, I.Sh, A.A. Starobinsky, Eur.J.Phys. to be published. arXiv:1509.08882.*

QFT in curved space

Books: *Birrell-Davies (1980); Buchbinder-Odintsov-I.Sh. (1992)*

Recent review: *I.Sh., Class.Q. Grav., gr-qc/0801.0216*

Relevant diagrams for the vacuum sector



All possible covariant counterterms have the same structure as

$$S_{vac} = S_{EH} + S_{HD}\,, \qquad S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R + 2\Lambda)\,,$$

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1C^2 + a_2E + a_3\Box R + a_4R^2\}\,.$$

Conformal anomaly

M.J. Duff, NPB (1977), see also Class. Quantum. Grav. (1994)

The Noether identity for the local conformal symmetry

$$\left[-2\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\,\Phi\,\frac{\delta}{\delta\Phi}\right]\,S(g_{\mu\nu},\,\Phi)\;=\;0$$

produces on shell $$-\,\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}}\;=\;T_{(vac)}{}^{\mu}_{\mu}\;=\;T^{\mu}_{\mu}\;=\;0\,.$$

At quantum level $S_{vac}(g_{\mu\nu})$ **is replaced by the EA** $\Gamma_{vac}(g_{\mu\nu})$.

$$\langle T^{\mu}_{\mu}\rangle\;=\;\beta_1 C^2 + \beta_2 E + a'\Box R\,,$$

where $a' = \beta_3$**.**

$$\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} = \frac{1}{360(4\pi)^2}\begin{pmatrix} +3N_0 + 18N_{1/2} + 36N_1 \\ -N_0 - 11N_{1/2} - 62N_1 \\ +2N_0 + 12N_{1/2} - 36N_1 \end{pmatrix}$$

One can use $\langle T^\mu_\mu \rangle$ to obtain equation for the finite 1-loop EA

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} \;=\; \langle T^\mu_\mu \rangle \;=\; \frac{1}{(4\pi)^2}\, \big(\omega C^2 + bE + c\Box R \big)\,.$$

Riegert; Fradkin & Tseytlin, PLB-1984.

$$\Gamma_{ind} \;=\; S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2} \int_x R^2(x) + \int_x \Big\{ \frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+ \frac{\omega}{8\pi\sqrt{-b}}\,\psi\, C^2 + \varphi\,\Big[\frac{\sqrt{-b}}{8\pi}\,(E - \frac{2}{3}\,\Box R) - \frac{\omega}{8\pi\sqrt{-b}}\, C^2 \Big] \Big\}\,. \qquad (1)$$

Here $S_c[\bar{g}_{\mu\nu}] = S_c[g_{\mu\nu}]$ is an unknown conformal functional and

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,R_{;\mu}\,\nabla^\mu\,.$$

is Paneitz operator (1983). The solution (1) is exact in case $S_c[\bar{g}_{\mu\nu}]$ is irrelevant, e.g., for the FRW metrics.

The above form of EA is the best one for Γ_{ind}.
I.Sh. and A.Jacksenaev, Phys. Lett. B (1994)

Similar expression has been independently introduced by
P. Mazur & E. Mottola, 1997-1998.

The expression (1) **proved to be very useful and reliable instrument to explore quantum effects.**

In particular, it was used to classify vacuum states in the vicinity of the Schwarzschild black hole,

R. Balbinot, A. Fabbri & I.Sh., PRL **83***; NPB* **559** *(1999).*

The output fits pretty well with the standard results obtained by other methods, see. e.g.,
S.M. Christensen & S.A. Fulling, PRD 15 (1977).

Generalization for the Reissner-Nordstrom black hole,
P.R. Anderson, E. Mottola & R. Vaulin, PRD **76** *(2007).*

A similar example is related to the quantum contributions to the gravitational wave equation on the cosmological background. The result fits nicely with other calculational methods, see, e.g.,

A.A. Starobinski, JETP Lett. 30 (1979); Let.Astr.Journ. 9 (1983).

