

Cosmological Nonlinear Density and Velocity Power Spectra

J. Hwang

UFES Vitória

November 11, 2015

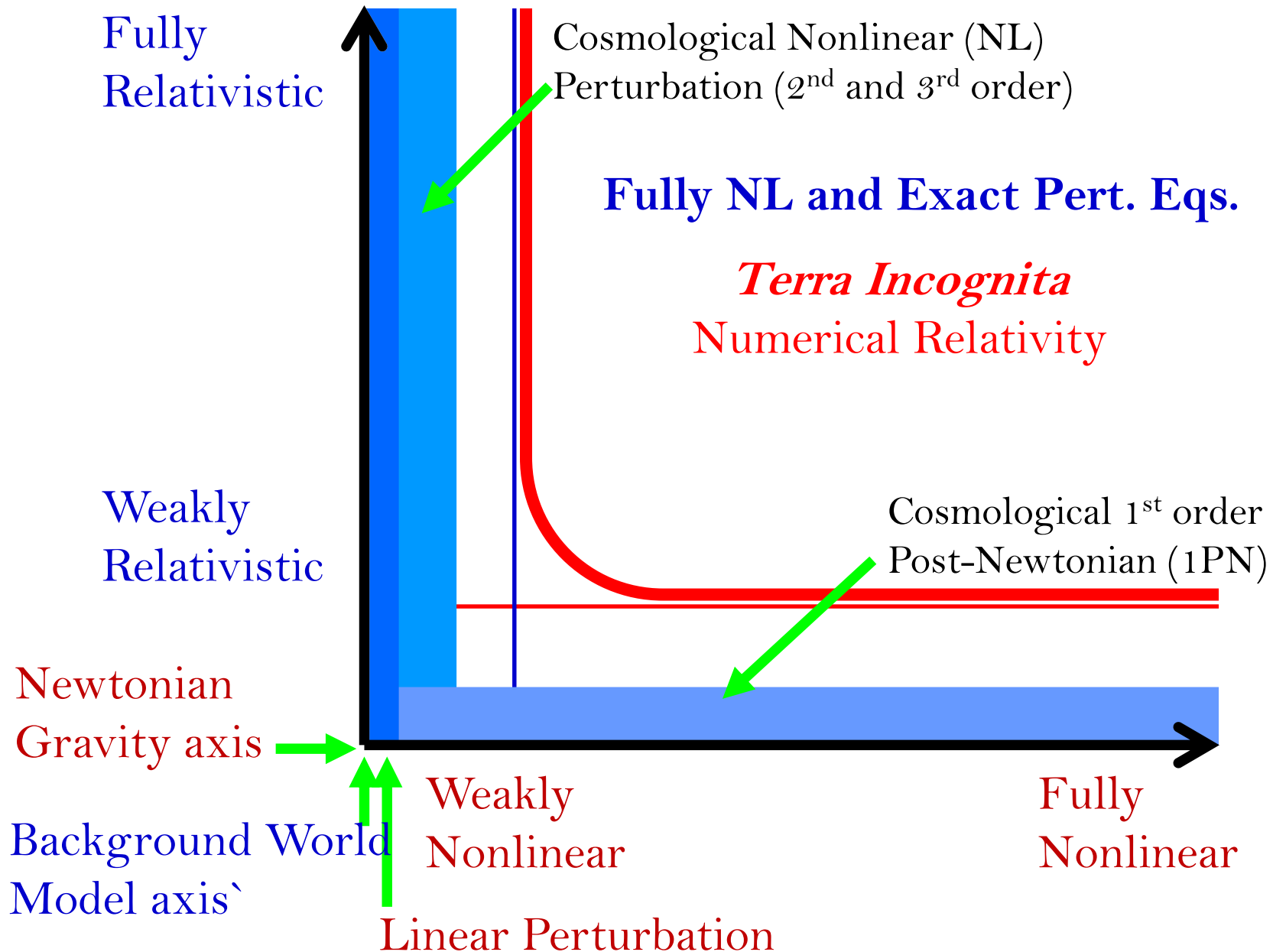
Perturbation method:

- ❖ Perturbation expansion
 - ❖ All perturbation variables are small
 - ❖ Weakly nonlinear
 - ❖ Strong gravity; fully relativistic
 - ❖ Valid in all scales
 - ❖ Fully nonlinear and Exact perturbations
- Excluding TT perturbation

Post-Newtonian method:

- ❖ Abandon geometric spirit of GR: recover the good old absolute space and absolute time
- ❖ Newtonian equations of motion with GR corrections
- ❖ Expansion in strength of gravity $\frac{\delta\Phi}{c^2} \sim \frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \ll 1$
- ❖ Fully nonlinear
- ❖ No strong gravity; weakly relativistic
- ❖ Valid far inside horizon
- ❖ Case of the Fully nonlinear and Exact perturbations

Relativistic vs. Nonlinear Pert.



