Introduction to Inflation

Lecture 2: Models of inflation and origin of structure

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Outline

Slow-roll field inflation

- *Slow-roll inflation* $\Rightarrow\ \dot\varphi^2 \ll V$
-

$$
\begin{aligned}
H^2 &= \frac{8\pi G}{3}\left(V+\frac{1}{2}\dot\varphi^2\right)\\
\ddot\varphi + 3H\dot\varphi &= -V'
\end{aligned}
$$

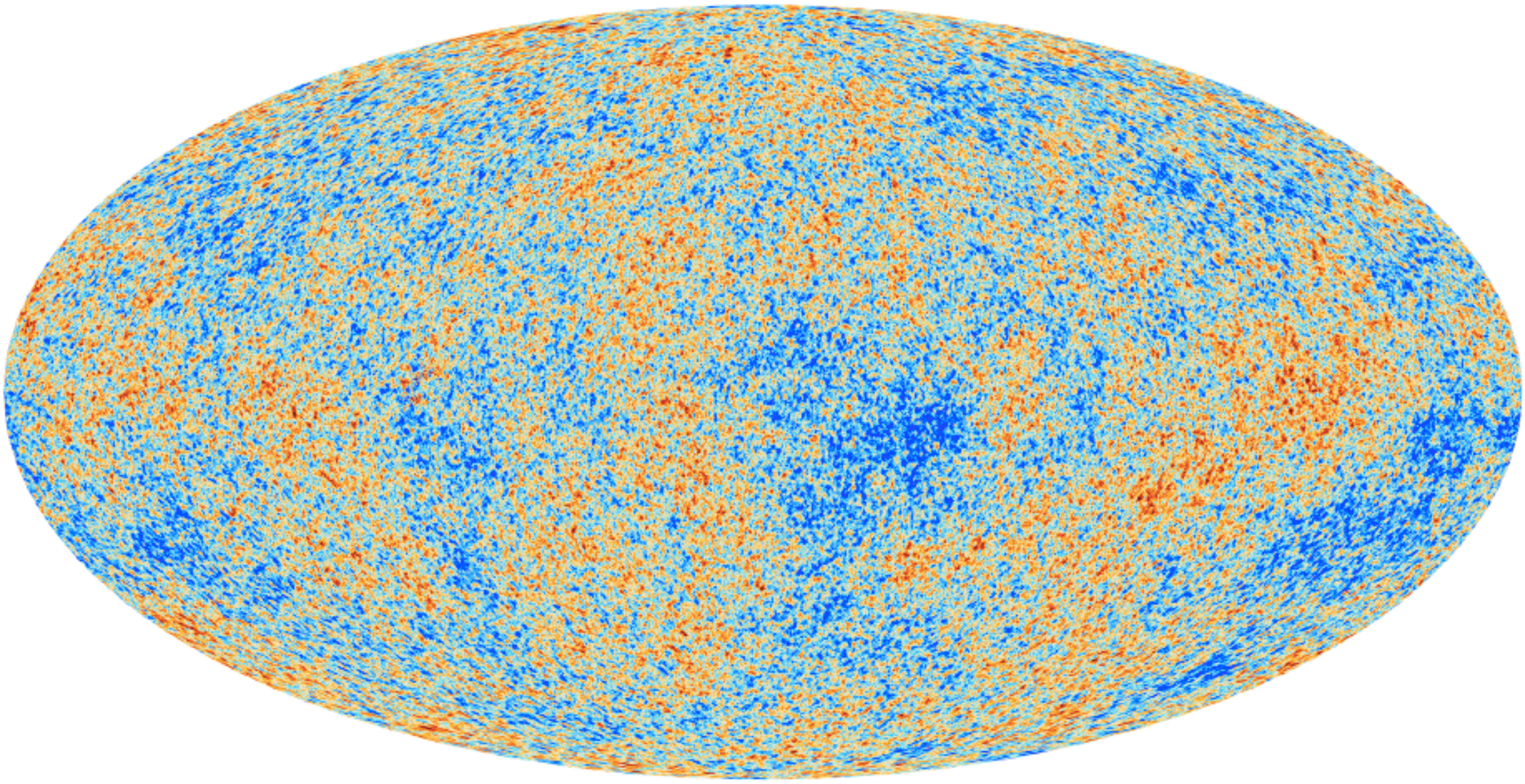
Slow-roll approximation:

$$
\begin{aligned}
H^2 \simeq \frac{8\pi G}{3}V \quad &\Leftrightarrow \quad \dot\varphi^2 \ll V\\
3H\dot\varphi \simeq -V' \quad &\Leftrightarrow \quad |\ddot\varphi| \ll 3H|\dot\varphi|
\end{aligned}
$$

