

Introduction to Inflation

Lecture 1: Homogeneous cosmology and scalar fields

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Verao Quantico, March 2017

why we like inflation

- physical model for origin of primordial universe (horizon structure, spatial flatness, inhomogeneities)
- explores ultra-high ($\gg \text{TeV}$ scale) physics
- combines gravity and quantum mechanics



an inflationary cosmologist's agenda:

1. prove inflation really happened
 - search for a smoking gun
2. show how inflation happened
 - what is the correct inflation model?
3. explore inflationary phenomenology
 - (non-)Gaussianity, (non-)adiabaticity, black holes, etc
 - particle production and reheating after inflation
4. understand the inflationary landscape
 - multiverse, measure problem, anthropic arguments, alternatives

Outline

Homogeneous cosmology

- spacetime metric
- Einstein equations
- problem of initial conditions

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Scalar field cosmology

- equations of motion
- inflation
- slow-roll inflation

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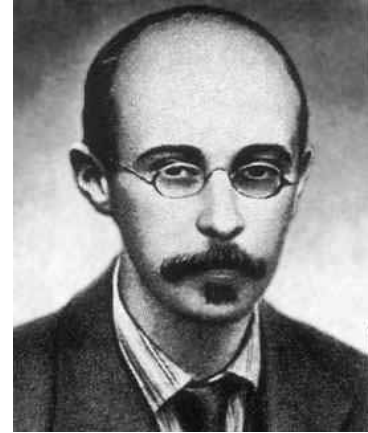
Reheating

- ending inflation

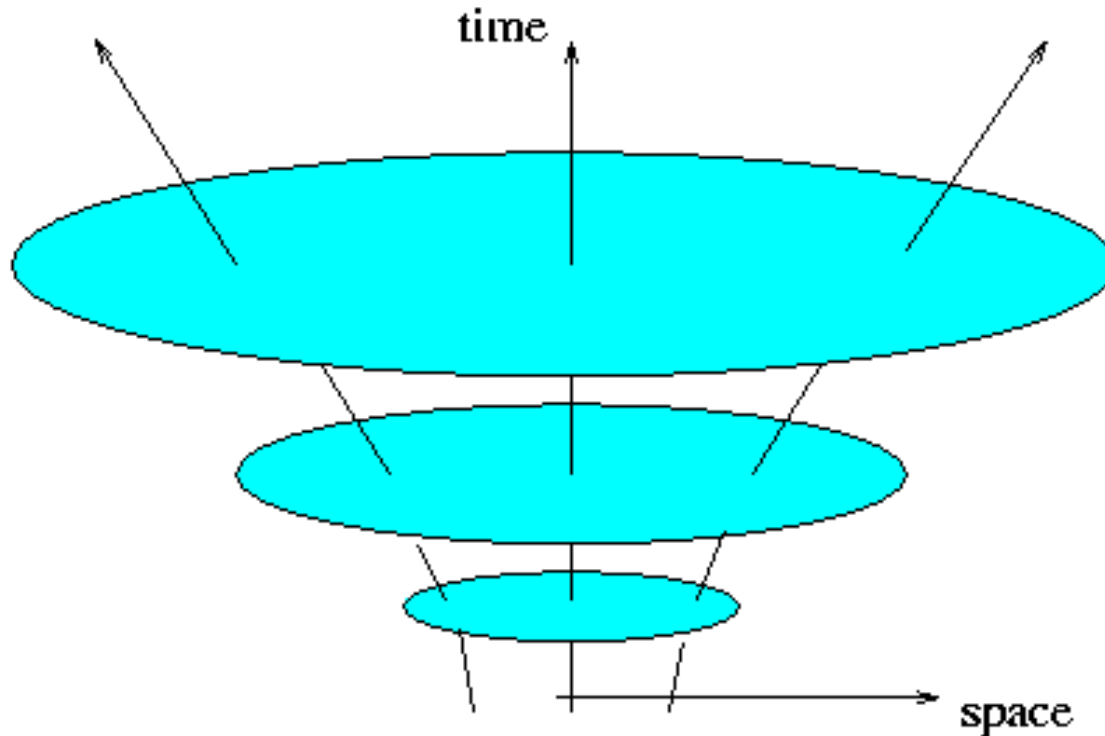
Reheating

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Friedmann's dynamic cosmology



- slice up 4D spacetime into expanding 3D space with uniform matter density ρ and spatial curvature K/a^2



- t

- ▶ 4D spacetime split into 1+3
- ▶ Friedmann-Lemaitre-Robertson-Walker (FLRW) line element:

$$ds^2 = -c^2 dt^2 + a^2(t) dX^2$$

- ▶ time + homogeneous and isotropic space
- ▶ dynamical scale factor, $a(t)$, where $a_0 = 1$ today
- ▶ maximally-symmetric 3-space, curvature K

$$dX^2 = \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

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$$dX^2 = \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