Fully NL & Exact Pert. Theory

HJ & Noh, MNRAS **433** (2013) 3472

Noh, JCAP **07** (2014) 037

The linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible using the methods of Hawking (1966). ... One could then judge the domain of validity of the linear treatment and, more important, gain some insight into the non-linear effects.

Sachs and Wolfe (1967)

Convention: (Bardeen 1988)

Decomposition, possible
to NL order (York 1973)

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a^2 \left(\beta_{,i} + B_i^{(v)} \right) d\eta dx^i + a^2 \left[(1 + 2\varphi) g_{ij}^{(3)} + 2\gamma_{,i|j} + C_{i|j}^{(v)} + C_{j|i}^{(v)} + 2C_{ij}^{(t)} \right] dx^i dx^j,$$

$$\chi \equiv a\beta + a^2 \dot{\gamma}, \quad \Psi_i^{(v)} \equiv B_i^{(v)} + a \dot{C}_i^{(v)},$$

Spatial gauge condition

No anisotropic stress

$$\tilde{T}_{ab} = \tilde{\mu} \tilde{u}_a \tilde{u}_b + \tilde{p} (\tilde{u}_a \tilde{u}_b + \tilde{g}_{ab}) + \tilde{\pi}_{ab},$$

$$\tilde{\mu} \equiv \mu + \delta\mu \equiv \mu (1 + \delta), \quad \tilde{p} \equiv p + \delta p, \quad \tilde{u}_i \equiv a v_i, \quad v_i \equiv -v_{,i} + v_i^{(v)},$$

Spatial gauge: $\gamma \equiv 0 \equiv C_i^{(v)},$

$$\chi_i \equiv \chi_{,i} + a \Psi_i^{(v)} = a \left(\beta_{,i} + B_i^{(v)} \right)$$

Temporal gauge still not taken yet!



Complete spatial gauge fixing.

Remaining variables are spatially gauge-invariant to fully NL order! $\therefore$ Lose no generality!

Metric:

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a\chi_i d\eta dx^i + a^2 (1 + 2\varphi) g_{ij}^{(3)} dx^i dx^j,$$

Energy-momentum tensor:

$$\tilde{T}_{ab} = \tilde{\varrho} c^2 \tilde{u}_a \tilde{u}_b + \tilde{p} (\tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b), \quad \tilde{\varrho} \rightarrow \tilde{\varrho} \left(1 + \frac{1}{c^2} \tilde{\Pi}\right),$$

Internal energy

Four-vector:

$$\tilde{u}_i \equiv a \frac{v_i}{c}, \quad \frac{1}{\widehat{\gamma}} v_i = \widehat{v}_i = \frac{1}{\mathcal{N}} \left[(1 + 2\varphi) \bar{v}_i - \frac{c}{a} \chi_i \right],$$

Coordinate three-velocity of fluid

Fluid three-velocity measured by Eulerian observer

$$\widehat{\gamma} \equiv \sqrt{1 + \frac{v^k v_k}{c^2(1 + 2\varphi)}} = \frac{1}{\sqrt{1 - \frac{\widehat{v}^k \widehat{v}_k}{c^2(1 + 2\varphi)}}} \quad \text{: Lorentz factor}$$

Scalar- & vector-type decomposition:

$$\chi_i = c\chi_{,i} + \chi_i^{(v)}, \quad \widehat{v}_i \equiv -\widehat{v}_{,i} + \widehat{v}_i^{(v)}, \quad \chi^{(v)i}{}_{,i} \equiv 0 \equiv \widehat{v}^{(v)i}{}_{,i}.$$

Metric convention:

$$\tilde{g}_{00} = -a^2 (1 + 2\alpha), \quad \tilde{g}_{0i} = -a\chi_i, \quad \tilde{g}_{ij} = a^2 (1 + 2\varphi) \delta_{ij},$$

Inverse metric:

$$\tilde{g}^{00} = -\frac{1}{a^2} \frac{1 + 2\varphi}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2}, \quad \tilde{g}^{0i} = -\frac{1}{a^2} \frac{\chi^i / a}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2},$$
$$\tilde{g}^{ij} = \frac{1}{a^2(1 + 2\varphi)} \left(\delta^{ij} - \frac{\chi^i \chi^j / a^2}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2} \right). \quad \textbf{Exact!}$$

Using the ADM and the covariant formalisms the rest are simple algebra. We do not even need the connection!