S.W. Hawking, T. Hertog, H.S. Reall, PRD (2001), hep-th/0010232.

These two examples show that one may expect to ignore S_c, even in the cases when it is supposed to be relevant.

- **Cosmological application: Starobinsky Model, which is based on quantum effects.**

Fischetti, Hartle and Hu, (1978);
Starobinsky, (1980-1983) ;
Mukhanov, Chibisov, (1982) ;
Anderson, Vilenkin, ... (1983-1986)
Hawking, Hertog and Real, (2001) .

Modified Starobinsky model

Fabris, Pelinson, Solà, I.Sh., ...

- **Cosmological Model based on the action**

$$S_{total} \;=\; -\,\frac{M_P^2}{16\pi}\,\int d^4x\sqrt{-g}\,(R+2\Lambda)+S_{matter}+S_{vac}+\bar{\Gamma}_{ind}\,.$$

Equation of motion for $a(t),\quad dt=a\,(\eta)\,d\eta,\quad k=0$

$$\frac{\ddddot{a}}{a}+\frac{3\dot{a}\dddot{a}}{a^2}+\frac{\ddot{a}^2}{a^2}-\left(5+\frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3}-\frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}-\frac{2\Lambda}{3}\right)\;=\;0\,,$$

$k=0$. **Particular solutions (Starobinsky, PLB-1980)**

$$a(t)\;=\;a_0e^{Ht}\,,$$

where Hubble parameter $H=\dot{a}/a$ **is**

$$H^2\;=\;-\,\frac{M_P^2}{32\pi b}\left(1\pm\sqrt{1+\frac{64\pi b}{3}\,\frac{\Lambda}{M_P^2}}\,\right).$$

Case of $\Lambda\neq 0$ **in** *A. Pelinson, I.Sh., F. Takakura, NPB-2003.*

For $0 < \Lambda \ll M_P^2$ **there are two solutions:**

$$H \approx \sqrt{\Lambda/3}\,; \qquad\qquad (IR)$$

$$H \approx \sqrt{-\frac{M_P^2}{16\pi b} - \frac{\Lambda}{3}} \approx \frac{M_P}{\sqrt{-16\pi b}}\,. \qquad\qquad (UV)$$

Perturbations of the conformal factor

$$\sigma(t) \to \sigma(t) + y(t).$$

The criterion for a stable (UV) **inflation is**

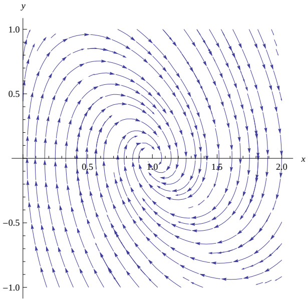
$$c > 0 \quad \Longleftrightarrow \quad N_1 < \frac{1}{3} N_{1/2} + \frac{1}{18} N_0\,,$$

in agreement with Starobinsky (1980).

The original Starobinsky model is based on the unstable case and involves special choice of initial data. This situation can be improved further by using the stable version and an appropriate transition scheme.

In the unstable phase there are very different solutions, some of them violent (hyperinflation). How can we know that the transition from stable to unstable phase really happens?

A. Pelinson et al, NPB(PS) (2003): **Phase portrait of a stable case:**



$$\text{Starobinsky (1980)}: \;\; x = \left(\frac{H}{H_0}\right)^{\frac{3}{2}}, \quad y = \frac{\dot{H}}{2\sqrt{H_0^3\,H}}, \quad dt = \frac{dx}{3H_0\,x^{2/3}\,y}.$$

Transition. **Suppose at UV** $(H \gg M_F)$ **there is SUSY, e.g., MSSM,**

$$N_1 = 12\,, \qquad N_{1/2} = 32\,, \qquad N_0 = 104\,.$$

This provides stable inflation, because

$$\frac{1}{3}\,N_{1/2} \,+\, \frac{1}{18}\,N_0 \,>\, N_1 \qquad \Longrightarrow \qquad c > 0\,.$$

For realistic SUSY model inflation is independent on initial data.