- ▶ alternative (conformal) time coordinate, $d\tau = c dt/a$:

$$ds^2 = a^2(\tau) [-d\tau^2 + dX^2]$$

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- ▶ cosmologists use natural units:

$$c = 1 \quad , \quad \hbar = 1 \quad , \quad k_{\text{Boltzmann}} = 1$$

so that

- ▶ time=length
 - ▶ energy=mass=frequency=temperature
- ▶ only dimensional unit left is Planck
mass/energy/temperature/time/length:

$$M_P = E_P = t_P^{-1} = L_P^{-1} = G^{-1/2} = 10^{19} \text{ GeV}$$

this is the quantum gravity scale...

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FLRW Einstein equations

- ▶ relate scale factor dynamics to density ρ and pressure P
- ▶ Evolution equation:

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3P) \quad (1)$$

- ▶ Friedmann constraint:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \quad (2)$$

- ▶ Continuity equation (energy conservation):

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + P) \quad (3)$$

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FLRW Einstein equations

- ▶ rewrite in terms of Hubble expansion rate: $H(t) = \dot{a}/a$
- ▶ Evolution equation:

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P) \quad (4)$$

- ▶ Friedmann constraint:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \quad (5)$$

- ▶ Continuity equation (energy conservation):

$$\dot{\rho} = -3H(\rho + P) \quad (6)$$

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- ▶ conformal Hubble rate: $\mathcal{H} = (da/d\tau)/a = \dot{a} = aH$

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Cosmological solutions

- ▶ Matter-domination (non-relativistic particles):

$$P_m = 0 \quad \Rightarrow \quad \rho_m \propto a^{-3} \quad \Rightarrow \quad a \propto t^{2/3} \text{ for } K = 0$$

- ▶ Radiation-domination (relativistic particles):

$$P_r = \rho_r/3 \quad \Rightarrow \quad \rho_r \propto a^{-4} \quad \Rightarrow \quad a \propto t^{1/2} \text{ for } K = 0$$

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- ▶ for g_* relativistic species in thermal equilibrium

$$\rho_r = g_* \frac{\pi^2}{30} T^4 = \frac{3H^2}{8\pi G} = \frac{3}{32\pi G t^2}$$

$$\Rightarrow \quad \frac{t}{1 \text{ sec}} = g_*^{-1/2} \left(\frac{1 \text{ MeV}}{T} \right)^2$$

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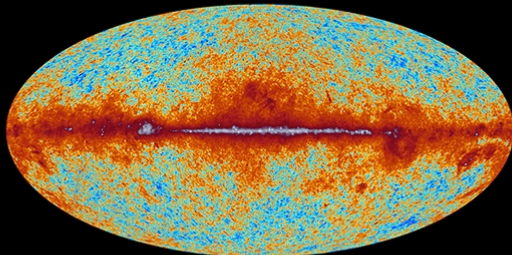
- ▶ Matter-radiation equality (when $\rho_m = \rho_r$):

$$1 + z_{\text{eq}} = \frac{a_0}{a_{\text{eq}}} = \frac{\rho_{m,0}}{\rho_{r,0}} \simeq 3400$$