Fully Nonlinear Perturbation Equations without taking temporal gauge condition:

Definition of κ :

$$\kappa \equiv 3 \frac{\dot{a}}{a} \left(1 - \frac{1}{\mathcal{N}} \right) - \frac{1}{\mathcal{N}(1+2\varphi)} \left[3\dot{\varphi} + \frac{c}{a^2} \left(\chi^k{}_{|k} + \frac{\chi^k \varphi_{,k}}{1+2\varphi} \right) \right].$$

ADM energy constraint:

$$\begin{aligned} & -\frac{3}{2} \left(\frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3c^2} \tilde{\mu} + \frac{\bar{K}c^2}{a^2(1+2\varphi)} - \frac{\Lambda c^2}{3} \right) + \frac{\dot{a}}{a} \kappa + \frac{c^2 \Delta \varphi}{a^2(1+2\varphi)^2} \\ & = \frac{1}{6} \kappa^2 - \frac{4\pi G}{c^2} (\tilde{\mu} + \tilde{p}) (\hat{\gamma}^2 - 1) + \frac{3}{2} \frac{c^2 \varphi^{[i} \varphi_{,i}}{a^2(1+2\varphi)^3} - \frac{c^2}{4} \bar{K}_j^i \bar{K}_i^j. \end{aligned}$$

ADM momentum constraint:

$$\begin{aligned} & \frac{2}{3} \kappa_{,i} + \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \left[\frac{1}{2} \left(\Delta \chi_i + \chi^k{}_{|ik} \right) - \frac{1}{3} \chi^k{}_{|ki} \right] + \frac{8\pi G}{c^4} (\tilde{\mu} + \tilde{p}) a \hat{\gamma}^2 \hat{v}_i \\ & = \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \left\{ \left(\frac{\mathcal{N}_{,j}}{\mathcal{N}} - \frac{\varphi_{,j}}{1+2\varphi} \right) \left[\frac{1}{2} \left(\chi^j{}_{|i} + \chi_i{}^{[j} \right) - \frac{1}{3} \delta_i^j \chi^k{}_{|k} \right] \right. \\ & \quad \left. - \frac{\varphi_{,j}}{(1+2\varphi)^2} \left(\chi_i \varphi_{,j} + \frac{1}{3} \chi_j \varphi_{,i} \right) + \frac{\mathcal{N}}{1+2\varphi} \nabla_j \left[\frac{1}{\mathcal{N}} \left(\chi^j \varphi_{,i} + \chi_i \varphi^{[j} - \frac{2}{3} \delta_i^j \chi^k \varphi_{,k} \right) \right] \right\}. \end{aligned}$$

Trace of ADM propagation:

$$\begin{aligned} & -3 \left[\frac{1}{\mathcal{N}} \left(\frac{\dot{a}}{a} \right) + \frac{\dot{a}^2}{a^2} + \frac{4\pi G}{3c^2} (\tilde{\mu} + 3\tilde{p}) - \frac{\Lambda c^2}{3} \right] + \frac{1}{\mathcal{N}} \dot{\kappa} + 2 \frac{\dot{a}}{a} \kappa + \frac{c^2 \Delta \mathcal{N}}{a^2 \mathcal{N}(1+2\varphi)} \\ & = \frac{1}{3} \kappa^2 + \frac{8\pi G}{c^2} (\tilde{\mu} + \tilde{p}) (\hat{\gamma}^2 - 1) - \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \left(\chi^i \kappa_{,i} + c \frac{\varphi^{[i} \mathcal{N}_{,i}}{1+2\varphi} \right) + c^2 \bar{K}_j^i \bar{K}_i^j. \end{aligned}$$

