V José Plínio Baptista School of Cosmology

Compact Objects

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Guarapari - Espírito Santo, Brazil

Quasiblack holes or Gravitationally intense stars (Synge) or Ultracompact objects in general relativity

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Outline

Compact objects - from Wikipedia

- Mathematics: Compact objects, also referred to as finitely presented objects, or objects of finite presentation, are objects in a category satisfying a certain finiteness condition.
- Astronomy: All compact objects have a high mass relative to their radius, giving them a very high density, compared to ordinary atomic matter.
  The term compact star (or compact object) refers collectively to **white dwarfs**, **neutron stars**, and **black holes**.
  It would grow to include **exotic stars** if such hypothetical, dense bodies are confirmed to exist.
  On June 2020, astronomers reported narrowing down the source of Fast Radio Bursts (FRBs), which may now plausibly include "**compact-object mergers** and **magnetars** arising from normal core collapse supernovae."

Compacteness – mass density

- Astronomy: All compact objects have a high mass relative to their radius, giving them a very high density, compared to ordinary atomic matter.
  - Picanha (beef): $\sim 1.13 \times 10^3$ kg/m^3
  - Concrete: $\sim 2.4 \times 10^3$ kg/m^3
  - Osmium (heaviest atom): 2.25×10^4 kg/m^3
  - Nucleus: 2.0×10^{17} — 4.0×10^{17} kg/m^3
  - Sun: $\sim 1.4 \times 10^5$ kg/m^3
  - White dwarfs: 1.0×10^7 — 1.0×10^{10} kg/m^3
  - Neutron stars: 8×10^{16} — 6×10^{17} kg/m^3
  - Black hole: ? (singularities)

Ultracompact objects - intense stars

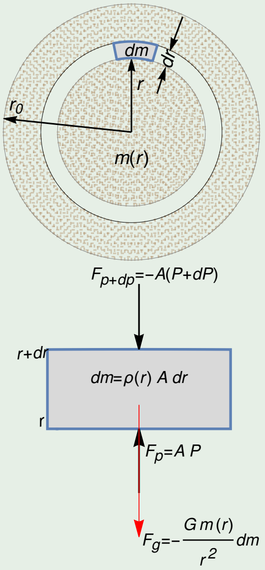
- Gravitationally intense stars – According to J. L. Synge (Mon. Not. R. Astr. Soc. 131, 1966):
  Spherical objects whose mass-to-radius ratio is larger than 1/3, $m/r_0 > 1/3$ (geometric units: $G = 1 = c$).
- The **photon sphere** for the Schwarzschild black hole spacetime is at $r = 3m$.
- Any object whose radius is smaller than the **photon sphere** (or **light ring**) are black hole mimickers.
  – Exotic stars, gravastars, quasiblack holes, regular black holes, etc.

Compactness – mass to radius ratio

Sphere of radius R, mass $m = GM/c^2$

- Sun ($M \sim 2 \times 10^{30}$ kg, $R \sim 7 \times 10^5$km): $m/R \sim 2 \times 10^{-6}$
- White dwarfs ($M_\odot$, $R \sim 7 \times 10^3$km): $m/R \sim 1 \times 10^{-3}$
- Neutron stars ($M_\odot$, $R \sim 10$km): $m/R \sim 0.1$
- Buchdahl bound: $m/R \leq 4/9$
- Schwarzschild black holes: $m/r_h = 1/2$
- Andréasson bound: $m/R \leq 1.$
- Regular black holes (core): $m/R \geq 1$

Hydrostatic equilibrium – Newton



- Spherical star (perfect fluid): density $\rho(r)$, pressure $p(r)$.
- Potential $\phi(r)$: $\nabla^2\phi = 4\pi\, G\rho(r)$.
- Take a fluid element located at the distance r from the center, with mass $dm = \rho(r)\, dV$.
- The mass inside r is $m(r) = \int_0^r 4\pi\, r^2 \rho(r)\, dr$.
  - Radial equilibrium:
    $$A\,p - A\,(p+dp) - \frac{Gm(r)}{r^2}\rho(r)\,A\,dr = 0.$$
  - $\Longrightarrow \dfrac{dp(r)}{dr} = -\dfrac{G\,m(r)}{r^2}\rho(r).$
  - Any increase in $\rho(r)$ may be balanced by an increase in the central pressure p_c.
  - **No compactness bounds:** m/r_0 arbitrary.

Newtonian fluid sphere - incompressible fluid

- Sphere of an incompressible perfect fluid ($\rho = \mathrm{const.} = \rho_0$, p).
- Solve Poisson equation $\nabla^2\phi = 4\pi G\rho_0$:

$$\phi(r) = \begin{cases} -\dfrac{G\,M}{2r_0}\left(3 - \dfrac{r^2}{r_0^2}\right), & 0 \leq r \leq r_0, \\ -\dfrac{GM}{r}, & r \geq r_0, \end{cases} \tag{1}$$

- Solve the equilibrium equation:

$$p(r) = \begin{cases} \dfrac{3GM^2}{8\pi r_0^4}\left(1 - \dfrac{r^2}{r_0^2}\right), & 0 \leq r \leq r_0, \\ 0, & r \geq r_0. \end{cases} \tag{2}$$

- $M = \dfrac{4\pi\rho_0\, r_0^3}{3}$.
- "Problems" just in the limit $r_0 \to 0$ (if the mass M is fixed).