Fine!

But why should inflation end? **Already for MSM** $(\,N_{1,\,1/2,\,0} = 12, 24, 4)$, $c < 0$, **inflation is unstable.**

Natural interpretation:
I.Sh. Int.J.Mod.Ph.D. (2002); A. Pelinson et al NPB (2003).

All sparticles are heavy $\Rightarrow$ **decouple when** H **becomes smaller than their masses.**

Direct calculations confirmed that the transition $c > 0 \Longrightarrow c < 0$ **is smooth, indicating a possibility of a smooth graceful exit.**

• Using anomaly for deriving EA of massive fields.

Why the energy scale H decreases during inflation?
In the exponential phase Hubble parameter $H(t) = const$.

Also: Using anomaly-induced EA for massive fields is "incorrect".

How can we take masses of the fields into account?

Consider a reliable Ansatz for the EA of massive fields.
J.Solà, I.Sh. PLB - 2002

The idea is to construct the conformal formulation of the SM and use it to derive EA for massive fields.

Replacing dimensional parameters by the new scalar χ:

$$m_{s,f} \to \frac{m_{s,f}}{M}\,\chi\,, \quad M_P^2 \to \frac{M_P^2}{M^2}\,\chi^2\,, \quad \Lambda \to \frac{\Lambda}{M^2}\,\chi^2\,.$$

M is related to a scale of conformal symmetry breaking.
Massive terms replaced by Yukawa and χ^4- interactions.

The new theory is conformal invariant

$$\sigma = \sigma(x), \qquad \begin{cases} \chi \to \chi\, e^{-\sigma}\,, & g_{\mu\nu} \to g_{\mu\nu}\, e^{2\sigma} \\ \varphi \to \varphi\, e^{-\sigma}\,, & \psi \to \psi\, e^{-3/2\,\sigma} \end{cases}$$

The conformal symmetry comes together with a new scalar χ, absorbing conformal degree of freedom. Fixing $\chi \to M$ we come back to original formulation.
The conformal anomaly becomes

$$\langle T \rangle = -\Big\{ wC^2 + bE + c\Box R + \frac{f}{M^2}\,[R\chi^2 + 6(\partial\chi)^2] + \frac{g}{M^4}\,\chi^4 \Big\},$$

f **and** g **are** β**-functions for** $(16\pi G)^{-1}$ **and** $\rho_\Lambda = \Lambda/8\pi G$ **,**

$$f = \sum_i \frac{N_f}{3\,(4\pi)^2}\, m_f^2\,, \qquad \tilde{f} = \frac{16\pi f}{M_P^2}\,,$$

$$g = \frac{1}{2(4\pi)^2} \sum_s N_s\, m_s^4 - \frac{2}{(4\pi)^2} \sum_f N_f\, m_f^4\,,$$

N_f **and** N_s **are multiplicities of the fields.**

Cosmological implications

$$S_t = S_{matter} + S^*_{EH} + S_{vac} + \bar{\Gamma}\,.$$

The equation of motion for $\Lambda = 0,\ \ g = 0$

$$a^2\,\ddot{\ddot{a}} + 3\,a\,\dot{a}\,\dddot{a} - \left(5 + \frac{4b}{c}\right)\dot{a}^2\,\ddot{a} + a\,\ddot{a}^2 - \frac{M_P^2}{8\pi c}\left(a^2\ddot{a} + a\dot{a}^2\right)\,[1 - \tilde{f}\cdot\ln a] \;=\; 0\,,$$

Let us solve by $M_P^2 \to M_P^2\,[1 - \tilde{f}\cdot\ln a]$**,**

$$\dot{\sigma} = H = H_o\sqrt{1 - \tilde{f}\sigma(t)}\,, \qquad H_o = \frac{M_P}{\sqrt{-16b}}\,.$$