Tracefree ADM propagation (with linear tensor perturbation):

$$\begin{aligned}
 & \underbrace{\ddot{h}_{ij} + 3H\dot{h}_{ij} - c^2 \frac{\Delta - 2\bar{K}}{a^2} h_{ij}}_{\text{Tensor-type to linear order}} + \left(\frac{1}{\mathcal{N}} \frac{\partial}{\partial t} + 3\frac{\dot{a}}{a} - \kappa + \frac{c\chi^k}{a^2 \mathcal{N}(1+2\varphi)} \nabla_k \right) \left\{ \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \right. \\
 & \times \left[\frac{1}{2} \left(\chi^i{}_{|j} + \chi_j{}^{|i} \right) - \frac{1}{3} \delta_j^i \chi^k{}_{|k} - \frac{1}{1+2\varphi} \left(\chi^i \varphi_{,j} + \chi_j \varphi^{|i} - \frac{2}{3} \delta_j^i \chi^k \varphi_{,k} \right) \right] \Big\} \\
 & - \frac{c^2}{a^2(1+2\varphi)} \left[\frac{1}{1+2\varphi} \left(\nabla^i \nabla_j - \frac{1}{3} \delta_j^i \Delta \right) \varphi + \frac{1}{\mathcal{N}} \left(\nabla^i \nabla_j - \frac{1}{3} \delta_j^i \Delta \right) \mathcal{N} \right] \\
 & = \frac{8\pi G}{c^2} (\tilde{\mu} + \tilde{p}) \left[\frac{\hat{\gamma}^2 \hat{v}^i \hat{v}_j}{c^2(1+2\varphi)} - \frac{1}{3} \delta_j^i (\hat{\gamma}^2 - 1) \right] + \frac{c^2}{a^4 \mathcal{N}^2 (1+2\varphi)^2} \left[\frac{1}{2} \left(\chi^{i|k} \chi_{j|k} - \chi_{k|j} \chi^{k|i} \right) \right. \\
 & + \frac{1}{1+2\varphi} \left(\chi^{k|i} \chi_k \varphi_{,j} - \chi^{i|k} \chi_j \varphi_{,k} + \chi_{k|j} \chi^k \varphi^{|i} - \chi_{j|k} \chi^i \varphi^{|k} \right) + \frac{2}{(1+2\varphi)^2} \left(\chi^i \chi_j \varphi^{|k} \varphi_{,k} - \chi^k \chi_k \varphi^{|i} \varphi_{,j} \right) \Big] \\
 & - \frac{c^2}{a^2(1+2\varphi)^2} \left[\frac{3}{1+2\varphi} \left(\varphi^{|i} \varphi_{,j} - \frac{1}{3} \delta_j^i \varphi^{|k} \varphi_{,k} \right) + \frac{1}{\mathcal{N}} \left(\varphi^{|i} \mathcal{N}_{,j} + \varphi_{,j} \mathcal{N}^{|i} - \frac{2}{3} \delta_j^i \varphi^{|k} \mathcal{N}_{,k} \right) \right].
 \end{aligned}$$

Covariant energy conservation:

$$\begin{aligned}
 & \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N} \hat{v}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] \tilde{\mu} + (\tilde{\mu} + \tilde{p}) \left\{ \mathcal{N} \left(3\frac{\dot{a}}{a} - \kappa \right) \right. \\
 & + \frac{(\mathcal{N} \hat{v}^k)_{|k}}{a(1+2\varphi)} + \frac{\mathcal{N} \hat{v}^k \varphi_{,k}}{a(1+2\varphi)^2} + \frac{1}{\hat{\gamma}} \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N} \hat{v}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] \hat{\gamma} \Big\} = 0.
 \end{aligned}$$

Covariant momentum conservation:

$$\begin{aligned}
 & \frac{1}{a\hat{\gamma}} \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N} \hat{v}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] (a\hat{\gamma} \hat{v}_i) + \hat{v}^k \nabla_i \left(\frac{c\chi_k}{a^2(1+2\varphi)} \right) + \frac{c^2}{a} \mathcal{N}_{,i} - \left(1 - \frac{1}{\hat{\gamma}^2} \right) \frac{c^2 \mathcal{N} \varphi_{,i}}{a(1+2\varphi)} \\
 & + \frac{1}{\tilde{\mu} + \tilde{p}} \left\{ \frac{\mathcal{N} c^2}{a\hat{\gamma}^2} \tilde{p}_{,i} + \hat{v}_i \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N} \hat{v}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] \tilde{p} \right\} = 0.
 \end{aligned}$$