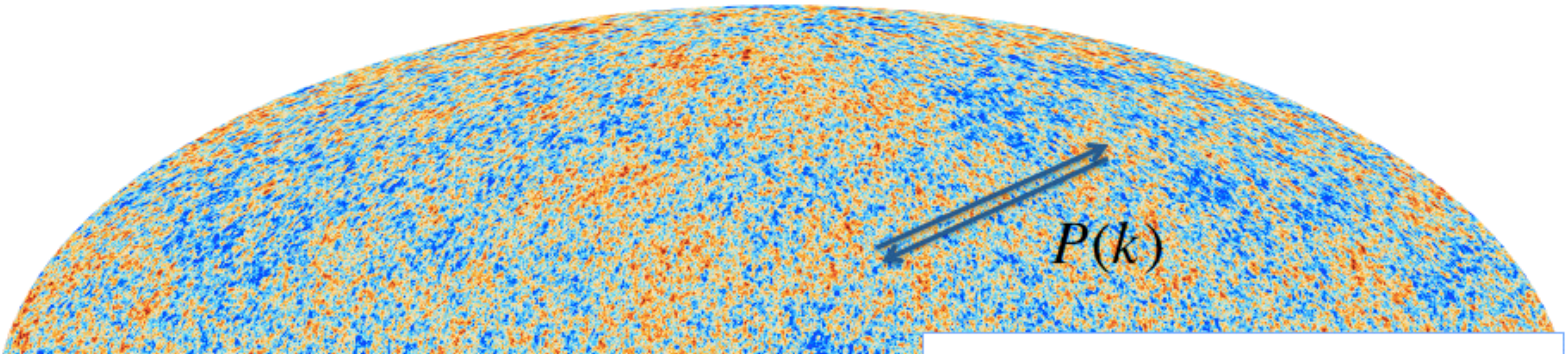
where slow-roll parameters are small: $\epsilon \ll 1$ and $|\eta| \ll 1$

$$
\begin{aligned}
\epsilon &\simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2\\
\eta &\simeq \frac{1}{8\pi G}\left(\frac{V''}{V}\right)
\end{aligned}
$$

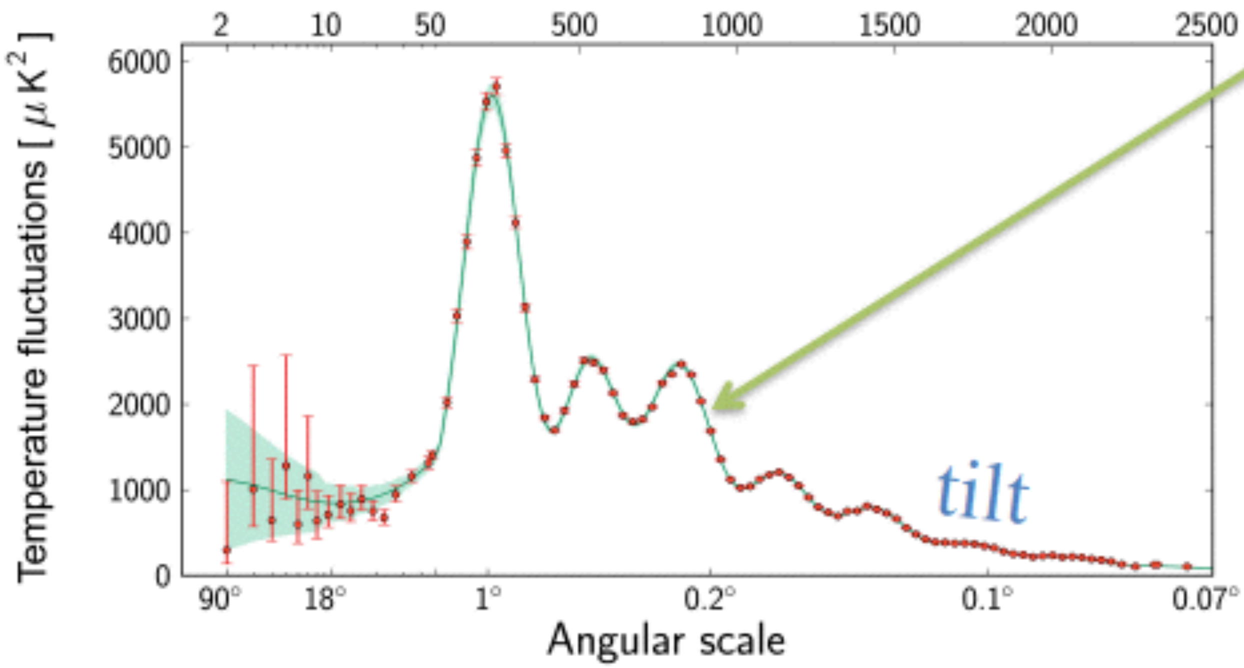
Precision cosmology Ade et al 2015



Precision cosmology Ade et al 2015



Angular power spectrum:



6-parameter model:

4-parameter cosmology:

- $H_0 = 67.8 \pm 0.9$
- $\Omega_{matter} = 0.308 \pm 0.012$
- $\Omega_\Lambda = 0.692 \pm 0.012$
- Reionisation z = 8.8 ± 1.7

2-parameter ***initial fluctuations***:

- Amplitude $\Delta T/T = 0.000047$
- Tilt = -0.032 ± 0.006

Outline

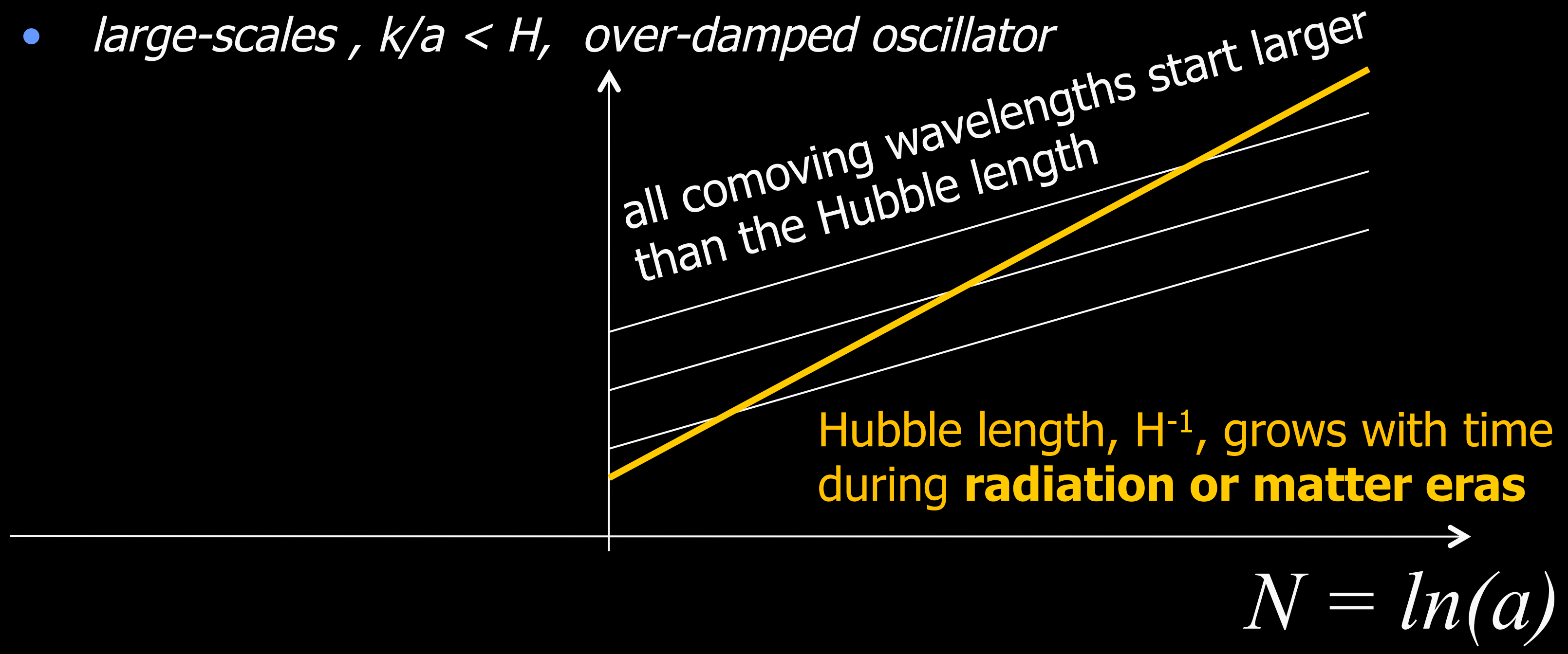
previously
slow-roll inflation
Perturbations
Scalar field perturbations
Gauge-invariant perturbations
Power spectra
Observational bounds
Models of inflation
Large-field inflation
Small-field inflation
Natural inflation
Higgs inflation
Starobinsky inflation
Reheating
ending inflation

Wave equation in FRW cosmology:

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + (k/a)^2\delta\varphi = 0$$

Characteristic timescales for waves, comoving wavemode k

- *small-scales , k /a > H , under-damped oscillator*
- *large-scales , k/a < H, over-damped oscillator*



