The asymptotic structure of gravity III

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In order to better understand its role in the quantum theory, where physical states are usually defined on spacelike (Cauchy) hypersurfaces,

it is desirable to understand its action on spacelike (Cauchy) hypersurfaces, and thus, at spatial infinity.

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So : which picture should be trusted?

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The purpose of this last lecture is to explain how the infinite-dimensional BMS group appears at spatial infinity.

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The definition of phase space must be completed by boundary conditions on the canonical variables (g_{ij} and π^{ij}).

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[In the anti-de Sitter case, one would have similar conditions, with the background metric replaced by the AdS metric and the Poincaré group replaced by the AdS group.]

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$$\int d^3x \pi^{ij} \dot{g}_{ij} \sim \ln r.$$

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$$h_{ij} \equiv g_{ij} - \delta_{ij} = \frac{\overline{h}_{ij}(\mathbf{n}^k)}{r} + O(\frac{1}{r^2}), \quad \overline{h}_{ij}(-\mathbf{n}^k) = \overline{h}_{ij}(\mathbf{n}^k)$$

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This can be seen by following the general prescriptions given in the first lecture.

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The idea is to allow a "parity-twisted component" of a specific form in the leading orders in the asymptotic expansions of the metric and its conjugate momentum.

This parity-twisted component takes the form of a variation of the fields generated by a $\mathcal{O}(1)$ diffeomorphism (rewritten in Hamiltonian form).

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$$U_{ij} = \partial_i \zeta_j + \partial_j \zeta_i = O(\frac{1}{r}), \quad \zeta^i = \zeta^i(\mathbf{n}^k) = O(1), \quad \zeta^i(-\mathbf{n}^k) = \zeta^i(\mathbf{n}^k),$$

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(The particular form $\zeta_i = \partial_i(rU)$ is needed to ensure that the boosts are canonical transformations - see literature for technical derivations.)

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$$\pi^{ij} = V^{ij} + p^{ij}$$
$$V^{ij} = \partial^i\partial^j V - \delta^{ij}\mathring{\triangle} V, \quad V = V(\mathbf{n}^k) = O(1), \quad V(-\mathbf{n}^k) = V(\mathbf{n}^k)$$
$$p^{ij} = \frac{(\overline{\pi}^{ij})^{odd}(\mathbf{n}^k)}{r^2} + O(\frac{1}{r^3}), \quad (\overline{\pi}^{ij})^{odd}(-\mathbf{n}^k) = -(\overline{\pi}^{ij})^{odd}(\mathbf{n}^k)$$

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The twisting is thus characterized by two functions of the angles, one odd (U) and one even (V).

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The zero mode of T and the first spherical harmonic component of W describe translations.

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The algebra of the generators can be easily verified to be the BMS algebra, written in a different basis.

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There is thus complete agreement with the null infinity results once the twist in the parity conditions is allowed.

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where T_i and S_α are respectively the generators of the standard translations and of the pure supertranslations.

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These features originate from the fact that the Poincaré subalgebra is not an ideal.

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Can one provide a definition of the angular momentum free from this supertranslation ambiguity?

Not within the original BMS algebra (this is a theorem due to Sachs).

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The logarithmic supertranslations are diffeomorphisms that grow like $\ln r$ at infinity and which preserve by construction the new boundary conditions.

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The physical logarithmic supertranslations are thus parametrized by the same class of functions as the pure supertranslations. We denote them L_α.

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[The simplicity in form of the central term results from the redefinition $L^\alpha = \sigma^{\alpha\beta} L_\beta$.]

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This is achieved by Darboux-like (nonlinear) redefinitions.

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One can also check

$$\{\tilde{M}_a, \tilde{M}_b\} = f_{ab}^{c}\tilde{M}_c \quad , \quad \{\tilde{M}_a, T_i\} = R_{ai}{}^{j} T_j\,. \tag{5.4}$$

so that the Poincaré algebra is unchanged.

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The new Lorentz generators are supertranslation invariant (ambiguity has been eliminated).

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Integrability of the charges associated with these field-dependent transformations is not an issue since we are working with the generators throughout.

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Once it is included, one finds a perfect match with the results obtained at null infinity.

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THANK YOU!