with

$$\mathcal{N} \equiv \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{a^2(1 + 2\varphi)}} \equiv 1 + \delta\mathcal{N}, \quad \overline{K}_j^i \overline{K}_i^j = \frac{1}{a^4 \mathcal{N}^2 (1 + 2\varphi)^2} \left\{ \frac{1}{2} \chi^{i|j} (\chi_{i|j} + \chi_{j|i}) - \frac{1}{3} \chi^i{}_{|i} \chi^j{}_{|j} \right. \\ \left. - \frac{4}{1 + 2\varphi} \left[\frac{1}{2} \chi^i \varphi^{|j} (\chi_{i|j} + \chi_{j|i}) - \frac{1}{3} \chi^i{}_{|i} \chi^j{}_{|j} \varphi_{,j} \right] + \frac{2}{(1 + 2\varphi)^2} \left(\chi^i \chi_i \varphi^{|j} \varphi_{,j} + \frac{1}{3} \chi^i \chi^j \varphi_{,i} \varphi_{,j} \right) \right\}.$$

To Background order:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \varrho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\varrho + 3 \frac{p}{c^2} \right) + \frac{\Lambda c^2}{3}, \quad \dot{\varrho} + 3 \frac{\dot{a}}{a} \left(\varrho + \frac{p}{c^2} \right) = 0,$$

ADM energy-constraint

Trace of ADM propagation

Covariant E-conservation

To linear order **without** taking temporal gauge condition: (Bardeen 1988)

Extrinsic curvature

Definition of kappa, $\mathbf{K}^i_i$

$$\kappa \equiv 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi.$$

ADM energy-conservation, $\mathbf{G}^0_0$

$$H\kappa + \frac{\Delta + 3K}{a^2}\varphi = -4\pi G\delta\mu,$$

ADM momentum-conservation, $\mathbf{G}^0_i$

$$\kappa + \frac{\Delta + 3K}{a^2}\chi = 12\pi G(\mu + p)av,$$

ADM propagation, trace, $\mathbf{G}^i_i$

$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = 4\pi G(\delta\mu + 3\delta p),$$

ADM propagation, tracefree, $\mathbf{G}^i_j - \dots$

$$\dot{\chi} + H\chi - \varphi - \alpha = 8\pi G\Pi.$$

Energy-conservation, $\mathbf{T}^c_{0;c}$

$$\delta\dot{\mu} + 3H(\delta\mu + \delta p) = (\mu + p) \left(\kappa - 3H\alpha + \frac{1}{a}\Delta v \right),$$

Momentum-conservation, $\mathbf{T}^c_{i;c}$

$$\frac{[a^4(\mu + p)v]^\cdot}{a^4(\mu + p)} = \frac{1}{a}\alpha + \frac{1}{a(\mu + p)} \left(\delta p + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right).$$

Meanings of variables:

Lapse function:

$$N = a \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{a^2(1 + 2\varphi)}} \equiv a\mathcal{N}$$

Lapse

Intrinsic curvature:

$$R^{(h)} = \frac{2}{a^2(1 + 2\varphi)^2} \left(-2\varphi^{[k}{}_{|k} + 3\frac{\varphi^{[k} \varphi_{,k}}{1 + 2\varphi} \right) + \frac{1}{a^2(1 + 2\varphi)} R^{(3)}, \quad R^{(3)} = 6\bar{K}.$$

Perturbed curvature

Trace of extrinsic curvature:

$$K = -\frac{1}{\mathcal{N}(1 + 2\varphi)} \left[3(H + \dot{\varphi} + 2H\varphi) + \frac{1}{a^2} \chi^k{}_{|k} + \frac{\chi^k \varphi_{,k}}{a^2(1 + 2\varphi)} \right]$$

$$\equiv -3H + \kappa,$$

Trace-free extrinsic curvature:

$$\bar{K}_j^i \equiv K_j^i - \frac{1}{3} \delta_j^i K = -\frac{1}{a^2 \mathcal{N}(1 + 2\varphi)} \left[\frac{1}{2} \left(\chi^i{}_{|j} + \chi_j{}^{[i} \right) - \frac{1}{3} \delta_j^i \chi^k{}_{|k} \right. \\ \left. - \frac{1}{1 + 2\varphi} \left(\chi^i \varphi_{,j} + \chi_j \varphi^{,i} - \frac{2}{3} \delta_j^i \chi^k \varphi_{,k} \right) \right].$$

Shear

Expansion scalar

Acceleration vector

Shear tensor

Rotation tensor

$$\tilde{\theta}^{(n)} = -K = 3H - \kappa, \quad \tilde{a}_i^{(n)} = \frac{\mathcal{N}_{,i}}{\mathcal{N}}, \quad \tilde{\sigma}_{ij}^{(n)} = -\bar{K}_{ij}, \quad \tilde{\omega}_{ij}^{(n)} = 0$$