Hydrostatic equilibrium - Relativity

$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ (static-spherical fluid distribution) $\longrightarrow$ TOV equation:

$$\frac{dp(r)}{dr} = -\frac{G}{r^2}\left(m(r) + \frac{4\pi r^3 p(r)}{c^2}\right)\left(\rho(r) + \frac{p(r)}{c^2}\right)\left(1 - \frac{2G\, m(r)}{r\, c^2}\right)^{-1}$$

- After some limit, an increase in $\rho(r)$ cannot be balanced by an increase in p_c (p contributes to the RHS of the equilibrium equation).
- **Compactness bound** (Buchdahl, 1959): $m/r_0 < 4/9$
- If $m/r_0 > 4/9 \Longrightarrow$ no equilibrium.

Relativistic effects on a perfect fluid

- Length contraction: $dr \Longleftrightarrow dr \times \sqrt{1 - v^2(r)/c^2}$
- Volume contraction: $dV \Longleftrightarrow dV \times \sqrt{1 - v^2(r)/c^2}$
- "Inertial mass": $\Longrightarrow$ Energy: $m(r) \Longleftrightarrow m(r)/\sqrt{1 - v^2(r)/c^2}$
- Passive mass density: $\rho \Longrightarrow \rho(r) + p(r)/c^2$
- Active gravitational mass:
  $m(r) \Longrightarrow m + 3 \times$ "mass from pressure" $= m(r) + 4\pi r^3 p(r)/c^2$

Schwarzschild interior solution

Schwarzschild interior solution (1916), see S. Antoci, arXiv:physics /9912033

- Sphere of an incompressible perfect fluid ($\rho_m = \mathrm{const.} = \frac{1}{8\pi}\frac{3}{R^2}$, p):
- $ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2$.
- Solve Einstein equations ($G = 1 = c$).

$$A(r) = \begin{cases} \left(1 - \dfrac{r^2}{R^2}\right)^{-1}, & 0 \le r \le r_0, \\ \left(1 - \dfrac{2m}{r}\right)^{-1}, & r \ge r_0, \end{cases}$$

$$B(r) = \begin{cases} \left(\dfrac{3}{2}\sqrt{1 - \frac{r_0^2}{R^2}} - \dfrac{1}{2}\sqrt{1 - \frac{r^2}{R^2}}\right)^2, & 0 \le r \le r_0, \\ 1 - \dfrac{2m}{r}, & r \ge 0. \end{cases}$$

Schwarzschild interior solution (1916)

Fluid quantities

- $$M(r) = \begin{cases} \frac{1}{2}\frac{r^3}{R^2}, & 0 \le r \le r_0, \\ m = M(r_0) = \frac{1}{2}\frac{r_0^3}{R^2}, & r \ge r_0. \end{cases}$$

$$8\pi\rho_m(r) = \begin{cases} \frac{3}{R^2}, & 0 \le r \le r_0, \\ 0, & r \ge r_0, \end{cases}$$

$$8\pi p(r) = \begin{cases} -1 + \frac{2\sqrt{1-\frac{r^2}{R^2}}}{3\sqrt{1-\frac{r_0^2}{R^2}}-\sqrt{1-\frac{r^2}{R^2}}}, & 0 \le r \le r_0 \\ 0, & r \ge r_0. \end{cases}$$

Schwarzschild interior solution

The central pressure

$$8\pi p_c = -1 + \frac{2}{3\sqrt{1 - \frac{r_0^2}{R^2}} - 1}$$

diverges if $\frac{r_0^2}{R^2} \to 8/9$. Or, if $\frac{m}{r_0} \to \frac{4}{9}$.

- No equilibrium solutions if $\frac{m}{r_0} \geq \frac{4}{9}$.
- No equilibrium solutions for $r_0 \leq \frac{9}{8} r_h$.

Buchdahl bound

Compact objects

- Does the result for Schwarzschild incompressible fluid model hold in general?
- Buchdahl bound (PRD, 1959) - Perfect fluid (ρ, p), spherical symmetry, $\rho > 0$ (non-increasing outward), $p \geq 0$. Equilibrium requires

$$\frac{r_0}{m} \geq \frac{9}{4}.$$

- Violation $\rightarrow$ black holes and singularities.
- How general is this result?
  – Specific energy conditions must hold.
  – Is spherical symmetry important?
- Many authors tried generalization of Buchdahl's result: Mak, Dobson & Harko, 2001; Bohmer & Harko, 2007; Karageorgis & Stalker, 2008; Andréasson, 2008, 2009.

Compactness bounds: generalization - Electric charge

Andréasson, CMP, 2009

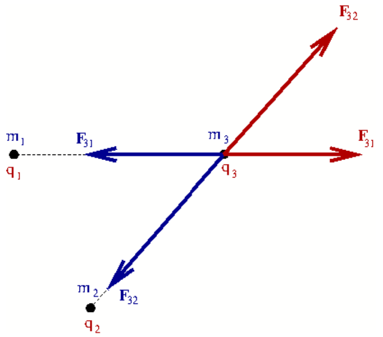
- Charged spherical spheres have a sharp bound.
- Buchdahl-Andréasson bound – Non-isotropic charged fluid (ρ, p_r p_t), spherical symmetry, $p_r + 2p_t \leq \rho$, $\rho \geq 0$, $q/m < 1$, $q/r < 1$:

$$\frac{r_0}{m} \geq \frac{9}{\left(1 + \sqrt{1 + 3\, q^2/r_0^2}\right)^2},$$

  r_0, m, q = radius, mass and charge of the configuration.
- Buchdahl bound for $q = 0$: $\frac{r_0}{m} \geq \frac{9}{4}$.
- Highest compactness: ($q \longrightarrow m$, $r_0 \to m$): $\frac{r_0}{m} \to 1$.
- Saturated by (charged) spherical thin shells.
- In the limit $r_0/m \to 1$, this bound includes QUASIBLACK HOLES.

