(Super-)Graphity

Black Holes and their Analogues
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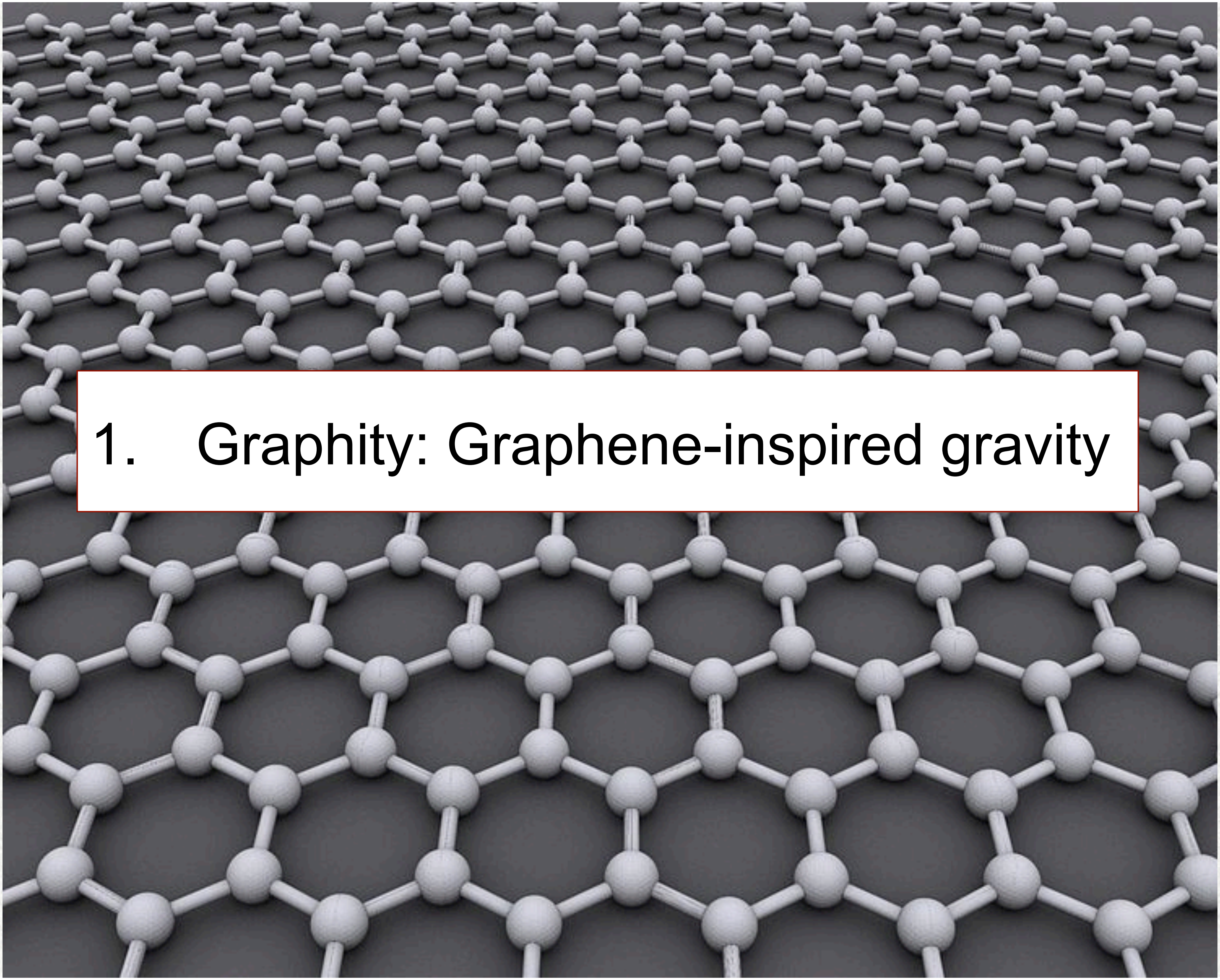
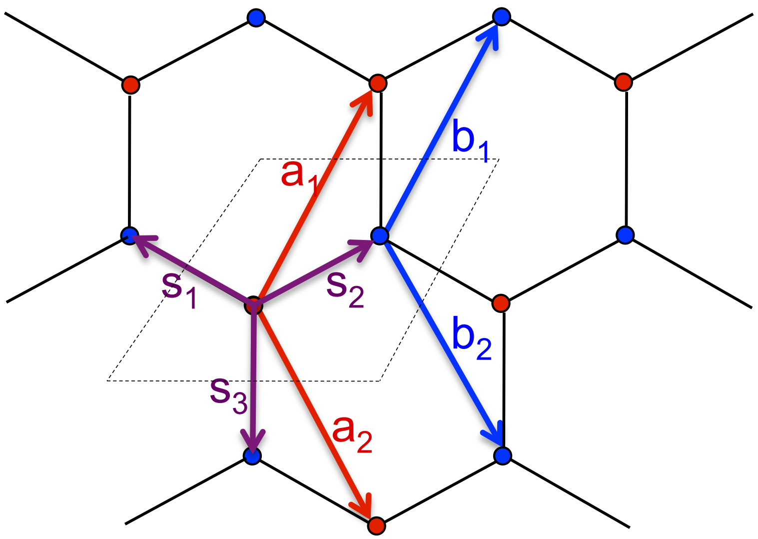