Perturbed expansion

Temporal gauge (slicing, hypersurface):

comoving gauge :	$v \equiv 0,$	$\longleftrightarrow \hat{v} \equiv 0$	from third order in
zero-shear gauge :	$\chi \equiv 0,$		the presence of vector-type
uniform-curvature gauge :	$\varphi \equiv 0,$		perturbation
uniform-expansion gauge :	$\kappa \equiv 0,$		
uniform-density gauge :	$\delta \equiv 0,$		
synchronous gauge :	$\alpha \equiv 0.$		Fully NL formulation Not available

Applicable to NL orders!



Complete gauge fixing.

Remaining variables are gauge-invariant to fully NL order!

Zero-pressure Irrotational Fluid

HJ & Noh, MNRAS **433** (2013) 3472

Noh, JCAP **07** (2014) 037

Conservation equations:

Energy-conservation to NL order:

Covariant energy-conservation

$$0 \stackrel{\text{orange}}{=} \tilde{T}_{a;b}^b \tilde{u}^a = -\tilde{\tilde{\mu}} - (\tilde{\mu} + \tilde{p}) \tilde{\theta},$$

$$0 = \tilde{T}_{a;b}^b \tilde{n}^a = \frac{1}{N} \underbrace{\tilde{T}_{0;b}^b}_{=0} - \frac{N^i}{N} \tilde{T}_{i;b}^b = -\frac{1}{N} (\tilde{u}_0 - N^i \tilde{u}_i) \tilde{T}_{a;b}^b \tilde{u}^a - \frac{1}{N} \left(\frac{\tilde{u}^i}{\tilde{u}^0} + N^i \right) \tilde{T}_{a;b}^b \tilde{h}_i^a,$$

Energy-conservation

ADM energy-conservation

Covariant energy-conservation:

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N}\widehat{v}^i + \frac{c}{a}\chi^i \right) \nabla_i \right] \tilde{\mu} + (\tilde{\mu} + \tilde{p}) \left\{ (3H - \kappa)\mathcal{N} + \frac{(\mathcal{N}\widehat{v}^i)_{|i}}{a(1+2\varphi)} \right. \\ \left. + \frac{\mathcal{N}\widehat{v}^i\varphi_{,i}}{a(1+2\varphi)^2} + \frac{1}{\widehat{\gamma}} \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N}\widehat{v}^i + \frac{c}{a}\chi^i \right) \nabla_i \right] \widehat{\gamma} \right\} = 0.$$

Covariant energy-conservation:

$$\begin{aligned}
 & \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\cancel{\mathcal{N}\hat{v}^i} + \frac{c}{a}\chi^i \right) \nabla_i \right] \tilde{\mu} + (\tilde{\mu} + \cancel{\tilde{p}}) \left\{ (3H - \kappa)\mathcal{N} + \frac{(\cancel{\mathcal{N}\hat{p}^i})_{|i}}{a(1+2\varphi)} \right. \\
 & \quad \left. + \frac{\cancel{\mathcal{N}\hat{p}^i}\varphi_{,i}}{a(1+2\varphi)^2} + \frac{1}{\cancel{\hat{\gamma}}} \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N}\hat{v}^i + \frac{c}{a}\chi^i \right) \nabla_i \right] \cancel{\hat{\gamma}} \right\} = 0.
 \end{aligned}$$

Comoving gauge + irrotational
Zero-pressure

Comoving gauge + irrotational ($v^i = 0$) + zero-pressure:

$$\underbrace{\left(\frac{\dot{\mu}}{\mu} + 3H \right)}_{= 0, \text{ Background order}} (1 + \delta) + \dot{\delta} - \kappa = \delta\kappa - \frac{\chi^{,i}\delta_{,i}}{a^2(1+2\varphi)}, \quad \textbf{Exact!}$$

To 12th-order perturbation, say:

$$\begin{aligned}
 \dot{\delta} - \kappa = & \delta\kappa - \frac{1}{a^2}\chi^{,i}\delta_{,i} \left(1 - 2\varphi + 4\varphi^2 - 8\varphi^3 + 16\varphi^4 - 32\varphi^5 \right. \\
 & \left. + 64\varphi^6 - 128\varphi^7 + 256\varphi^8 - 512\varphi^9 + 1024\varphi^{10} \right),
 \end{aligned}$$

