The asymptotic structure of gravity
Part I: General Considerations

Marc Henneaux

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Introduction

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These lectures will explain these recent results and discuss their connection with null infinity.

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(Main collaborators : Oscar Fuentealba and Cédric Troessaert)

Part I : General considerations

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from which the whole canonical structure derives.

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This function is determined up to a constant.

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In particular, while all smooth functions define canonical transformations in classical mechanics,

this might not be the case in field theory.

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Here,

$$A_i = \frac{\overline{A}_i}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad \pi^i = \frac{\overline{\pi}^i}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right).$$

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$G[\epsilon]$ (with surface term included) is a well-defined Hamiltonian generator even if ϵ does not vanish at infinity (Regge-Teitelboim criterion (Regge-Teitelboim 1974)).

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There is a surface term in the symplectic form!

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Asymptotic symmetries

Poisson manifolds and nonlinear realizations

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and involves a surface term even though the bulk term contains no derivative! (Beyond RT)

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which should not be factored out.

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The set of proper gauge symmetries form therefore an ideal.

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The physical content of asymptotic symmetries is entirely captured by their asymptotic behaviour, hence the terminology.

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These generators define a Poisson manifold.

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The generators of improper gauge transformations provide therefore observables.

This is why the energy and the other Poincaré charges are observables in gravity, even though they are not invariant under all diffeomorphisms.

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[If p is a constant of the motion, p^2 or any function of p is also a constant of the motion but it is clearly not independent from p even though not a multiple of p.]

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($f^A dQ_A$ is exact, "integrability")

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[If f_{AB} in $\{Q_A, Q_B\} = f_{AB}(Q)$ is invertible, one can go to Darboux coordinates where $f_{AB} = \delta_{AB}$.]

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