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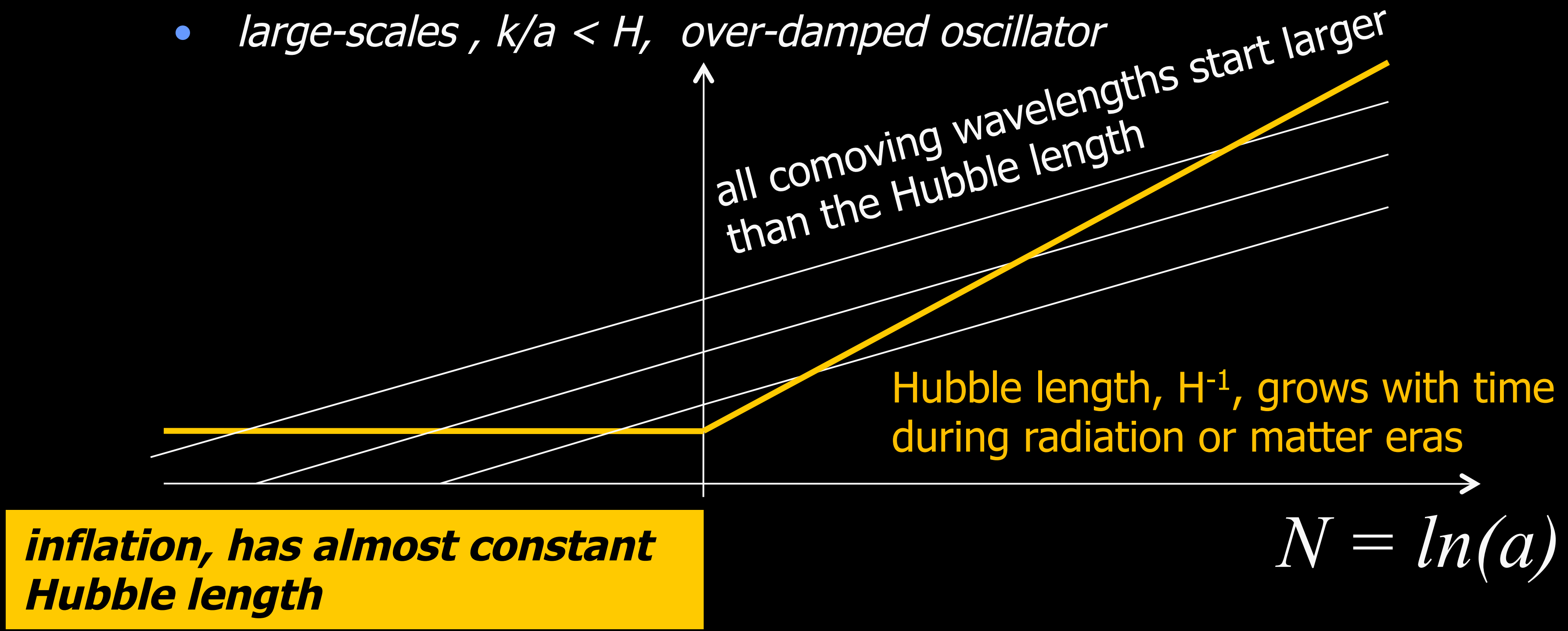

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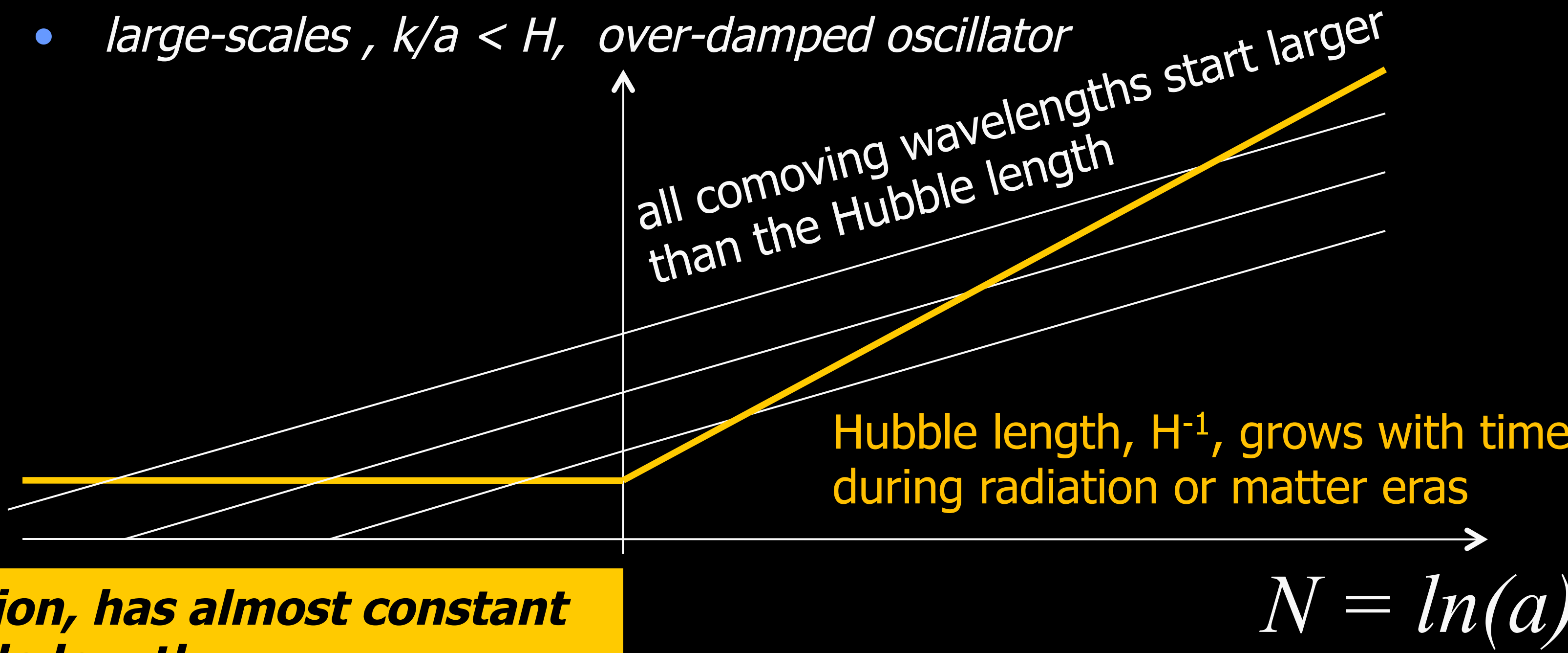
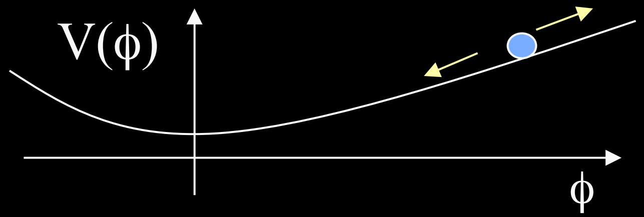