Zero-pressure fluid in the comoving gauge

Exact equations:

Covariant energy-conservation:

$$\dot{\delta} - \kappa - \delta\kappa + \frac{1}{a^2}\chi^{,i}\delta_{,i} = \frac{2\varphi\chi^{,i}\delta_{,i}}{a^2(1+2\varphi)},$$

Trace of ADM propagation:

$$\begin{aligned} \dot{\kappa} + 2H\kappa - 4\pi G\delta\mu - \frac{1}{3}\kappa^2 + \frac{1}{a^2}\chi^{,i}\kappa_{,i} - \frac{1}{a^4}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right] &= \frac{2\varphi\chi^{,i}\kappa_{,i}}{a^2(1+2\varphi)} - \frac{4\varphi(1+\varphi)}{a^4(1+2\varphi)^2}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right] \\ &+ \frac{2}{a^4(1+2\varphi)^3}\left\{\frac{2}{3}(\Delta\chi)\chi^{,i}\varphi_{,i} - 2\chi^{,ij}\chi_{,i}\varphi_{,j} + \frac{1}{1+2\varphi}\left[\frac{1}{3}(\chi^{,i}\varphi_{,i})^2 + \chi^{,i}\chi_{,i}\varphi^{,j}\varphi_{,j}\right]\right\}, \end{aligned}$$

ADM momentum constraint:

$$\left(\kappa + \frac{\Delta}{a^2}\chi\right)_{,i} = \frac{2\varphi\Delta\chi_{,i}}{a^2(1+2\varphi)} + \frac{1}{a^2(1+2\varphi)^2}\left[2(\Delta\chi)\varphi_{,i} + \frac{1}{2}\chi^{,k}\varphi_{,ik} - \chi_{,ik}\varphi^{,k} + \frac{3}{2}\chi_{,i}\Delta\varphi - \frac{3}{2}\frac{1}{1+2\varphi}\left(\chi_{,i}\varphi_{,k} + \frac{1}{3}\chi_{,k}\varphi_{,i}\right)\varphi^{,k}\right].$$

**RHS = pure Einstein's gravity corrections,
starting from the third order, all involving φ**

Definition of kappa + ADM momentum constraint:

$$[\ln(1+2\varphi)]_{,i} = \frac{1}{a^2(1+2\varphi)^2}\left[\chi^{,k}\varphi_{,ik} + \chi_{,i}\Delta\varphi - \frac{1}{1+2\varphi}(\chi_{,i}\varphi_{,k} + 3\chi_{,k}\varphi_{,i})\varphi^{,k}\right],$$

Identify: $\kappa \equiv -\frac{1}{a}\nabla \cdot \mathbf{u}$

Linear-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Relativistic/Newtonian correspondence to second order.

This equation is valid to fully nonlinear order in Newtonian theory.

Second-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla \cdot (\delta\mathbf{u})] + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}),$$

Pure relativistic correction appearing from third order.
All involving φ .

Third-order:

$$\begin{aligned} \ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = & -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla \cdot (\delta\mathbf{u})] + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) \\ & + \frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)] \cdot \nabla\delta\} - \frac{4}{a^2}\nabla \cdot \left[\varphi \left(\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla \cdot \mathbf{u} \right) \right] \\ & + \frac{2}{3a^2}\varphi\mathbf{u} \cdot \nabla(\nabla \cdot \mathbf{u}) + \frac{\Delta}{a^2}[\mathbf{u} \cdot \nabla(\Delta^{-1}X)] - \frac{1}{a^2}\mathbf{u} \cdot \nabla X - \frac{2}{3a^2}X\nabla \cdot \mathbf{u}, \\ X \equiv & 2\varphi\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla \cdot [\mathbf{u} \cdot \nabla(\nabla\varphi) + \mathbf{u}\Delta\varphi]. \end{aligned}$$

Power spectra:

$$f(\mathbf{k}) = \int d^3x f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\langle \delta(\mathbf{k}_1, t) \delta(\mathbf{k}_2, t) \rangle \equiv (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) P_\delta(k_1, t)$$

$$\langle \theta(\mathbf{k}_1, t) \theta(\mathbf{k}_2, t) \rangle \equiv (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) P_\theta(k_1, t).$$