Charged matter (Is it possible $r_0/m > 9/4$.

Electrically charged matter – Newton-Coulomb



The resulting force on each and every particle vanishes if ${q_i}^2 = G\, {m_i}^2$
$\Longrightarrow\ q_i = \pm\sqrt{G}\, m_i\, (Gauss, k_e = 1), \quad i = 1,\, 2,\, ...,\, N.$

Charged fluid: $\dfrac{dp(r)}{dr} = -\dfrac{G\, m(r)}{r^2}\rho_m(r) + \rho_e(r)\dfrac{q(r)}{r^2}$

Charged DUST ($p = 0$): $\rho_e = \pm\sqrt{G}\rho_m$

Is it possible $r_0/m < 9/4$ in GR?

Electrically charged matter – Einstein-Maxwell

–Modified TOV equation ($c = 1$)

$$\frac{dp(r)}{dr} = -\frac{G}{r^2}\Big[p(r) + \rho_m(r)\Big] \frac{\left[4\pi p(r) r^3 + m(r) - \frac{q^2(r)}{r}\right]}{\left[1 - \frac{2m(r)}{r} + \frac{q^2(r)}{r^2}\right]} + \frac{q(r)}{r^2}\frac{\rho_e(r)}{\sqrt{1 - \frac{2m(r)}{r} + \frac{q^2(r)}{r^2}}},$$

Equilibrium of charged DUST ($p = 0$): $\rho_e = \pm\sqrt{G}\rho_m$

Bonnor stars

Charged dust, $\rho_e = \rho_m$

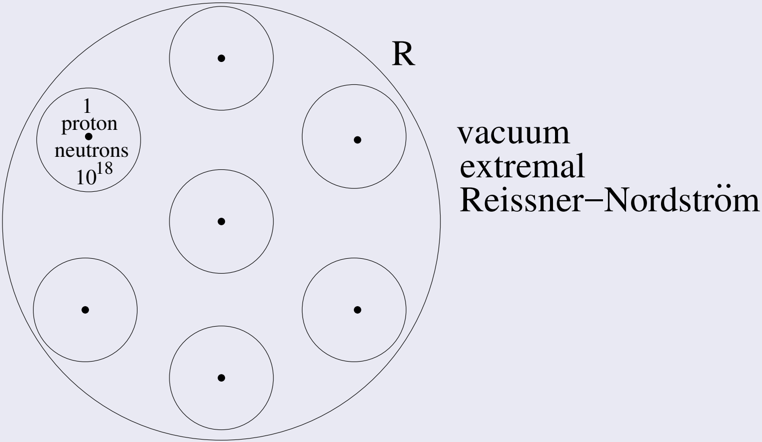


Figure: A Bonnor star made of clouds of extremal matter in the interior joined to a vacuum extremal Reissner-Nordström solution.

Bonnor stars

Charged dust, $\rho_e = \rho_m$

- Exact solutions: Weyl 1917, Majumdar 1947, Papapetrou 1947, Das 1962, De & Raychaudary 1968, Bonnor 1974, 1980, many others.
- Static spacetime:

$$ds^2 = -U^{-2}(x^i)dt^2 + U^2(x^i)\left[h_{jk}dx^j dx^k\right].$$

- Electric (static) potential ϕ proportional to the metric potential U: $U = \pm\phi + \text{const.}$ $(G = 1 = c)$
- $\nabla^2 U = 4\pi U^3 \rho_m$
- Neutral equilibrium (charged matter distributions of any shape).

Bonnor stars

Spheres of charged dust

- Arbitrary $U = U(r)$ and $\rho_e(r) = \rho_m(r), \quad r \leq r_0$.
- Extremal Reissner-Nordström in the exterior region, $r \geq r_0$.
- Admits very compact objects: $\dfrac{r_0}{m} \to 1$.
- Also $r_0 \to r_h$: quasiblack holes (Lemos & Weinberg, PRD, 2004).
- Very compact spheres with $m = q = r_+ = r_-(= r_h)$.
- These are solutions showing that the Buchdahl bound may be surpassed, without violating energy conditions.

Path to a quasiblack hole configuration

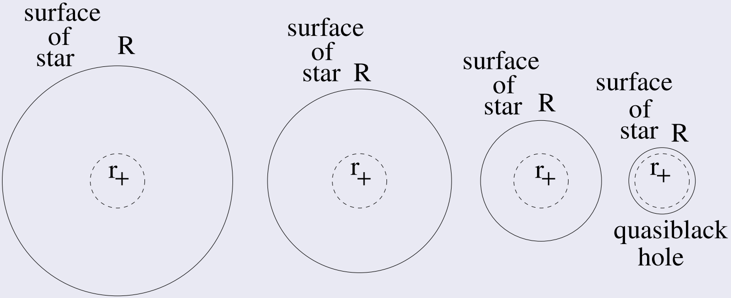


Figure: Sequence of star configurations to a quasiblack hole where $R = r_+$ (Lemos & Zaslavskii, IJMPD, 2020).