1. Graphity: Graphene-inspired gravity

Graphene:

1 hexagonal lattice = 2 triangular elementary lattices



Red lattice: a_1, a_2

Blue lattice: b_1, b_2

Nearest neighbors: s_1, s_2, s_3

Electrons are described by fermionic occupation states on each sublattice: 2 sublattices → 2-component fermions

$$\psi^{\alpha} = \begin{bmatrix} a \\ b \end{bmatrix}_{\vec{r}}, \quad \psi_{\alpha}^{\dagger} = \begin{bmatrix} a^{\dagger} & b^{\dagger} \end{bmatrix}_{\vec{r}}, \quad \alpha = 1,2$$

Hopping to nearest sites implies changing sublattice,

→ multiplication by $\vec{\gamma} = (\sigma_x, \sigma_y)$.

The Hamiltonian for nearest neighbor hopping becomes, in the continuum limit, that of a free Dirac spinor:

$$H = -iv_f \int (\psi^\dagger \vec{\gamma} \cdot \vec{\nabla} \psi + h.c.) d^2x$$

[P.R.Wallace, 1947]

This can also be recognized as the Hamiltonian for a Dirac particle whose action is

$$I = -i \int (\psi^\dagger \gamma^\mu \partial_\mu \psi + h.c.) d^2x$$

Here the “speed of light”, $v_f \sim 10^3$km/s, has been set =1

N.B. Fermionic matter requires a metric structure:

$$\not{\partial} = \gamma^\mu \partial_\mu \equiv \varepsilon_{abc} e^a e^b \gamma^c d$$

where $e^a = e^a_\mu dx^\mu$ is the vielbein.

Alternatively, one may use $\gamma^\mu = E^\mu_a \gamma^a$, where

$$E^\mu_a e^b_\mu = \delta^b_a$$

This is because spin-½ fermions are zero-forms (scalars under general coordinate transformations).

N.B. Fermions are always combined as $e^a \psi$:

$$\overline{\psi} \gamma^\mu \partial_\mu \psi + \ldots = (\overline{\psi} e^a) \varepsilon_{abc} \gamma^b d(e^c \psi) + \ldots$$

Graphene is a system of fermions in a $s=½$ representation of the $SO(1,2)$ group.