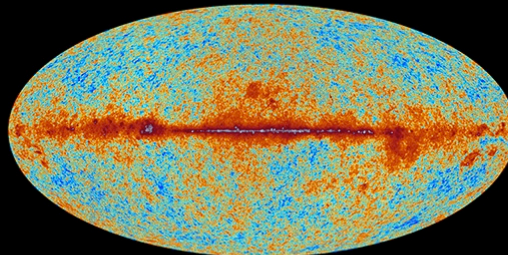


planck

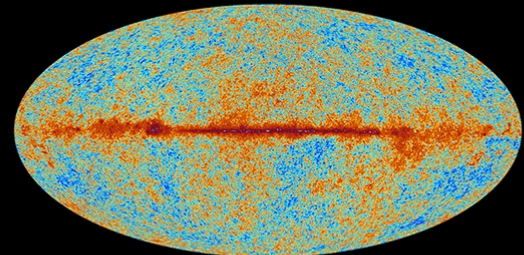
The sky as seen by Planck



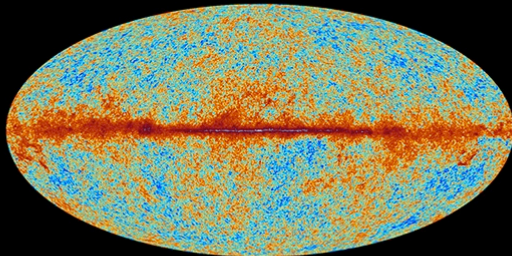
30 GHz



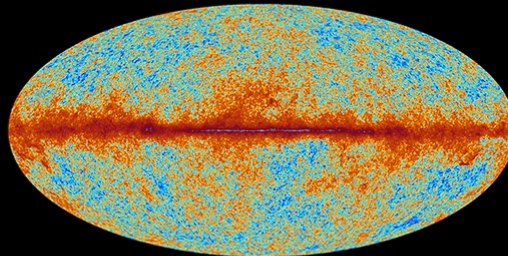
44 GHz



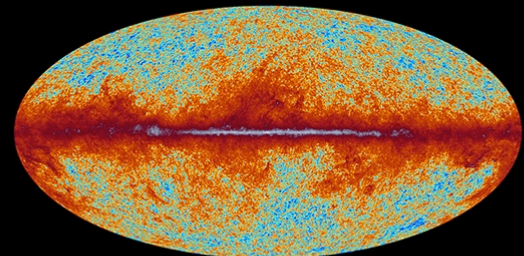
70 GHz



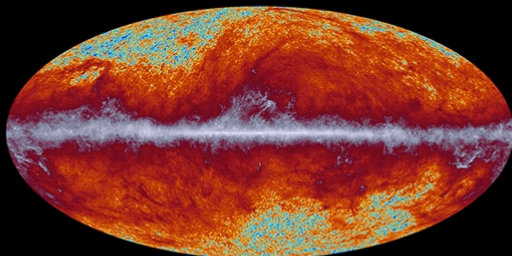
100 GHz



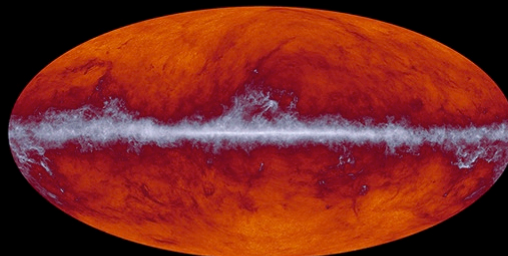
143 GHz



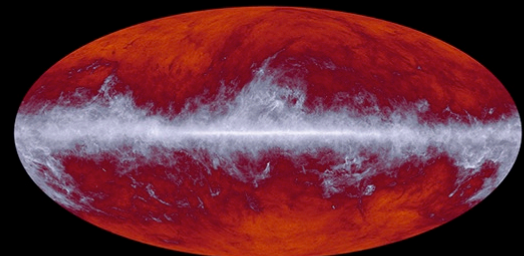
217 GHz



353 GHz

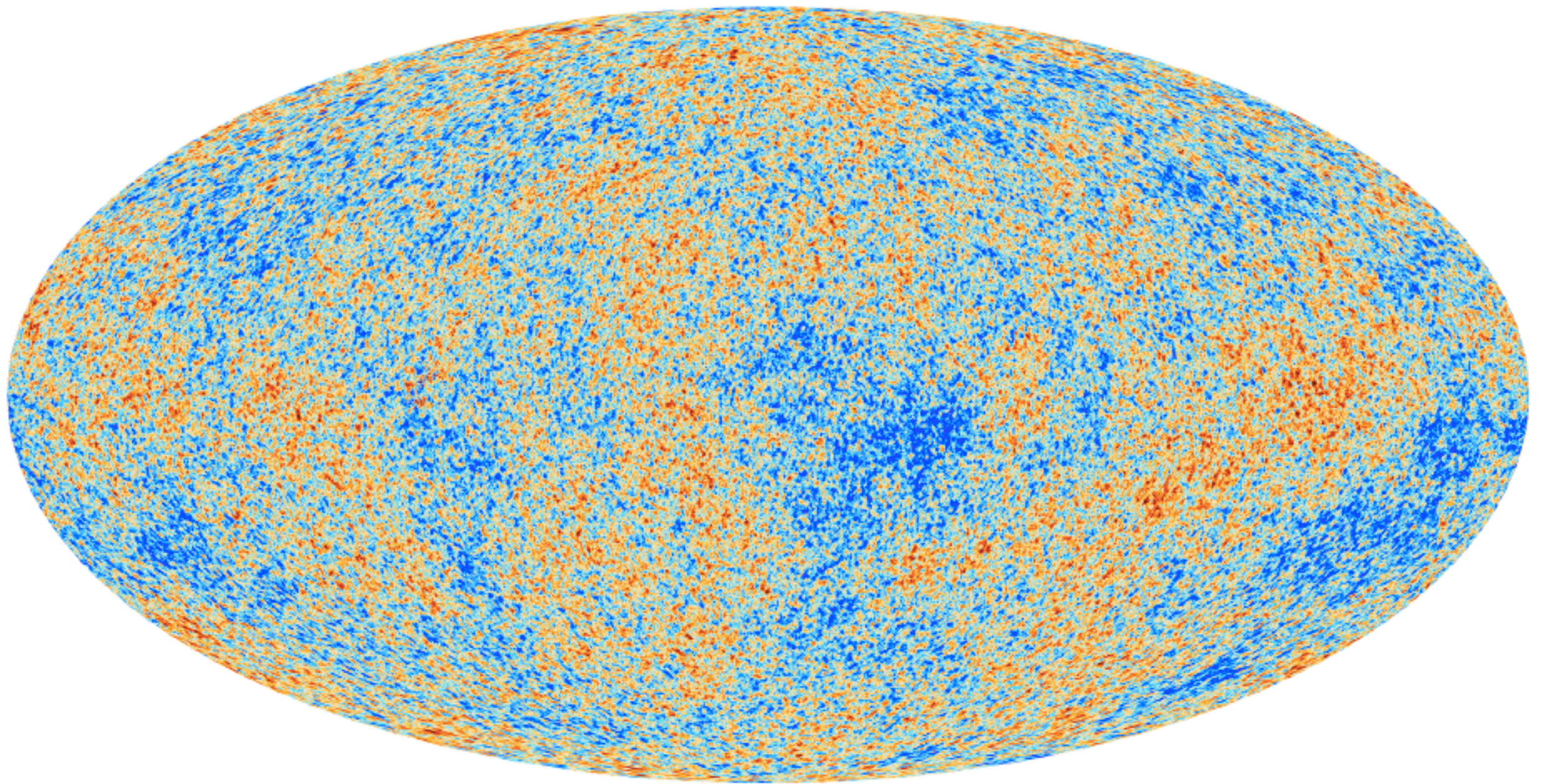


545 GHz

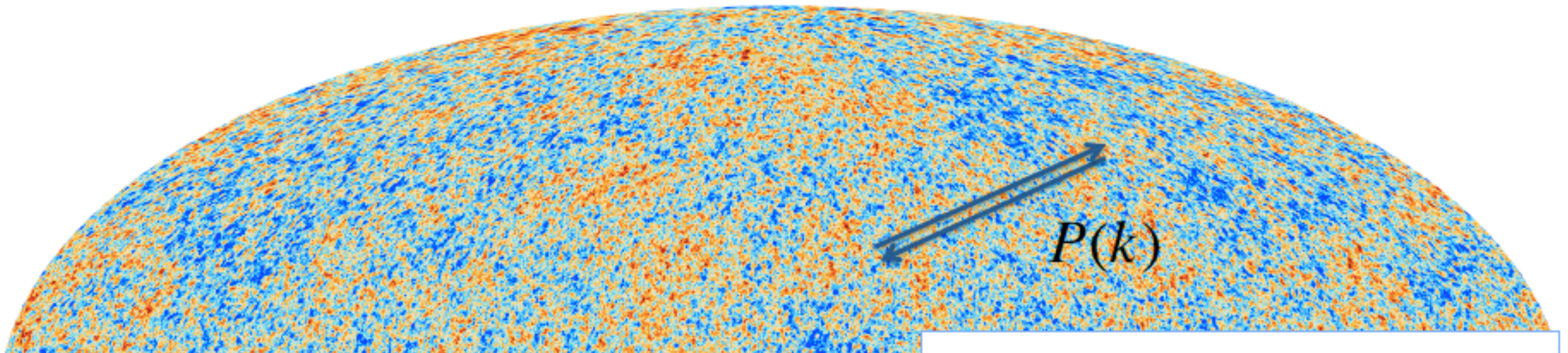


857 GHz

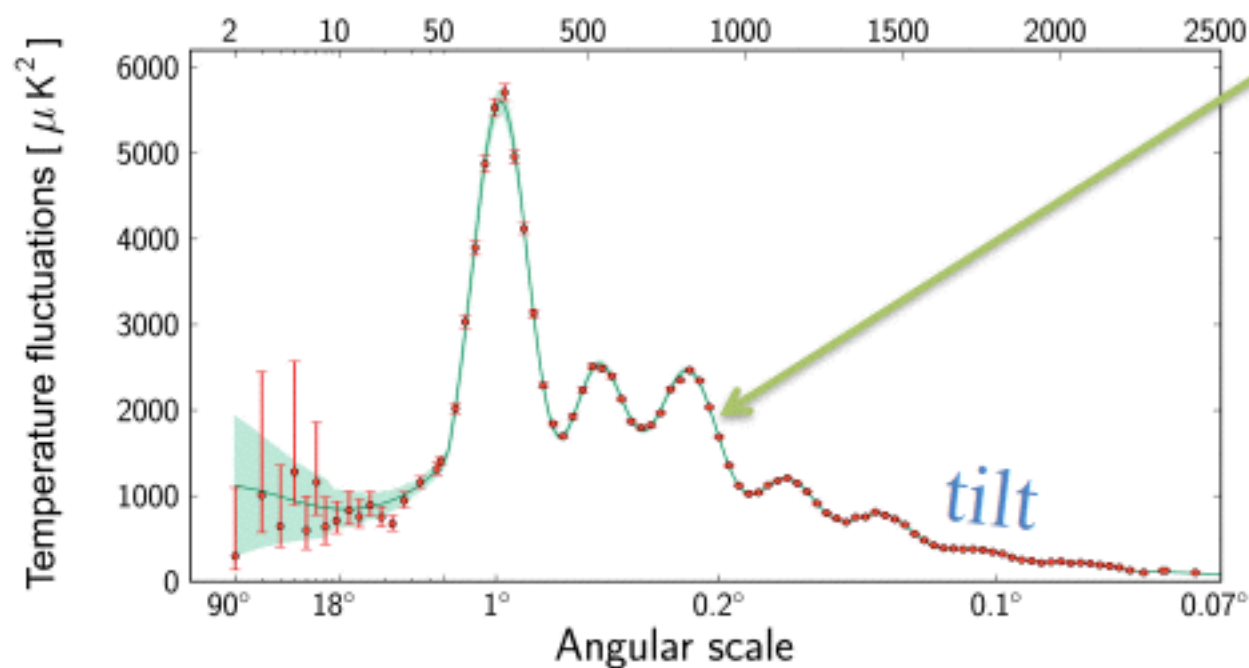
Precision cosmology Ade et al 2015



Precision cosmology Ade et al 2015



Angular power spectrum:



6-parameter model:

4-parameter cosmology:

- $H_0 = 67.8 \pm 0.9$
- $\Omega_{\text{matter}} = 0.308 \pm 0.012$
- $\Omega_{\Lambda} = 0.692 \pm 0.012$
- Reionisation $z = 8.8 \pm 1.7$

2-parameter *initial fluctuations*:

- Amplitude $\Delta T/T = 0.000047$
- Tilt = -0.032 ± 0.006

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Problem of initial conditions in the hot big bang cosmology

1. Flatness problem

- ▶ *Why is the Universe (almost) spatially flat?*

$$\frac{|K|}{H_0^2} < 0.005 \quad (95\% \text{ c.l. Planck 2015})$$

2. Horizon problem

- ▶ *Why is the cosmic microwave background so isotropic?*

3. Smoothness problem

- ▶ *Where do the small ($\sim 10^{-5}$) primordial density and temperature fluctuations come from?*

- ▶ Causal particle horizon:
(coordinate distance travelled by photons in expanding universe since $a = 0$)

$$\begin{aligned} L &= \int dX \\ &= \int_0^t \frac{c dt'}{a(t')} \\ &= \tau \end{aligned}$$

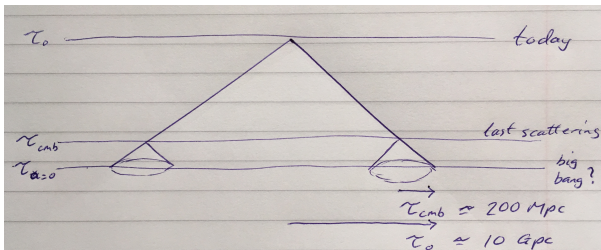


Figure : horizon problem

- spacetime metric
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ending inflation

2. Horizon problem