Path to a quasiNONblack hole configuration

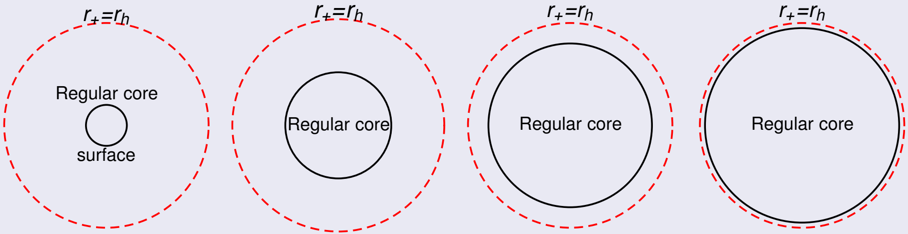


Figure: Sequence of regular BH configurations to a quasiNONblack hole.

Quasiblack holes: Definitions [1]

Static metric

$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega$

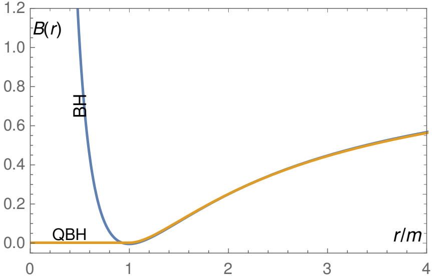
- (a) function $\dfrac{1}{A(r)}$ attains a minimum at some $r_* \neq 0$, such that $\frac{1}{A(r_*)} = \epsilon$, with $\epsilon \ll 1$.

  Invariant definition: replace $\dfrac{1}{A(r)}$ by $(\nabla r)^2$, ∇ is the gradient.
- (b) For a small $\epsilon > 0$ the configuration is regular everywhere, with a nonvanishing $B(r)$;
- (c) $\lim_{\epsilon\to 0} B(r) = 0$ for all $r \leq r_*$. In this limit r_* is the horizon radius r_+, $r_* = r_+$.

These three features define a quasiblack hole.

[1] Lemos & Zaslavskii, Int.J.Mod.Phys.D 29 (2020) 11, 2041019; e-Print: 2007.00665 [gr-qc]

Quasiblack hole – metric potentials



The metric potentials $B(r)$ and $A^{-1}(r)$ as functions of r for an extremal quasiblack hole and for an extremal black hole. In the region $r < r_+$ the functions are totally different while in the exterior they are the same.

Quasiblack holes - Further properties

1. A quasiblack hole is on the verge of forming an event horizon, instead, a quasihorizon appears.
2. Quasiblack holes with finite stresses must be extremal to the outside.
3. The curvature invariants of extremal and nonextremal quasiblack hole spacetime remain regular everywhere.
4. A free-falling observer in a quasiblack hole spacetime finds in its frame infinite tidal forces at the quasihorizon. This shows some form of degeneracy, i.e., a combination of features typical of regular and singular systems, at the quasihorizon.
5. Both in an extremal and in a nonextremal quasiblack hole spacetime, outer and inner regions become mutually impenetrable and disjoint. An interesting example is the Lemos-Weinberg solution, where the interior is a Bertotti-Robinson spacetime, the quasihorizon region is extremal Bertotti-Robinson, and the exterior is extremal Reissner-Nordström.
6. There are infinite redshift whole 3-regions.
7. For far away observers a quasiblack hole spacetime is indistinguishable from that of a black hole. Quasiblack holes are black hole mimickers.

Quasiblack hole – Carter-Penrose diagram

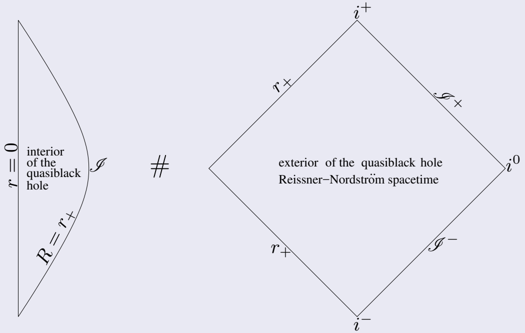


Figure: Carter-Penrose diagram for a generic quasiblack hole, extremal or nonextremal. The interior is a spacetime with matter, has the timelike $r = 0$ origin, and is bounded by a timelike line at radius $R = r_+$ that acts like an infinity scri $\mathscr{I}$. The exterior region is identical to the outer Reissner-Nordström black hole region. Interior and exterior are mutually impenetrable and disjoint. r_+ plays the role of a kind of singular boundary. The symbol $\#$ indicates the connected sum of disjoint spacetimes (Lemos & Zaslavskii, IJMPD, 2020)

Charged fluid spheres: Guilfoyle solutions

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

Exterior = Reissner-Nordström metric:
$B(r) = 1/A(r) = 1 - 2m/r + q^2/r^2$.