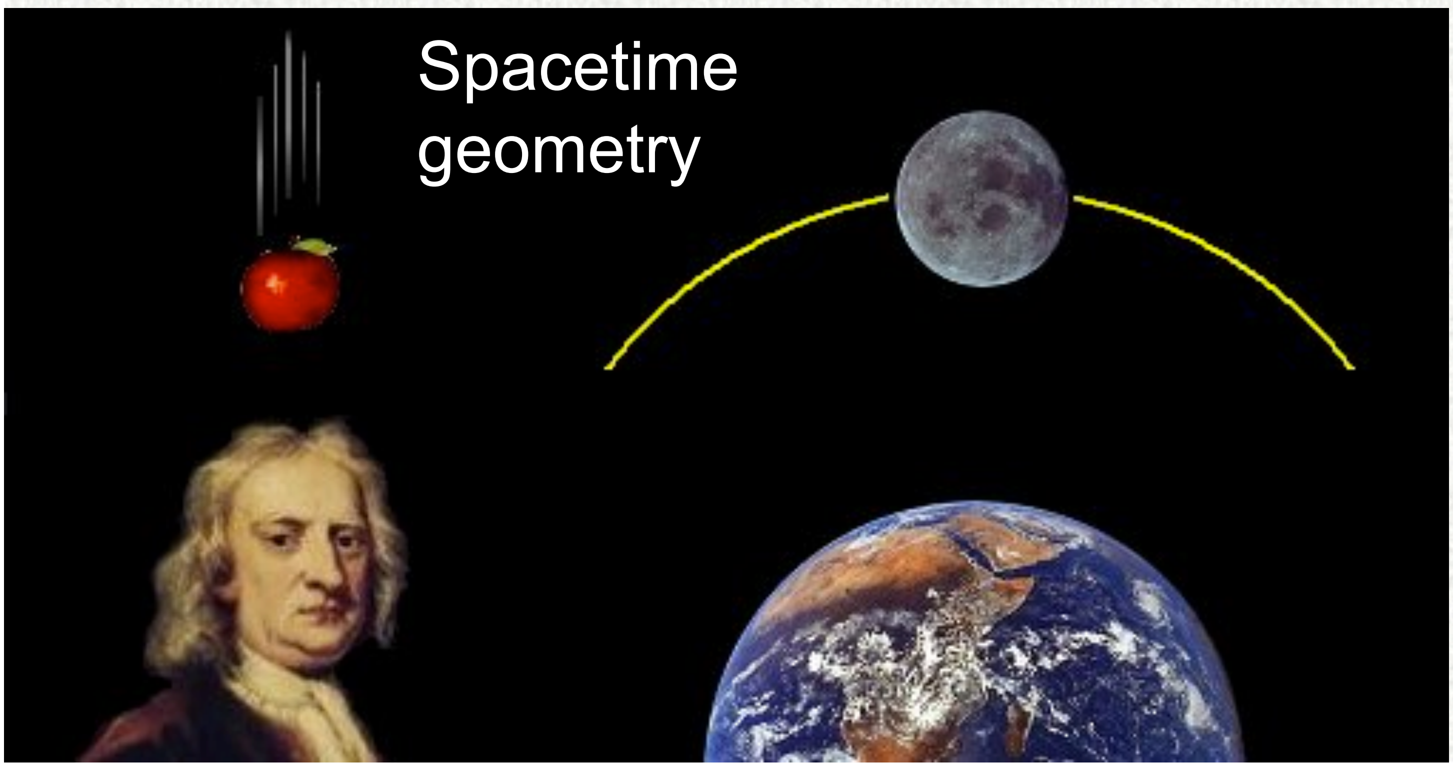
The substrate is a (2+1) dimensional spacetime endowed with local $SO(1,2)$ symmetry, just like a locally Lorentz-invariant (curved!) spacetime.

The only difference is that the "speed of light" is the Fermi velocity, $v \sim c/300$.

→ Experimental graphity

2. Beyond free electrons

Fermions (matter) couple to



Recipe: in the presence of gauge fields

$$\mathbf{A}_\mu = A_\mu^k (\mathrm{T}_k)_s^r \quad ,$$

or in a curved geometry, just add the right minimal couplings.

The three-dimensional graphene action, including Lorentz, and internal gauge couplings, is

$$I_F = -\int \overline{\psi}(i\gamma^\mu \nabla_\mu + m)\psi\sqrt{-g}\,d^3x$$

where $\nabla_\mu \equiv \partial_\mu - i\mathbf{A}_\mu + \frac{1}{4}\omega^{ab}_{\ \ \mu}\Gamma_{ab}$.

Here m is the fermion mass (*Kekulé distorsion*).

The spinor components in graphene don't describe *spin*. They only account for the existence of two sublattices:

$$\psi^{\alpha} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \alpha = 1,2$$

In order to include spin, an additional $SU(2)$ index is required: $\psi = \psi^{\alpha}_{r}$, $\psi^{\dagger} = \psi^{\dagger r}_{\alpha}$, $\mathbf{A}_{\mu} = A^{k}_{\mu}(\mathrm{T}_{k})^{r}_{s}$

A spinning electron in graphene with nontrivial geometry is described by the minimal couplings:

$$\nabla_{\mu} \equiv \partial_{\mu} - i\mathbf{A}_{\mu} + \tfrac{1}{4}\omega^{ab}_{\mu}\Gamma_{ab}$$

$SU(2)$ connection

$SO(1,2)$ connection

The relevant fields for graphene are:

$\psi \ (\overline{\psi})$: spin-½ fermion [conduction electrons]

$\mathbf{A}$: $SU(2)$ connection [spin gauge freedom]

$\omega^a{}_b$: Lorentz connection [affine structure]

e^a : Dreibein [metric structure]

All of them could in principle enter as dynamical fields in the action. The connections $\mathbf{A}$ and $\omega^a{}_b$ can be economically described by their corresponding 3D Chern-Simons actions.