- ▶ Causal particle horizon:
(coordinate distance travelled by photons in expanding universe since $a = 0$)

$$\begin{aligned} L &= \int dX \\ &= \int_0^t \frac{c dt'}{a(t')} \\ &= \tau \end{aligned} \tag{7}$$

- ▶ in a matter-dominated cosmology ($a \propto t^{2/3} \propto \tau^2$):

$$\frac{L_{\text{cmb}}}{L_0} = \frac{\tau_{\text{cmb}}}{\tau_0} = \sqrt{\frac{1}{1 + z_{\text{cmb}}}} \simeq \frac{1}{30}$$

2. Horizon problem

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- scales $> 2^\circ$ on the cosmic microwave background sky are not in causal contact, *so how can one explain their properties?*

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Solving the problem of initial conditions

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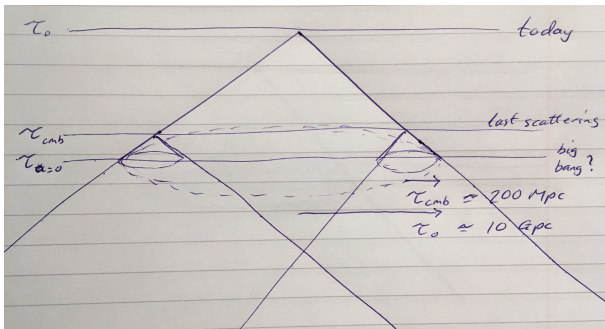
- ▶ *Inflation* = accelerated expansion: $\ddot{a} > 0$
- ▶ from evolution equation (4): $\rho + 3P < 0$