Interior solution

$$B(r) = a\,[\phi(r) + b]^2 + c \text{ (Guilfoyle relation)}$$

$$8\pi\,\rho_{\rm m}(r) + \frac{Q^2(r)}{r^4} = \frac{3}{R^2} = \text{const.} \text{ (Cooperstock-de La Cruz-Florides)},$$

$$A(r) = \left(1 - \frac{r^2}{R^2}\right)^{-1},$$

$$Q(r) = 4\pi\int_0^r \rho_{\rm e}(r)\frac{r^2dr}{\sqrt{1-\frac{r^2}{R^2}}} = \frac{r^2}{\sqrt{B(r)}}\sqrt{1-\frac{r^2}{R^2}}\;\frac{d\phi(r)}{dr},$$

$$B(r) = \left[\frac{2-a}{a^{1+1/a}}\,F(r)\right]^{2a/(a-2)}, \quad F(r) = k_0R^2\sqrt{1-\frac{r^2}{R^2}} - k_1,$$

Guilfoyle solutions: Class Ia

$$8\pi\rho_{\mathrm{m}}(r) = \frac{3}{R^2} - \frac{a}{(2-a)^2}\frac{k_0^2\, r^2}{F^2(r)}\,,$$

$$Q(r) = \frac{\epsilon\sqrt{a}}{2-a}\frac{k_0\, r^3}{F(r)},$$

$$8\pi p(r) = -\frac{1}{R^2} + \frac{a}{(2-a)^2}\frac{k_0^2\, r^2}{F^2(r)} + \frac{2k_0\, a}{2-a}\frac{\sqrt{1-r^2/R^2}}{F(r)}\,,$$

$$k_0 = \frac{|q|\, a^{2/a}}{r_0^3}\left(\frac{m}{q} - \frac{q}{r_0}\right)^{1-2/a}\,,$$

$$k_1 = \sqrt{1-\frac{r_0^2}{R^2}}\left[k_0\, R^2 - \frac{a^{1+1/a}}{2-a}\left(1-\frac{r_0^2}{R^2}\right)^{-1/a}\right].$$

Boundary surface $r = r_0$ (implying $p(r_0) = 0$)

$$\Rightarrow \frac{1}{R^2} = \frac{1}{r_0^3}\left(2m - \frac{q^2}{r_0}\right).$$

Guilfoyle Class Ia solutions

Parameters and relations - A plethora of objects

- Five parameters (a, R, r_0, m, q) Two relations:

$$m = \frac{r_0}{2}\left(\frac{r_0^2}{R^2} + \frac{q^2}{r_0^2}\right).$$

$$a = \frac{r_0^2}{4q^2}\left(\frac{r_0^2}{R^2} - \frac{q^2}{r_0^2}\right)^2\left(1 - \frac{r_0^2}{R^2}\right)^{-1}.$$

- Stars satisfying energy conditions (EC) (Guilfoyle, 1999)
- Quasiblack hole with pressure satisfying EC (L & Z, PRD, 2010)
- Regular black holes with a de Sitter core (L & Z, PRD, 2011)
- Saturates the Buchdahl-Andréasson limit (L & Z, CQG, 2015)
- Regular black holes with a core of phantom charged matter (L & Z, PRD, 2015)
- Plethora of objects (L & Z, PRD, 2017)
- Stability analysis (Masa, L & Z, 2021, to be published)

Guilfoyle solution saturates the Andréasson bound

Central pressure

$$8\pi p_c = \lim_{r\to 0} 8\pi p(r) = -\frac{1}{R^2} + \frac{2k_0\, a}{2-a}\,\frac{1}{k_0 R^2 - k_1}$$

$$k_0 = \frac{|q|\, a^{2/a}}{r_0^3}\left(\frac{m}{q} - \frac{q}{r_0}\right)^{1-2/a},$$

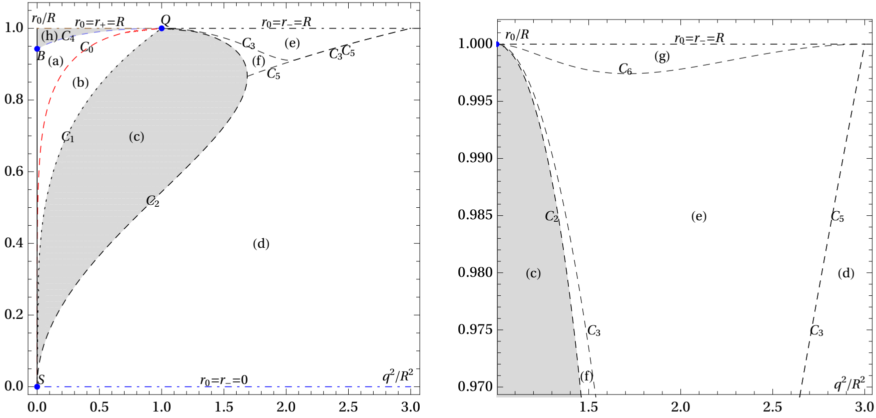
$$k_1 = \sqrt{1-\frac{r_0^2}{R^2}}\left[k_0\, R^2 - \frac{a^{1+1/a}}{2-a}\left(1-\frac{r_0^2}{R^2}\right)^{-1/a}\right].$$

– Taking the limit $p_c \to \infty \Longrightarrow k_0 R^2 - k_1 = 0$. Solving for m/r_0 in terms of q/r_0:

$$\frac{r_0}{m} = \frac{9}{\left(1+\sqrt{1+\frac{3q^2}{r_0^2}}\right)^2}.$$

– Exactly the Andréasson result.