inflation, has almost constant Hubble length

require N > 50 during inflation for largest observable modes today to be within Hubble scale during inflation

Vacuum fluctuations

Hawking '82, Starobinsky '82, Guth & Pi '82



- *small-scale/underdamped zero-point fluctuations (k>aH)*

$$\left\langle \delta\varphi_k \delta\dot{\varphi}^*_{k'} \right\rangle = i\delta\left(k-k'\right)$$

- *large-scale/overdamped perturbations in growing mode (k<aH)*

linear evolution $\Rightarrow$ Gaussian random field

$$\Rightarrow \quad \mathcal{P}\left(\delta\varphi\right)_{k=aH} \approx \frac{4\pi k^3 \left|\delta\varphi_k^2\right|}{(2\pi)^3} = \left(\frac{H}{2\pi}\right)^2$$

fluctuations of any light scalar fields (m<3H/2) `frozen-in' on large scales

Scalar field perturbations

- Background cosmology has preferred time-slicing:

$$\varphi = \varphi_0(t)$$

but perturbed field does not:

$$\varphi = \varphi_0(t) + \delta\varphi(t, x^i)$$

- Time-slicing (gauge) freedom:

$$t \to \tilde{t} = t + \delta t(t, x^i)$$

- Scalar field $\tilde{\varphi} = \varphi$ invariant at same physical point:

$$\varphi_0(\tilde{t}) + \delta\tilde{\varphi}(\tilde{t}, x^i) = \varphi_0(t) + \delta\varphi(t, x^i)$$

- but field perturbation transforms at first order:

$$\delta\tilde{\varphi} = \delta\varphi - \dot{\varphi}_0 \delta t$$

Scalar perturbations

- Gauge-dependent scalar field perturbation:

$$\delta\tilde{\varphi} = \delta\varphi - \dot{\varphi}_0 \delta t$$

- Gauge-dependent metric/curvature perturbation:

$$\tilde{\psi} = \psi + H\delta t$$

- Gauge-invariant field perturbation:

$$Q = \delta\varphi + \frac{\dot{\varphi}_0}{H}\psi$$

- Gauge-invariant curvature perturbation:

$$R = \psi + \frac{H}{\dot{\varphi}_0}\delta\varphi = \frac{H}{\dot{\varphi}_0} Q$$

  - R remains constant for adiabatic perturbations on large (super-Hubble) scales

Tensor perturbations

- Gauge-independent at first order:

$$h_{ij} = \left[h^{+}_{\vec{k}} q^{+}_{ij}(\vec{k}) + h^{\times}_{\vec{k}} q^{\times}_{ij}(\vec{k}) \right] e^{i\vec{k}.\vec{x}}$$