2. Horizon problem

- Causal particle horizon:
(coordinate distance travelled by photons since $a = 0$)

$$\begin{aligned}\tau &= \int_0^t \frac{c dt'}{a(t')} \\ &= \int_{-\infty}^a \frac{d \ln a'}{\dot{a}(a')} \end{aligned} \quad (7)$$

- if $\dot{a}$ decreases as $a \rightarrow 0$ then the integral diverges
there is no particle horizon for $\ddot{a} > 0$



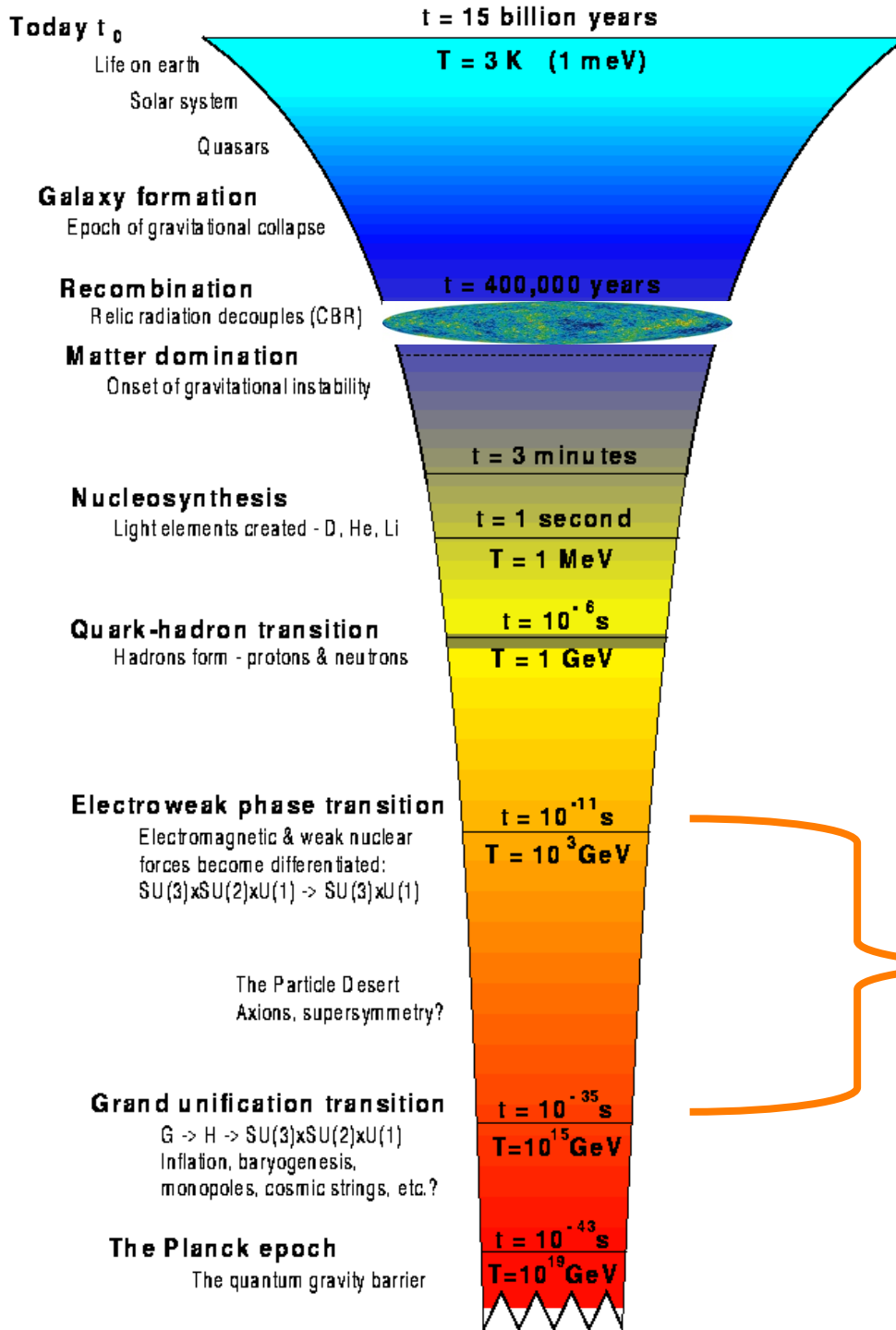
2. Horizon problem

- Causal particle horizon:
(coordinate distance travelled by photons since $a = 0$)

$$\begin{aligned}\tau &= \int_0^t \frac{c dt'}{a(t')} \\ &= \int_{-\infty}^a \frac{d \ln a'}{\dot{a}(a')}\end{aligned}\tag{8}$$