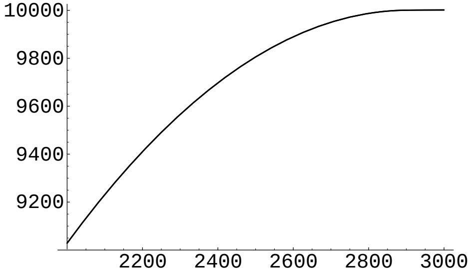
This leads to the simple solution

$$\sigma(t) \;=\; H_o\,t \;-\; \frac{H_0^2}{4}\,\tilde{f}\;t^2\,.$$

Remarkably, this formula fits with the numerical solution with a wonderful 10^{-6} **precision!**

$\tilde{f} > 0 \;\;\Rightarrow\;\;$ **we arrive at the tempered inflation!!**

Anomaly-induced inflation slows down if taking masses of quantum fields into account.



$$\sigma(t) = \ln a(t) \approx H_0 t - \frac{H_0^2}{4} \tilde{f}\, t^2 , \qquad H_0 \propto M_P$$

The total amount of e-folds may be as large as 10^{32}, but only 65 last ones, where $H \propto M_*$ (SUSY breaking scale) are relevant.

From the formal QFT viewpoint, there is no solution, because for the transition period, when

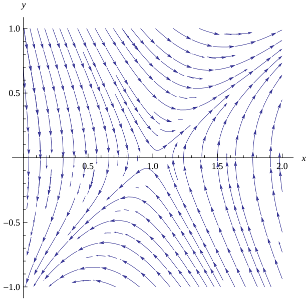
$$H \quad \sim \quad \text{masses of quantum matter fields}$$

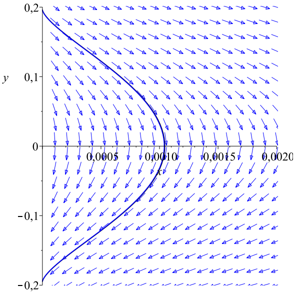
we have no method, approach, idea or approximation to perform calculations, except for dS space, which is useless here.

The simplest, purely phenomenological approach is to take a final point of the stable tempered inflation epoch ... and use it as initial point for the unstable phase.

A. Pelinson, T. de Paula, I.Sh., A. Starobinsky, arXive: 1509.08882; EPL - to appear.

The qualitative output of this phenomenological approach is positive, in the sense that the final point of the stable inflation (related to SUSY breakdown) belongs to the “right” integration curve of the unstable inflation.





One can check that this curve really ends up at the classical radiation-dominated solution.

This result gives us a chance to have a consistent inflation based on QFT results.

Getting the “right” coefficient of R^2

The transition from stable to unstable inflation is not sufficient to make the last consistent.

In order to have a successful inflation we need to go beyond the anomaly-induced effective action and provide the coefficient of the overall R^2-term to be about 5×10^8 as requested to control density perturbations after the inflation ends,

A.A. Starobinski, Let.Astr.Journ. 9 (1983).

It turns out that there is a chance to have a natural quantum mechanism to generate such a term.

Renormalization group (RG) in curved space tells us that the values of all parameters of the theory may run with the change of energy scale.

The RG for the coefficient a_4 in the vacuum action has a form

$$\mu \frac{da_4}{d\mu} = \beta_4 = l_1 + l_2 \xi + l_3 \xi^2 \,,$$

where the coefficients $l_{1,2,3}$ are given by power series in coupling constants, corresponding to the loop expansion.

We assume that the high energy GUT-like model (SUSY or not) includes gauge g, Yukawa h and four-scalar f couplings, hence $l_{1,2,3} = l_{1,2,3}(g, h, f)$.