- Each mode obeys damped wave equation:

$$\ddot{h}_{\vec{k}} + 3H\dot{h}_{\vec{k}} + \frac{k^2}{a^2} h_{\vec{k}} = 0$$

Tensor perturbations

- Simple equation for canonically normalised field $u_{\vec{k}} = ah_{\vec{k}}/\sqrt{32\pi G}$ in terms of conformal time

$$u''_{\vec{k}} + \left(k^2 - \frac{a''}{a}\right) u_{\vec{k}} = 0$$

where $a''/a \simeq (2-\epsilon)a^2H^2$

  - small scales (early times during inflation): $k^2 \gg (aH)^2$

$$u''_{\vec{k}} + k^2 u_{\vec{k}} = 0$$

    oscillations $u_{\vec{k}} \propto e^{\pm ik\tau}$

  - large scales (late times during inflation): $k^2 \ll (aH)^2$

$$u''_{\vec{k}} - \frac{a''}{a} u_{\vec{k}} = 0$$

    growing mode (squeezed state) $u_{\vec{k}} \propto a \;\Rightarrow\; h_{\vec{k}} \to \text{const}$

Scalar perturbations

- Klein-Gordon equation for Sasaki-Mukhanov variable, $v = aQ = zR$, in terms of conformal time:

$$v'' + \left(k^2 - \frac{z''}{z}\right) v = 0$$

where $z = a\dot{\varphi}/H$ and $z''/z \simeq (2 + 5\epsilon - 3\eta)a^2H^2$

  - small scales (early times during inflation): $k^2 \gg (aH)^2$

$$v''_{\vec{k}} + k^2 v_{\vec{k}} = 0$$

    oscillations $v_{\vec{k}} \propto e^{\pm ik\tau}$

  - large scales (late times during inflation): $k^2 \ll (aH)^2$

$$v''_{\vec{k}} - \frac{z''}{z} v_{\vec{k}} = 0$$

    growing mode (squeezed state) $v_{\vec{k}} \propto z \;\Rightarrow\; R_{\vec{k}} \to \text{const}$

Power spectra

- quantise canonically normalised fields on small scales $(k^2 \gg (aH)^2)$:
$$\langle \hat{u}_{\vec{k}_1} \hat{u}'_{\vec{k}_2} \rangle = \langle \hat{v}_{\vec{k}_1} \hat{v}'_{\vec{k}_2} \rangle = \frac{i\hbar}{2} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2)$$
- large scales $(k^2 \ll (aH)^2)$ described by *classical* fields:
$$\langle X_{\vec{k}_1} X_{\vec{k}_2} \rangle = \frac{P_X(k_1)}{4\pi k_1^3} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2)$$
  - tensors:
$$P_T(k) \simeq 64\pi G \left(\frac{H}{2\pi} \right)^2_{k=aH}$$
  - scalars:
$$P_R(k) \simeq \left[\left(\frac{H}{\dot{\varphi}} \right)^2 \left(\frac{H}{2\pi} \right)^2 \right]_{k=aH}$$
  - tensor-scalar ratio: $r \equiv P_T(k)/P_R(k) = 16\epsilon$

Weak scale dependence from slow roll

- Tensors

$$n_T \equiv \frac{d\ln P_T}{d\ln k} \simeq \left(\frac{d\ln H^2}{d\ln k}\right)_{k=aH} \simeq \left(\frac{d\ln H^2}{d\ln(aH)}\right)_{k=aH}$$

slow-roll time-dependence

$$\frac{d\ln H}{dt} = -\epsilon H \quad , \quad \frac{d\ln(aH)}{dt} = (1-\epsilon)H$$

gives (to first order in slow-roll)

$$n_T \simeq -2\epsilon$$

- Scalars (similar, exercise for the reader!)

$$n_R - 1 \equiv \frac{d\ln P_R}{d\ln k} \simeq -6\epsilon + 2\eta$$

Scale dependence of scale dependence (running)?

- Tensors

$$\alpha_T \equiv \frac{dn_T}{d\ln k} \simeq -4\epsilon(2\epsilon - \eta)$$

- Scalars

$$\alpha_R \equiv \frac{dn_R}{d\ln k} \simeq -8\epsilon(3\epsilon - 2\eta) - 2\xi^2$$

where

$$\xi^2 = \eta\left(2\epsilon - \frac{\dot{\eta}}{H\eta}\right) \sim M_P^3 \frac{V'''}{V}$$

- running is higher-order in slow-roll, $\alpha_R \sim (n_R - 1)^2$, hence expected to be negligible in slow-roll

Constraints from Planck 2015

- Scalar power spectrum

$$P_R(k = 0.05\,\mathrm{Mpc}^{-1}) = (2.207 \pm 0.763)\times 10^{-9}\ (68\%\ \mathrm{c.l.})$$