The Guilfoyle solutions: parameter space and stability



- Charged stars (a), tension overcharged stars (b) – stable in part
- (c): Singular objects
- (d): Regular phantom black holes - stable in part
- (e) and (f): Regular phantom black holes - unstable
- (g): Regular black holes - stable in part

Radial perturbations- Eulerian approach

- $f(r) \to f_0(r) + \delta f(r,t)$, for any function, $f_0(r)$ meaning the equilibrium solution.
- $\gamma =$ adiabatic index, e.g.,
  $\delta p = \gamma\left(\rho_m + p\right)\rho_m^{-1}\delta\rho_m, \qquad \zeta = r^2 B_0^{-1/2}\xi, \qquad \xi = \delta r.$

$$\left(F\zeta'\right)' + \left(H + \omega^2 W\right)\zeta = 0,$$

where

$$F = \gamma r^{-2} p_0 B_0^{3/2} A_0^{1/2},$$

$$\begin{aligned} H \;=\; & B_0^{3/2} A_0^{1/2}\left[\frac{1}{r^2\left(\rho_0 + p_0\right)}\left(\frac{Q_0 Q_0'}{4\pi r^4} - p_0'\right)^2 + \frac{4}{r^3}\left(\frac{Q_0 Q_0'}{4\pi r^4} - p_0'\right)\right. \\ & \left. -8\pi p_0\left(\rho_0 + p_0\right)\frac{A_0}{r^2} - \left(\rho_0 + p_0\right)\frac{A_0 Q_0^2}{r^6} - \frac{Q_0 Q_0'}{\pi r^7}\right], \end{aligned}$$

$$W = \left(\rho_0 + p_0\right) r^{-2} B_0^{1/2} A_0^{3/2}.$$

Beyond compactness bounds: Regular black holes

Avoiding singularities in GRT

- Just violate the singularity theorems (energy conditions)
- Cosmological singularities could be avoided by matter with a de Sitter equation of state $p = -\rho_m$ (Sakharov 1966, Gliner 1966, 1975).
- $p = -\rho_m$ is equivalent to a cosmological constant $T_{\mu\nu} = \Lambda g_{\mu\nu}$ (Zel'dovich 1968).
- The Bardeen regular black hole (1968): The first regular black hole - modification of the Reissner-Nordström metric (with a magnetic type of matter) which near the center tend to a de Sitter solution.
- Most of the subsequent regular black hole solutions are based on Bardeen's proposal. There has been a tremendous development on the implementation and on the analysis of the properties of regular black hole solutions (see e.g. Ansoldi 2008, arXiv:0802.0330, for a review).

The model: Spherical symmetry

de Sitter region (interior): Charged non-isotropic fluid with a de Sitter type equation of state

$+$

Timelike thin shell

$+$

Reissner-Nordström region (exterior)

(Masa, Oliveira, Zanchin, IJMPD 2018, PRD 2021)

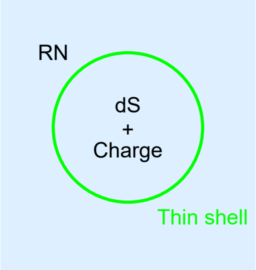


Figure: dS region: $p_r = -\rho_m$

Method

- Matching: Darmois-Israel junction conditions at the thin shell (radius a) - stationary solutions
- Stability around the stationary configurations: equation of motion of the shell

Inner solution

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{2}$$

Assumptions

$$p_r(r) = -\rho_m(r) \qquad \text{(Sakharov; Gliner)} \tag{3}$$

$$8\pi\rho_m(r) + \frac{Q^2(r)}{r^4} = \frac{3}{R^2} \qquad \text{(Cooperstock-de La Cruz; Florides),} \tag{4}$$

$$Q(r) = 4\pi\int_0^r \rho_e(\bar{r})\sqrt{A(\bar{r})}\bar{r}^2 d\bar{r} = q\left(\frac{r}{a}\right)^{3+n}; \quad Q(r=a) = q \tag{5}$$

Solution

$$B(r) = A^{-1}(r) = 1 - \frac{r^2}{R^2}, \tag{6}$$

$$8\pi\rho_m(r) = \frac{3}{R^2} - \frac{q^2}{a^4}\left(\frac{r}{a}\right)^{2(n+1)}, \qquad 8\pi p_t(r) = -\frac{3}{R^2} - \frac{q^2}{a^4}\left(\frac{r}{a}\right)^{2(n+1)}. \tag{7}$$

Exterior solution

Assumptions

$$\rho_m(r) = p_r(r) = p_t(r) = 0, \qquad Q(r) = q \tag{8}$$

Solution

$$B(r) = A^{-1}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \tag{9}$$

- m = black hole mass, q = black hole charge
- Horizons if $m \geq q$
- Inner horizon r_-, event horizon r_+, where $B(r) = (r - r_+)(r - r_-)/r^2$

Junction conditions: Darmois-Israel (NC 1966)

Two different spacetimes, $\mathcal{M}^+$ e $\mathcal{M}^-$, are matched by a thin shell.

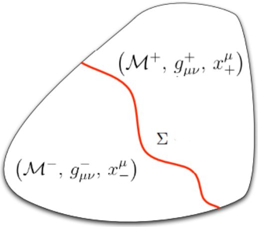
- h_{ab}: induced metric,
  $[h_{ab}] = 0$

- K_{ab}: extrinsic curvature

- S_{ab}: energy momentum tensor of the shell,

  $$8\pi S_{ab} = -\epsilon\left([K_{ab}] - h_{ab}\,[K]\right)$$

  $$[K_{ab}] = K^+_{ab} - K^-_{ab}$$



Junction at $r = a$

$$ds^2 = -B(r)\,dt^2 + \frac{1}{B(r)}dr^2 + r^2\left(d\theta^2 + sen^2\theta d\varphi^2\right). \tag{10}$$

$$B(r) = \begin{cases} B_{dS}(r) = 1 - \frac{r^2}{R^2}, & \text{for } r < a(\tau), \\ \\ B_{RN}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, & \text{for } r > a(\tau), \end{cases} \tag{11}$$

m: black holes mass, q: black holes charge, τ: proper time.