The three-dimensional action for spin-1/2 fermions in curved graphene is

$$I = \alpha_1 \int Tr[\tfrac{1}{2}\mathbf{A}d\mathbf{A} + \tfrac{1}{3}\mathbf{A}^3]$$ SU(2)-CS

$$+\alpha_2 \int [\tfrac{1}{2}\omega^a{}_b d\omega^b{}_a + \tfrac{1}{3}\omega^a{}_b\omega^b{}_c\omega^c{}_a]$$ SO(1,2)-CS

$$- \int \overline{\psi}(i|e|\Gamma^a E_a^\mu \nabla_\mu + e^a T_a)\psi d^3x$$ Dirac

where $\nabla_\mu \equiv \partial_\mu - i\mathbf{A}_\mu + \frac{1}{4}\omega^{ab}{}_\mu \Gamma_{ab}$, and α_1, α_2 are arbitrary dimensionless constants.

Recapping:

- Conduction electrons in graphene behave as 'massless Dirac fermions in 2+1dimensions'.
- Curved graphene requires coupling to Lorentz (spin) connection, ω^{ab}_{μ}.
- Fermions require a metric structure $e^a_{\cdot}$
- Spin requires $SU(2)$ connection $\mathbf{A}$.

It turns out that for $\alpha_1 = \alpha_2 = 1$ a miracle occurs...

3. (Super)graphity

If $\alpha_1 = \alpha_2 = 1$, the Lagrangian becomes a 3D Chern-Simons form,

$$L = \frac{1}{2}\left\langle \mathcal{A} d\mathcal{A} + \frac{2}{3}\mathcal{A}\mathcal{A}\mathcal{A} \right\rangle$$

where the connection one form $\mathcal{A}$ is

$$\mathcal{A} = \mathrm{A}^k \mathrm{T}_k + \frac{1}{2}\omega^{ab}\mathrm{J}_{ab} + \overline{\psi}^r_\alpha (\Gamma)^\alpha{}_\beta Q^\beta_r + \overline{Q}^r_\alpha (\Gamma)^\alpha{}_\beta \psi^\beta_r$$

$\boxed{SU(2)}$ $\boxed{SO(1,2)}$ $\boxed{Supersymmetry\ (!)}$

and the bracket $\langle \cdots \rangle$ is the symmetrized trace in the algebra spanned by T_k, J_{ab}, Q^β_r, and $\overline{Q}^r_\alpha$:

$su(1,2|2)$ Superalgebra

The connection

$$\mathcal{A} = \mathrm{A}^i \mathrm{T}_i + \tfrac{1}{2}\omega^{ab}\mathrm{J}_{ab} + \overline{\psi}^r_\alpha (\Gamma)^\alpha{}_\beta Q^\beta_r + \overline{Q}^r_\alpha (\Gamma)^\alpha{}_\beta \psi^\beta_r + \ldots$$

is a one-form. The gamma matrices and the vielbein frame play a crucial role:

$$\Gamma \equiv \Gamma_a e^a = \Gamma_a e^a_\mu dx^\mu$$

Spinors are irreducible representations of the Lorentz group defined in the tangent space (they are spacetime scalars).

The local frame (vielbein) establishes an isomorphism between the tangent space (Lorentz indices) and the base manifold (coordinate indices).