- if $\dot{a}$ decreases as $a \rightarrow 0$ then the integral diverges
there is no particle horizon for $\ddot{a} > 0$
- however
there exists a finite future *event horizon* for $\ddot{a} > 0$:

$$\tau_{\infty} - \tau = \int_t^{\infty} \frac{c dt'}{a(t')} = \int_{\ln a}^{\infty} \frac{d \ln a'}{\dot{a}(a')}$$

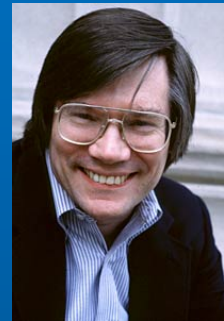


Inflation

- accelerated early expansion

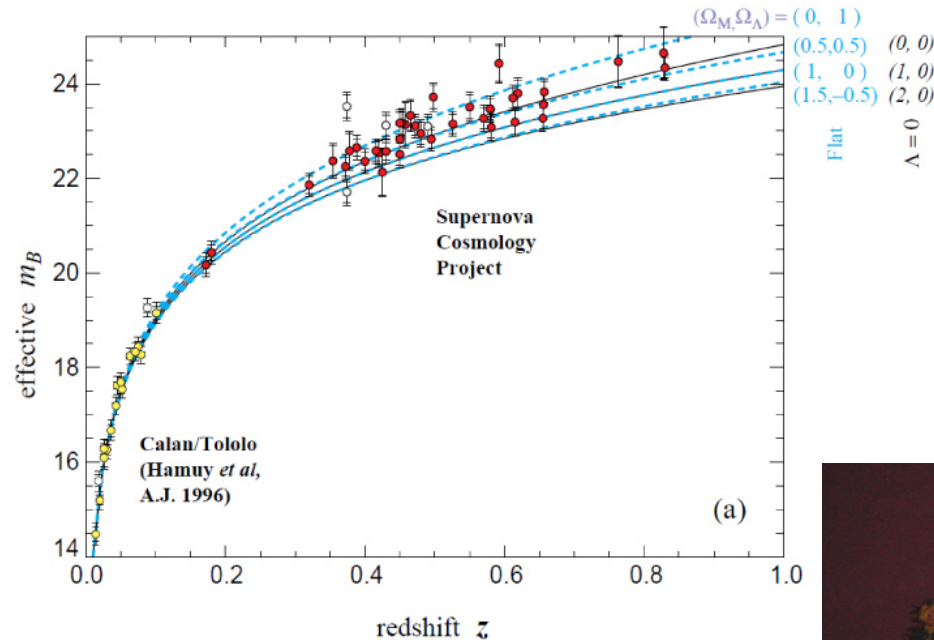
Starobinsky (1980)

Guth (1981)



- $10^3 - 10^{16}$ GeV
 > LHC
 < quantum gravity
- redshift $10^{15} - 10^{28}$
- size of our presently observable universe was 10^{11} m – 1 cm
- density $10^{30} - 10^{82}$ gcm $^{-3}$

Accelerated expansion in the *late* Universe



1998 supernovae
Hubble diagram
distance v redshift

Nobel prize **2011**



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Homogeneous scalar field cosmology

- ▶ FLRW spacetime

$$ds^2 = -dt^2 + a^2(t)dX^2$$

- ▶ Spatially homogeneous scalar field: $\varphi(t)$
- ▶ Klein-Gordon equation:

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi)$$

rolling down gradient, $V'(\varphi)$, with Hubble damping, H

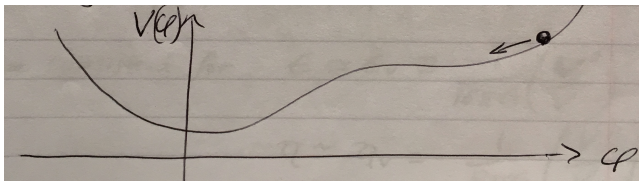


Figure : ball rolling down hill

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rolling down gradient, $V'(\varphi)$, with Hubble damping, H

- ▶ Energy density (kinetic+potential):

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

- ▶ Pressure (kinetic-potential):

$$P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

- Einstein equations:

$$H^2 = \frac{8\pi G}{3} \left(V + \frac{1}{2}\dot{\phi}^2 \right)$$

$$\dot{H} = -4\pi G\dot{\phi}^2$$

- define slow-roll parameter:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3\dot{\phi}^2}{2V + \dot{\phi}^2}$$