- Scalar tilt

$$n_R = 0.9645 \pm 0.0049$$

- Scalar running

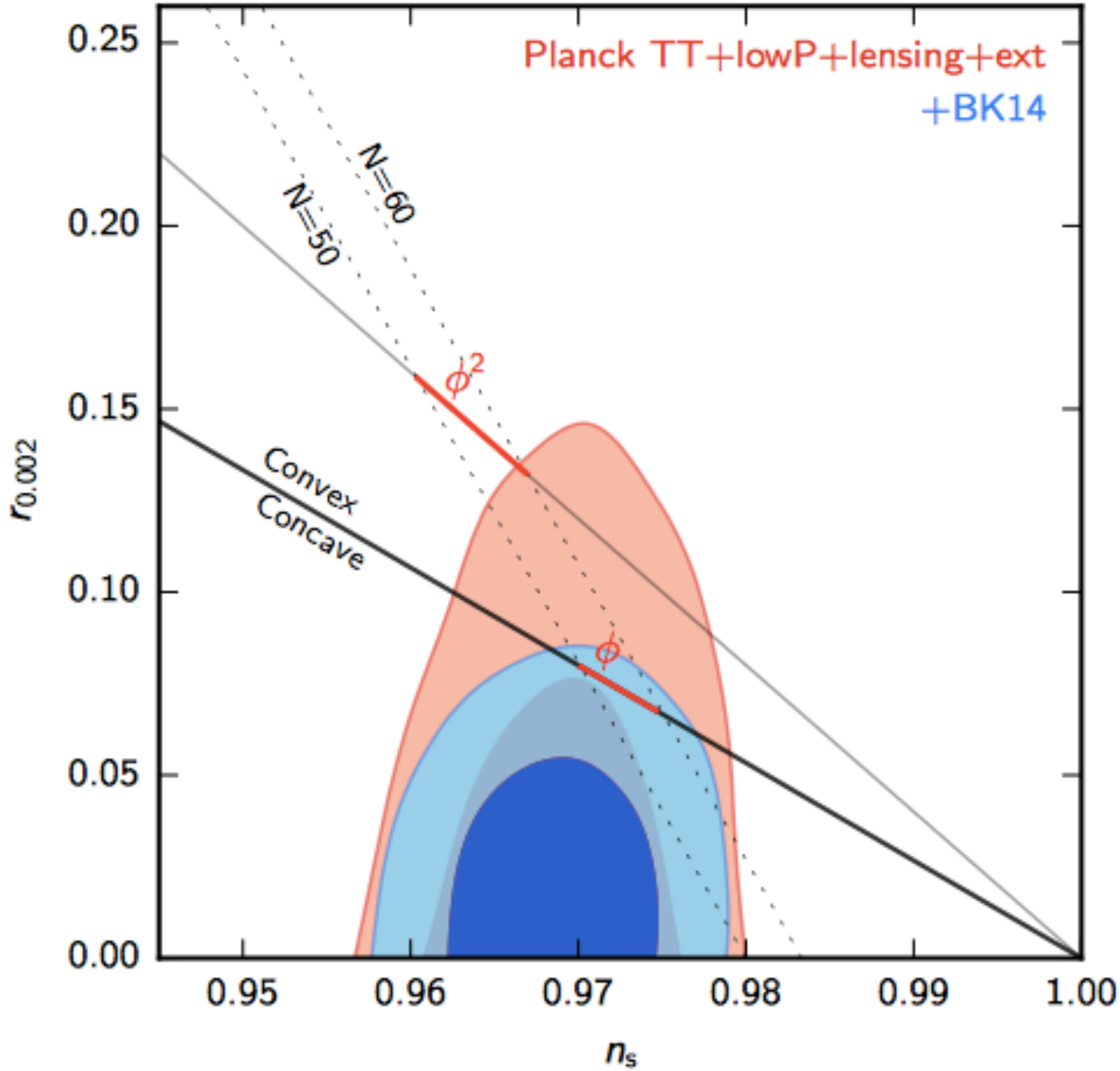
$$\alpha_R = -0.002 \pm 0.013$$

- Tensor-scalar ratio

$$r_{k=0.002\ \mathrm{Mpc}^{-1}} < 0.112\ (95\%\ \mathrm{c.l.})$$

- Planck+BICEP2+Keck

$$r_{k=0.002\ \mathrm{Mpc}^{-1}} < 0.09\ (95\%\ \mathrm{c.l.})$$



Planck TT+lowP+lensing+ext
+BK14
0.25
0.20
0.15
0.10
0.05
0.00
0.95
0.96
0.97
0.98
0.99
1.00
$r_{0.002}$
n_s
N=60
N=50
ϕ^2
ϕ
Convex
Concave