The superalgebra involves both $SO(1,2)$, $SU(2)$ and SUSY generators,

$SO(1,2)$: $\left[\mathrm{J}^{ab},\, \mathrm{J}^{cd}\right] = \eta^{bc}\mathrm{J}^{ad} - \eta^{ac}\mathrm{J}^{bd} + \eta^{ad}\mathrm{J}^{bc} - \eta^{bd}\mathrm{J}^{ac}$,

$SU(2)$: $\left[\mathrm{T}_i, \mathrm{T}_j\right] = \mathrm{i}\varepsilon_{ij}{}^{k}\mathrm{T}_k$, $\quad \left[\mathrm{T}_k,\, \mathrm{J}^{ab}\right] = 0$

SUSY: $\left\{Q^{\alpha}_{r},\, \overline{Q}^{s}_{\beta}\right\} = i\delta^{\alpha}_{\beta}\mathrm{T}_i(\sigma^{i})^{s}_{r} + \tfrac{1}{2}\delta^{s}_{r}\mathrm{J}^{ab}(\Gamma_{ab})^{\alpha}{}_{\beta}$

$$\left[\mathrm{J}^{ab},\, Q^{\alpha}_{r}\right] = \tfrac{1}{2}(\Gamma^{ab})^{\alpha}{}_{\beta}Q^{\beta}_{r}; \quad \left[\mathrm{J}^{ab},\, \overline{Q}^{r}_{\alpha}\right] = -\tfrac{1}{2}(\Gamma^{ab})^{\beta}{}_{\alpha}\overline{Q}^{r}_{\beta},$$

$$\left[\mathrm{T}_i, Q^{\alpha}_{r}\right] = i(\sigma_i)^{s}_{r}Q^{\alpha}_{s}; \quad \left[\mathrm{T}_i, \overline{Q}^{r}_{\alpha}\right] = -i(\sigma_i)^{r}_{s}\overline{Q}^{s}_{\alpha},$$

This $su(1,2|2)$ *algebra acts in the tangent space.*

The graphene system is invariant under local $SU(2)$ x $SO(1,2)$ (up to surface terms) and, for $\alpha_1 = \alpha_2 = 1$, also SUSY invariant:

$$\delta A^k{}_\mu = -\tfrac{i}{2}(\bar{\varepsilon}\Gamma_\mu \sigma^k \psi + \bar{\psi}\sigma^k \Gamma_\mu \varepsilon)$$

$$\delta\omega^a_\mu = \bar{\varepsilon}^r \Gamma^a \Gamma_\mu \psi_r + \bar{\psi}^r \Gamma^a \Gamma_\mu \varepsilon_r$$

$$\delta\psi^r = \tfrac{1}{3}\nabla\!\!\!/^r{}_s \varepsilon^s$$

$$\delta e^a_\mu = 0$$

N.B. The metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, remains invariant under this *unconventional* supersymmetry

Field equations [for $U(1)$, almost same for $SU(2)$]:

$$\delta A \quad : \quad F = \tfrac{i}{2}\varepsilon_{abc}\overline{\psi}\Gamma^a\psi e^b e^c$$

$$(F_{\mu\nu} = \varepsilon_{\mu\nu\lambda} j^\lambda)$$

$$\delta\omega \quad : \quad R^{ab} = -2\overline{\psi}\psi e^a e^b \quad (*)$$

$$\delta\overline{\psi} \quad : \quad [\not{\partial} + i\not{A} - \tfrac{1}{4}\Gamma^a \not{\omega}_{ab}\Gamma^b + e^a_\mu T_{a\nu\lambda}\varepsilon^{\mu\nu\lambda}]\psi = 0$$

$$\delta e \quad : \quad \overline{\psi}\varepsilon^a{}_{bc}\Gamma^c[\overleftarrow{d}e^b - e^b\overrightarrow{d} + 2iAe^b]\psi = 2|e|\overline{\psi}\psi T^a$$

- Standard equations for CS electrodynamics, gravity & spin ½ in 2+1 dimensions.
- ψ acquires "mass" from torsion: $m(x) = e^a T_a$
- From (*), $R^{ab}e_b = 0 \Rightarrow DT^a = 0$, and

$$T_a = \tau\varepsilon_{abc}e^b e^c, \quad m = 6\tau = \text{constant}$$

4. Vacuum graphity

Vacuum (matter-free) solutions:

$$\boxed{\psi = 0}$$

(1) $$\mathsf{F}^k = \cancel{\tfrac{i}{2}\varepsilon_{abc}\overline{\psi}\Gamma^a\sigma^k\psi e^b e^c} = 0$$

(2) $$R^{ab} = \cancel{-2\overline{\psi}^r\psi_r e^a e^b} = 0$$

(3) $$\cancel{[\not{\partial} + i\not{A} - \tfrac{1}{4}\Gamma^a\not{\omega}_{ab}\Gamma^b + \mu]\psi} = 0 \quad ✔$$

(4) $$\cancel{\overline{\psi}c^a{}_{bc}\Gamma^c[\overleftarrow{d}e^b - e^b\overrightarrow{d} + 2iAe^b]\psi} = \cancel{2|e|\overline{\psi}\psi T^a} \quad ✔$$

$\psi = 0 \Rightarrow$

- No gauge field
- Flat spacetime

? Not quite..