- The shell is composed by a perfect fluid: $S_b^a = diag\,(-\sigma, P, P)$
- Assumption: $\sigma = \omega P$, $\omega =$ constant.
- The shell is timelike if inside the de Sitter horizon R and the inner r_-, or outside r_+ of Reissner-Nordström metric ($a < R$ and $a < r_-$, or $a > r_+$).

Energy conservation and equation of motion of the shell

- Energy conservation $S^{a}_{b;a} = 0$

$$\frac{d\sigma}{da} = -\frac{2}{a}\left(\sigma + P\right), \tag{12}$$

$$\sigma = \frac{1}{4\pi}\left(\sqrt{1 - \frac{a^2}{R^2} + \dot{a}^2} - \sqrt{1 - \frac{2m}{a} - \frac{q^2}{a^2} + \dot{a}^2}\right). \tag{13}$$

- Equation of motion of the shell:

$$\dot{a}^2 + V(a) = -1, \tag{14}$$

$$V(a) = -\frac{a^2}{R^2} - \left[\frac{\left(\frac{a^3}{R^2} + \frac{q^2}{a} - 2m\right)\left(\frac{a_0}{a}\right)^{-2\omega}}{2M_0} - \frac{M_0\left(\frac{a_0}{a}\right)^{1+2\omega}}{2a_0}\right]^2, \tag{15}$$

where $M_0 = 4\pi a_0^2 \sigma(a_0)$.

Equilibrium solutions: Stationary shell

Stationary shell: $\dot{a} = 0$, $a = a_0$ =const, are found by solving the Eqs.

$$V(a_0) = -1, \quad \left.\frac{dV}{da}\right|_{a=a_0} = 0. \tag{16}$$

Basic condition

- The exterior metric is the Reissner-Nordström, (maybe) with event and Cauchy horizons:

$$\frac{r_\pm}{R} = \frac{m(a_0,\, q,\, \omega)}{R} \pm \sqrt{\frac{m^2(a_0,\, q,\, \omega)}{R^2} - \frac{q^2}{R^2}}. \tag{17}$$

- Four free parameters: R, ω, a_0, and q. From the boundary conditions, RBH with timelike matching iff $0 < a_0 < r_-$.
- Different class of solutions depending on ω, a_0/R, and q/R. For all $n \geq 0$.

Stability

– Power series of the effective potential around an equilibrium solution $(a = a_0)$:

$$V(a) = V(a_0) + (a - a_0)\left.\frac{dV}{da}\right|_{a=a_0} + \frac{(a-a_0)^2}{2}\left.\frac{d^2V}{da^2}\right|_{a=a_0} + \ldots \tag{18}$$

– Equilibrium solution ($a = a_0$ =const.): $V(a_0) = -1$, $\left.\frac{dV}{da}\right|_{a=a_0} = 0$.

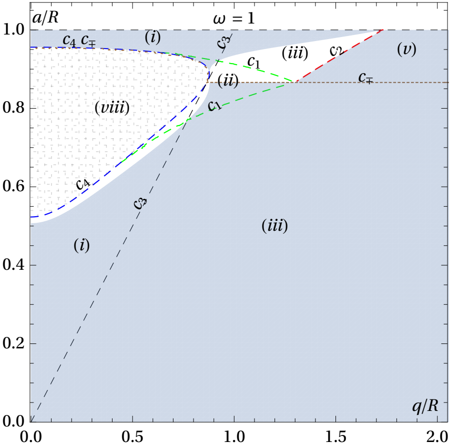
$$\Rightarrow V(a) = -1 + \frac{(a-a_0)^2}{2}\left.\frac{d^2V}{da^2}\right|_{a=a_0} + \mathcal{O}\left[(a-a_0)^3\right]. \tag{19}$$

–

- Stability (Instability) when:

$$\left.\frac{d^2V}{da^2}\right|_{a=a_0} > 0 \qquad \left(\left.\frac{d^2V}{da^2}\right|_{a=a_0} < 0\right) \tag{20}$$

For $\omega = 1$



RBHs Stability:

$$\sqrt{3}/2 \approx 0.8660 < \frac{a_0}{R} < 1$$

$$0.9207 < \frac{q}{R} < \sqrt{3}$$

$$0.9207 < \frac{m}{R} < 2$$

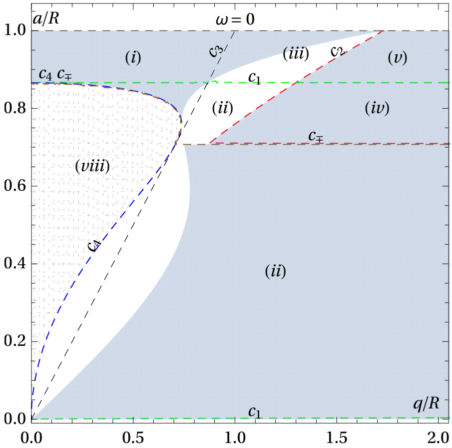
$$0 < \frac{M_0}{R} < 0.3594$$

$$\sqrt{3}/2 \approx 0.8660 < \frac{q}{m} < 1$$

Figure: Stability regions for $\omega = 1$ in the $(q/R,\ a_0/R)$ plane.