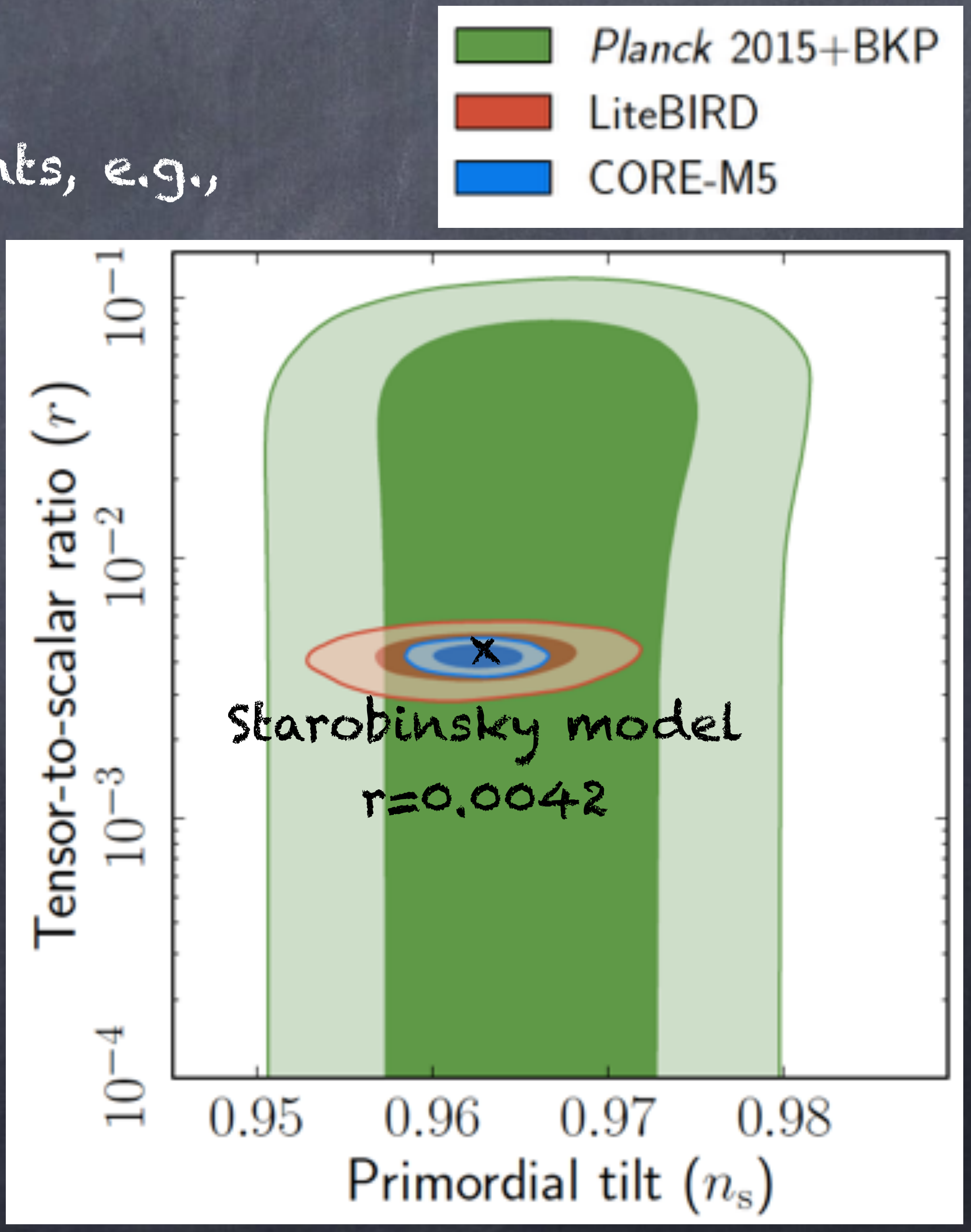
The search for a smoking gun:

- scale-invariant primordial density perturbations was a post-diction (Harrison/ Zel'dovich 1970)
    - now see deviation from scale-invariance predicted by slow-roll inflation
- primordial gravitational waves
    - an unexpected bonus feature - not seen yet

the search for primordial GWs

$$r = \frac{A_T^2}{A_S^2} = \frac{\langle h^2 \rangle}{2 \times 10^{-9}}$$

- current Planck+BICEP/Keck upper bound
  - $r < 0.07$, $V^{1/4} < 2 \times 10^{16}$ GeV
- several upcoming ground-based experiments, e.g.,
  - Polarbear-2: $r < 0.01$
  - Simons Array: $r < 0.006$
- future satellite experiments
  - arXiv:1612.08270
    - LiteBIRD (2024): $r < 0.0004$
    - CORE-M5 (2029): $r < 0.0002$
    - $V^{1/4} < 4 \times 10^{15}$ GeV



Summary

- Primordial perturbations from inflation:
  - initial conditions from sub-Hubble scale vacuum fluctuations
  - gauge-invariant field+metric fluctuations from inflaton field fluctuations
  - slow-roll induces weak scale-dependence
    - *confirmed by observations*
  - gravitational waves (tensors) also predicted
    - *but not yet seen*
- Observations can
  - constrain model parameters
  - discriminate between different kinds of inflation models