En general, $R^{ab} = 0 = \mathsf{F}$ admit nontrivial solutions

- *What is the most general spacetime geometry so that $R^a{}_b = 0, \quad \mathsf{F} = 0$?*

Does $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b = 0$

imply some restriction on the metric?

This question is not empty because R^{ab} depends on $\omega^a{}_b$, not on $g_{\mu\nu}$.

Geometry?

$$R^a{}_b = 0 \longrightarrow g_{\mu\nu} = \eta_{ab} e^a e^b = ?$$

Torsion: $T^a = De^a = de^a + \omega^a{}_b e^b$

Hence, $DT^a = DDe^a = R^a{}_b e^b = 0$

- Lorentz flatness ➔ covariantly constant torsion.
- Torsion-free spacetimes are particular cases.
- Can one integrate $DT^a = 0$?

What is the most general solution of

$$DT^a = D[de^a + \omega^a{}_b e^b] = 0 \ ?$$

Torsion-free

Contortion

Split $\omega^a{}_b = \tilde{\omega}^a{}_b + \kappa^a{}_b,$

$$R^{ab} = \tilde{R}^{ab} + \tilde{D}\kappa^{ab} + \kappa^a{}_c \kappa^{cb}$$

$$T^a = \kappa^a{}_b e^b$$

Hence $DT^a = 0 \Rightarrow \tilde{D}\kappa^a{}_b = 0 \Rightarrow \kappa^a{}_b = \tau \varepsilon^a{}_{bc} e^c.$

So, $R^a{}_b = 0 \quad \not\Rightarrow \quad \tilde{R}^a{}_b = 0$

[Lorentz flatness] [Riemann flatness]

In 3 dimensions the answer is

$$\boxed{T^a = \tau \varepsilon^a{}_{bc} e^b e^c}$$

where $-\infty < \tau < \infty$ is an integration constant.

Hence, $$R^a{}_b = \tilde{R}^a{}_b + \kappa^a{}_c \kappa^c{}_b + \tilde{D}\kappa^a{}_b$$
$$= \tilde{R}^a{}_b + \tau^2 e^a e_b$$

Finally, $$\boxed{R^a{}_b = 0 \Rightarrow \tilde{R}^a{}_b = -\tau^2 e^a e_b}$$

A 3D Lorentz-flat geometry is either *AdS_3*(!) or Minkowski space.

$$R^a{}_b = 0 \quad \Leftrightarrow \quad \tilde{R}^a{}_b = \Lambda e^a e_b, \;\; \Lambda \leq 0$$

3*D* Lorentz flat — Ads_3/ Minkowski

- $\Lambda = -\tau^2 = \frac{-1}{l^2}$ is an integration constant
- The sign of Λ is the Lorentzian signature

$$\eta_{ab} = (-,+,+)$$

- For euclidean signature this construction gives a 3-sphere.

Adams-Hopf Theorem:

"The only parallelizable spheres are S^1, S^3 and S^7"

A *3D* theory whose field equation is

$$R^a{}_b = 0$$

admits the black hole,* a locally AdS_3 solution,

$$\boxed{ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 d\phi^2,}$$

where** $f^2 \equiv \frac{r^2}{l^2} - M$ ($l =$ arbitrary length scale)

The dreibein are $e^0 = f dt,\ e^1 = f^{-1} dr,\ e^2 = r d\phi$

$$T^a = \frac{1}{l} \varepsilon^a{}_{bc} e^b e^c \Rightarrow DT^a \equiv 0 \quad ✔$$

* For $-1<M<0$, this describes point particles (conical defects)

** Can also add angular momentum

The *3D* torsion-free (metric) connection is

$$\tilde{\omega}^0{}_1 = \tfrac{r}{l^2}dt, \;\; \tilde{\omega}^1{}_2 = f d\phi, \;\; \tilde{\omega}^2{}_0 = 0.$$