White regions contain stable solutions. Stable RBH are found in region (iii). Stable charged gravastars, in region (i).

For $\omega = 0$



RBHs Stability:

$$\sqrt{3}/2 \approx 0.8660 < \frac{a_0}{R} < 1$$

$$0.9008 < \frac{q}{R} < \sqrt{3}$$

$$0.9008 < \frac{m}{R} < 2$$

$$0 < \frac{M_0}{R} < 0.3912$$

$$\sqrt{3}/2 \approx 0.8660 < \frac{q}{m} < 1$$

Figure: Stability regions for $\omega = 0$ (dust shell) in $(q/R,\ a_0/R)$-plane.

White region contains stable solutions. Stable extremal quasiblack holes are found on the line c_3 (inside the white portion of region (ii)).

Compact objects rotate

Ultracompact rotating objects: A simple model

Rotating de Sitter region (interior): Non-isotropic fluid with a de Sitter type equation of state $p_r = -\rho_m$
+ Thin shell or smooth matching
+ Kerr (Kerr-Newman) region (exterior)

Metric in Boyer-Lindquist coordinates

$$ds^2 = -\left(1 - \frac{2M(r)r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{4M(r)ra\sin^2\theta}{\Sigma} dt\, d\phi$$

$$+ \sin^2\theta \left(r^2 + a^2 + \frac{2M(r)ra^2\sin^2\theta}{\Sigma}\right) d\phi^2. \quad \text{(Azreg-Aïnou, PLB, 2014)}$$

- $M(r)$ = Mass function to be chosen appropriately.
- $\Delta = r^2 + a^2 - 2r\, M(r), \quad \Sigma = r^2 + a^2 \cos^2\theta$
- Parameters: Mass (m), rotation ($a = J/m$), matching (r_0).

Our choices

- $M(r) = \frac{5m}{2}\frac{r^3}{r_0^3}\left(1 - \frac{3\,r^2}{5r_0^2}\right)\Theta(r_0 - r) + m\left[1 - \Theta(r_0 - r)\right],$
  –$\Theta(x - x_0) =$ Heaviside step function.
- Smooth matching: No shell at the boundary surface.

Results

- $8\pi\rho_m = -8\pi p_r = \frac{15m}{r_0^3}\frac{r^4}{\Sigma^2}\left(1 - \frac{r^2}{r_0^2}\right)\Theta(r_0 - r)$
- $8\pi p_\theta = 8\pi p_\phi = \frac{15m}{r_0^3}\frac{r^2}{\Sigma}\left[\frac{r^2}{\Sigma}\left(1 - \frac{r^2}{r_0^2}\right) - \left(2 - \frac{3r^2}{r_0^2}\right)\right]\Theta(r_0 - r)$
- Exact solutions for rotating stars and regular black holes whose exterior regions are the Kerr metric.
- Work in progress (Masa & Zanchin, 2021).

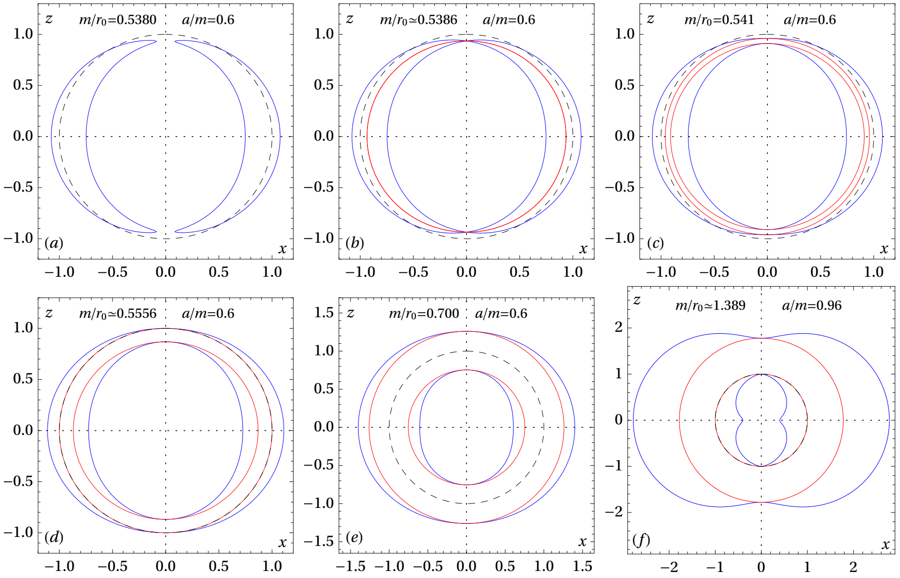


Figure: Important surfaces for $a/m = 0.6$ in the x–z plane, where $x = (r/r_0)\sin\theta$ and $z = (r/r_0)\cos\theta$. The blue lines represent the ergosurfaces, the red lines are the horizons, and the dashed lines indicate the boundary surfaces $r = r_0$.

Summary

- We presented a brief review of the Majumdar-Papapetrou system (Bonnor stars) with the aim of introducing the concept of quasiblack holes.
- Defined static quasiblack holes.
- Presented the plethora of objects in the Guilfoyle solution.
- Presented other simple models of compact objects
- Presented a simple model for rotating compact objects.

Thank You!