The RG equation for ξ has a general form

$$\mu \frac{d\xi}{d\mu} = \beta_\xi = l_4 + l_5 \xi \,, \quad \textbf{where} \quad l_{4,5} = l_{4,5}(g, h, f).$$

In the strong-coupling regime the running may significantly change both sign and magnitude of ξ and a_4, even at the short interval on the energy scale.

Conclusions.

- Integrating conformal anomaly is an efficient, economic and simple way to derive the the non-local part of the effective action of vacuum.

- There are many generalizations of the original method. One can even obtain some results for the classically non-conformal quantum fields, even massive ones, if treating masses as small perturbations.

- The main application of this method is the Starobinsky model, which is capable to provide a direct link between QFT methods and the background of cosmology.

Recent generalization.

Quantum effects of chiral fermion produce an imaginary contribution which violates parity,

$$\langle T^\mu_\mu \rangle \,=\, -\,\omega_1 C^2 - bE_4 - c\Box R - \epsilon P_4 \,,$$

where the Pontryagin density term appears,

$$P_4 \,=\, \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,R_{\mu\nu\rho\sigma}\,R_{\alpha\beta}{}^{\rho\sigma}\,, \qquad \epsilon = \frac{i}{48\cdot 16\pi^2}\,.$$

L.Bonora, S.Giaccari, B.de Souza, JHEP (2014), arXiv:1403.2606.

It is a relatively easy exercise to derive the corresponding anomaly-induced effective action.

S. Mauro, I.Sh., arXiv:1412.5002.

First, one can prove the conformal symmetry of this term

$$P_4 = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,C_{\mu\nu\rho\sigma}\,C_{\alpha\beta}{}^{\rho\sigma}\,.$$

After that we immediately arrive at

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2 + \int_x \Big\{\frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+\varphi\,\Big[\,\frac{\sqrt{-b}}{8\pi}\,\big(E-\frac{2}{3}\,\Box R\big)\,-\,\frac{1}{8\pi\sqrt{-b}}\,\big(\omega C^2+\epsilon P_4\big)\Big]+\frac{1}{8\pi\sqrt{-b}}\,\psi\,\big(\omega C^2+\epsilon P_4\big)$$

It is natural to change variables,

$$\chi = \frac{\psi-\varphi}{\sqrt{2}}\,,\qquad \xi = \frac{\psi+\varphi}{\sqrt{2}}\,,$$

Then the total gravitational action becomes

$$\Gamma_{grav} = S_{EH} + S_{HD} + S_c[g_{\mu\nu}] + \int_x \Big\{\xi\Delta_4\chi\,+\,k_1\big(E-\frac{2}{3}\,\Box R\big)\big(\xi-\chi\big)$$

$$+\,k_2\,\chi\,\big(\omega C^2+\epsilon P_4\big)\,+\,k_3 R^2\Big\}\,.$$

The coefficients are, as before,

$$k_1 = \frac{1}{8\pi}\sqrt{-\frac{b}{2}}\,, \qquad k_2 = \frac{1}{8\pi\sqrt{-2b}}\,, \qquad k_3 = -\frac{2b+3c}{36\,(4\pi)^2}\,,$$

The action

$$\Gamma_{grav} = S_{EH} + S_{HD} + S_c[g_{\mu\nu}] + \int_x \Big\{ \underline{\xi\Delta_4\chi} + k_1\big(E - \frac{2}{3}\Box R\big)(\xi - \chi)$$

$$+ k_2\,\chi\,\big(\omega C^2 + \underline{\epsilon P_4}\big) + k_3 R^2 \Big\}\,.$$

is a special case of the Chern-Simons modified general relativity,

R. Jackiw and S.Y. Pi, Phys. Rev. D 68 (2003), gr-qc/0308071.

A. Lue, L. Wang, M. Kamionkowski, Phys.Rev.Lett. 83 (1999) 1506.

S. Alexander, N. Yunes, Phys.Rept. 480 (2009) 1.

with a special form of the kinetic term.