This is not Riemann-flat: $\tilde{R}^{ab} = \frac{-1}{l^2} e^a e^b$ ✔

The Lorentz connection $\omega^a{}_b = \tilde{\omega}^a{}_b + \frac{1}{l}\varepsilon^a{}_{bc}e^c$ is

$$\omega^0{}_1 = \tfrac{r}{l}\left(\tfrac{dt}{l} + d\phi\right), \;\; \omega^1{}_2 = f\left(\tfrac{\varepsilon dt}{l} + d\phi\right), \;\; \omega^2{}_0 = f^{-1}\tfrac{dr}{l}$$

➔ $R^a{}_b = 0$ (Lorentz flat) ✔

This connection is locally “pure gauge”

$$\omega^a{}_b = (\Lambda^{-1})^a{}_c \, d\Lambda_b{}^c,$$ ✔

but not globally so, due to topology (r=0 removed).

In particular:

Any simply connected patch of a *2+1* black hole can be covered with a congruence of freely falling mutually inertial observers, as in Minkowski space.

5. SU(2) hair on the 2+1 BH

su(2), so(3), so(1,2) are isomorphic algebras.

$$\left.\begin{array}{c} R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b = 0 \\ \mathbf{F} = d\mathbf{A} + \mathbf{A}\mathbf{A} = 0 \end{array}\right\} \boxed{\text{Isomorphic}}$$

Both locally pure gauge:

$$\omega^a{}_b = (\Lambda^{-1})^a{}_c \, d\Lambda^c{}_b \longleftrightarrow \mathbf{A} = g^{-1} dg$$

$$\mathbf{A}^3 = \tfrac{r}{l}\left(\tfrac{dt}{l} + d\phi\right) \sim \omega^0{}_1$$

$$\mathbf{A}^2 = ih\left(\tfrac{dt}{l} + d\phi\right) \sim \omega^1{}_2$$

$$\mathbf{A}^1 = h^{-1} \tfrac{dr}{l} \qquad \sim \omega^2{}_0$$

$$\boxed{h^2 = \tfrac{r^2}{l^2} - Q}$$

The two solutions are related:

$$f(r) \leftrightarrow h(r), \quad M \leftrightarrow Q$$

(apart from a few factors of i).

Although the connections are locally pure gauge, they are globally nontrivial due to the topology of the spacetime manifold:

$$[R^2 - \{0\}] \times R$$

An important simplification results from the fact that there is no direct coupling between ω and $\mathbf{A}$.

SU(2)-charged static black holes:

$$ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 d\phi^2 \,, \qquad \boxed{f = \frac{r^2}{l^2} - M}$$

$$\mathbf{A} = h^{-1} \tfrac{dr}{l'} \sigma^1 + ih(\tfrac{dt}{l'} + d\phi)\sigma^2 + \tfrac{r}{l'}(\tfrac{dt}{l'} + d\phi)\sigma^3 \,, \qquad \boxed{h = \frac{r^2}{l'^2} - Q}$$

These solutions are characterized by mass (M), and $SU(2)$ charge (Q): Black holes ($M \geq 0$), AdS ($M = -1$), and point particles ($-1 < M < 0$).

- They can be boosted: $\begin{array}{l} M_0 , \;\; \to M, J \neq 0 \\ Q_0 , \;\; \to Q, K \neq 0 \end{array}$

- For some values of M, J, Q, K globally defined Killing spinors can be found. These BPS states are candidates to stable ground states.

SUMMARY

- Graphene → Dirac Eq.
  → Metric structure (Dreibein)
- Curved graphene → Minimal couplings to Lorentz connection

Geometry

- Dynamic Lorentz connection
  → Graphity
- Fermions + Lorentz + Spin [Internal SU(2) symmetry]
  → Supergraphity!

SUMMARY (contd.)

- Equations $R^{ab}e_b=0$

  → Cov. constant Torsion

  → Constant (Riemannian) negative curvature

  → Locally AdS

  → BHs, point particles, etc.

N.B.: Cosmological constant is a constant of integration

- Spin [Internal SU(2) symmetry]

  → SU(2) hairy BH

- The fun's just begun...

Happy Birthday, Professor Bronnikov!

Happy Birthday, GR!

References

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