The asymptotic structure of gravity II

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Part II : Electromagnetism

Introduction : the problem in a nutshell

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How can one see that at spatial infinity ?

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These conditions imply

$$F_{ik} = \frac{f_{ik}^{(1)}(\mathbf{n})}{r^2} + \frac{f_{ik}^{(2)}(\mathbf{n})}{r^3} + o(r^{-3})$$

for the magnetic field.

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One needs to strengthen the boundary conditions.

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because the leading orders of Lorentz parameters are parity-odd (boosts and rotations).

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From the point of view of containing the known solutions, these boundary conditions seem therefore acceptable.

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We must therefore both determine the asymptotic gauge transformations that preserve the boundary conditions and compute the value of their generators.

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The odd part ϵ_{odd} of the gauge parameter ϵ gives a zero contribution to the charges $G[\epsilon]$.

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By contrast, the gauge transformations with ϵ_{even} are improper gauge transformations that do change the physical state of the system.

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This seems to be an infinite-dimensional global symmetry with angle-dependent gauge transformations...

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Can one devise a “more liberal” strengthening of the original boundary conditions?

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"Parity-twisted boundary conditions" with a twist given by a gauge transformation.

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and it would be wrong to take the quotient by such transformations.

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With these new boundary conditions, the potential possesses, in addition to the even part characteristic of the previous boundary conditions, an odd part (equal to a pure gradient), which is called the "parity-twisted component".

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We impose $\partial_A \overline{\pi}^A = 0$, which is equivalent to the request that the leading order $O(r^{-1})$ in $\partial_i \pi^i$ be absent.

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To cure these problems, one needs to introduce the field A_0 and its conjugate π^0.

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It is this odd part which is the source of the difficulty.

Difficulties with boosts

Let Ω be the symplectic form and X be a vector field.

The transformation defined by X is canonical if $d_V(\iota_X\Omega) = 0$ (symplectic structure invariant). One then has $\iota_X\Omega = -d_V F$, where F is the canonical generator of the transformation.

In our case, if one performs the computation for the boosts, one finds

$$d(\iota_b\Omega) = \int d^3x\,\partial_m\left(\sqrt{g}\xi d_V F^{mi}\right) d_V A_i = \oint d^2x\sqrt{\bar{\gamma}}\, d_V\overline{A}_r\,\overline{D}^A(b\, d_V\overline{A}_A) \neq 0.$$

This term would be zero if $\overline{A}_A$ would be even, but we have seen that we must allow an odd part if we want non-trivial asymptotic $u(1)$ transformations.

It is this odd part which is the source of the difficulty.

The failure to be a canonical transformation is entirely due to a non-vanishing boundary term.

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We cure the problem by re-introducing A_0, with

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and its conjugate π^0. (Note that $\overline{\psi} \equiv \overline{A}_0$.)

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We add to the symplectic 2-form the surface term

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The transformation law $\delta_b A_0 = \mathcal{L}_b A_0$ implies that $\overline{\Psi}$ transforms under boosts as

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The constraint $\pi^0 \approx 0$ is first class and can be used to shift A_0,

$$\delta_\mu A_0 \equiv [A_0, G[\mu]] = \mu\,, \qquad G[\mu] = \int d^3x \mu \pi^0 \approx 0\,.$$

The generator $G[\mu]$ is well-defined and defines a proper gauge transformation provided μ vanishes at infinity (at least) as $\mathcal{O}(r^{-2})$ (which implies in particular $\delta_\mu \overline{A}_0 = 0$).

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This transformation is a non trivial symmetry and $\overline{A}_0$ a non-trivial "boundary field".

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One can show that the leading logarithmic term (which is $r^2 \log r$ for the field strengths) is eliminated by our parity conditions, but subdominant logarithms will be present.

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The parity conditions on the initial data also provides a justification of Strominger's matching conditions.

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The action of the Poincaré subalgebra characterising this semi-direct sum is :

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\delta_{(Y,b,T,W)}\overline{\mu} = Y^A \partial_A \overline{\mu} - \overline{D}_A(b\overline{D}^A\overline{\epsilon}), \quad \delta_{(Y,b,T,W)}\overline{\epsilon} = Y^A \partial_A \overline{\epsilon} - b\overline{\mu}.
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End of Part II