- Inflation, $\ddot{a} > 0 \Rightarrow \dot{H} + H^2 > 0 \Rightarrow \epsilon < 1 \Rightarrow \dot{\phi}^2 < V$

Slow-roll field inflation

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- ▶ Inflation, $\ddot{a} > 0 \Rightarrow \epsilon < 1 \Rightarrow \dot{\phi}^2 < V$
- ▶ *Slow-roll inflation* $\Rightarrow \epsilon \ll 1 \Rightarrow \dot{\phi}^2 \ll V$

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- ▶

$$H^2 = \frac{8\pi G}{3} \left(V + \frac{1}{2} \dot{\phi}^2 \right)$$
$$\ddot{\phi} + 3H\dot{\phi} = -V'$$

Slow-roll approximation:

$$H^2 \simeq \frac{8\pi G}{3} V \Rightarrow \dot{\phi}^2 \ll V \Rightarrow \epsilon \ll 1$$
$$3H\dot{\phi} \simeq -V' \Rightarrow |\ddot{\phi}| \ll 3H|\dot{\phi}| \Rightarrow |\eta| \ll 1$$

Slow-roll field inflation

- ▶ Inflation, $\ddot{a} > 0 \Rightarrow \epsilon < 1 \Rightarrow \dot{\phi}^2 < V$
- ▶ *Slow-roll inflation* $\Rightarrow \epsilon \ll 1 \Rightarrow \dot{\phi}^2 \ll V$
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$$H^2 = \frac{8\pi G}{3} \left(V + \frac{1}{2} \dot{\phi}^2 \right)$$
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Slow-roll approximation:

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consistency conditions for potential: $\epsilon_v \ll 1, |\eta_v| \ll 1$

$$\epsilon \simeq \epsilon_v \equiv \frac{1}{16\pi G} \left(\frac{V'}{V} \right)^2$$

$$\eta \simeq \eta_v \equiv \frac{1}{8\pi G} \left(\frac{V''}{V} \right)$$

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- note:
slow-roll parameters approximately constant in slow-roll
(an exercise for the reader!)

$$\frac{d\epsilon}{d \ln a} = \frac{\dot{\epsilon}}{H} \simeq 2\epsilon(-\eta + 2\epsilon) = \mathcal{O}(\epsilon^2)$$

Slow-roll field inflation: e-folds

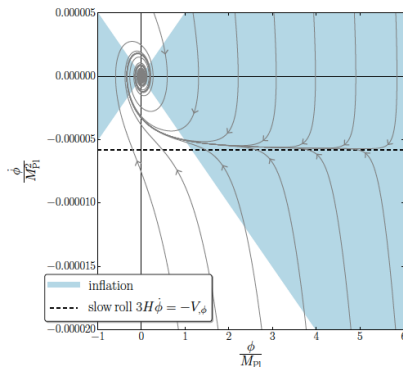
- useful approximation for growing mode (reduces 2nd-order ODE to 1st order)

$$H^2 \simeq \frac{8\pi G}{3} V$$

$$3H\dot{\phi} \simeq -V'$$

- e.g., phase-space diagram for $V(\phi) = m^2\phi^2/2$

Grain+Vennin (2017)



Slow-roll field inflation: e-folds

- useful approximation for growing mode
(reduces 2nd-order ODE to 1st order)

$$H^2 \simeq \frac{8\pi G}{3} V$$
$$3H\dot{\phi} \simeq -V'$$

- e.g., calculate the integrated logarithmic expansion:
e-folds

$$N = \ln\left(\frac{a_2}{a_1}\right) = \int_{t_1}^{t_2} H dt$$
$$\simeq \int_{\varphi_1}^{\varphi_2} \frac{H}{\dot{\phi}} d\varphi \simeq \int_{\varphi_2}^{\varphi_1} \sqrt{\frac{4\pi}{\epsilon_V}} \frac{d\varphi}{M_P} \quad \text{for } V' < 0$$

Slow-roll field inflation: e-folds

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- $N \geq 50$ requires $\Delta\varphi \sim 50\sqrt{\epsilon_V/4\pi}M_P$
so $\Delta\varphi \ll M_P$ requires $\epsilon_V \ll 0.01$

- ▶ Standard hot big bang cosmology has strong observational support
 - ▶ primordial nucleosynthesis of the light elements at $T \sim \text{MeV}$
 - ▶ cosmic microwave background well described by 6-parameter model
- ▶ Inflation can provide suitable initial conditions
 - ▶ spatially flat
 - ▶ statistically homogeneous and isotropic
 - ▶ *nearly scale-invariant, Gaussian, adiabatic primordial density perturbations*
- ▶ Scalar field potential energy can drive acceleration
 - ▶ *and produce primordial perturbations*

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