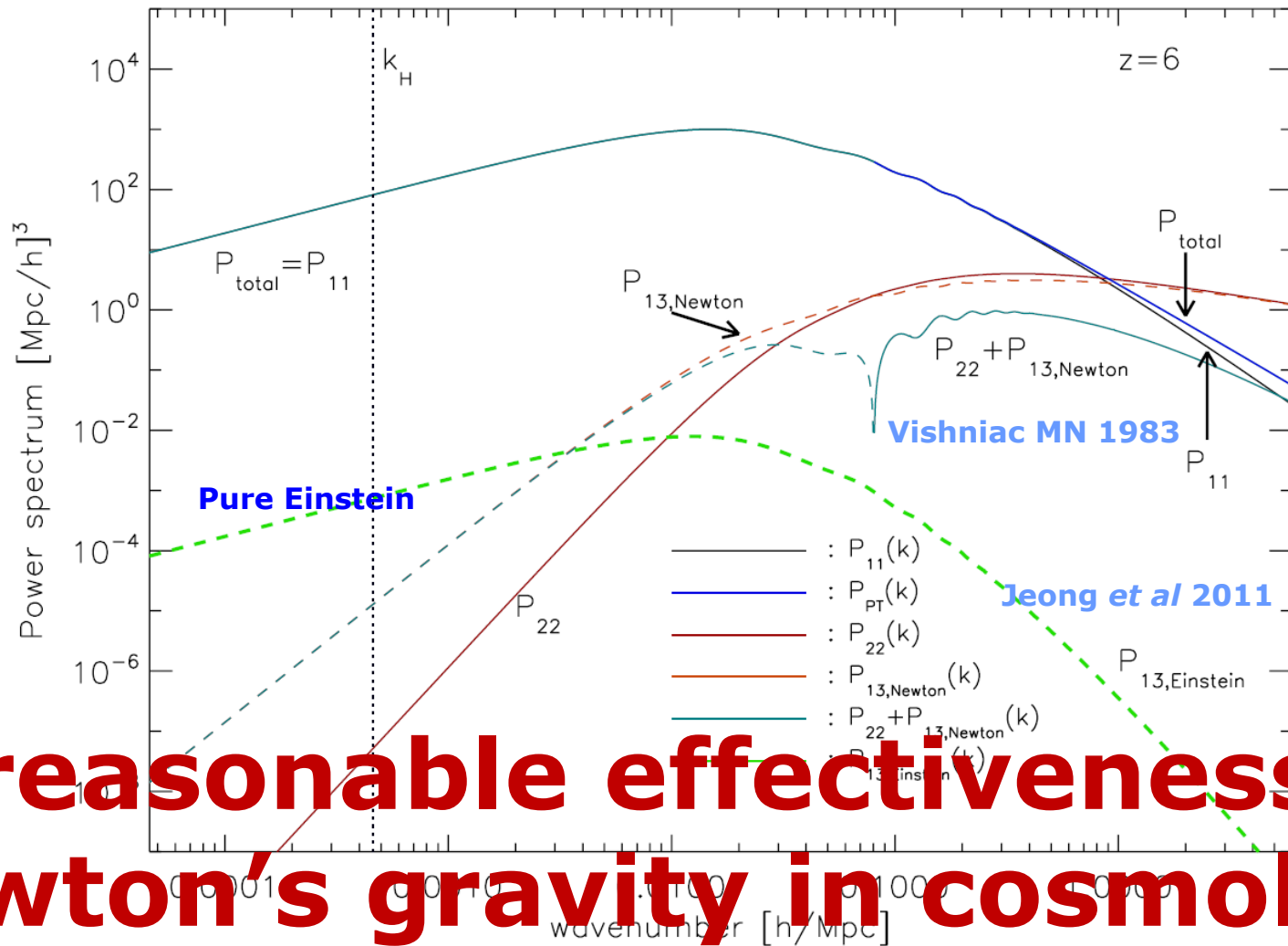
$$P \equiv P_{11} + P_{22} + P_{13}$$

$$\theta \equiv \frac{1}{a} \nabla \cdot \mathbf{u}$$

$$P_{13} = P_{13}^{\text{N}} + P_{13}^{\text{E}}$$

$$P_{13}^{\text{E}} = P_{13}^{\text{ES}} + P_{13}^{\text{EV}} + P_{13}^{\text{ET}}$$

Leading Nonlinear **Density** Power-spectrum in the Comoving gauge:



Unreasonable effectiveness of Newton's gravity in cosmology!

General Relativistic Continuity and Euler equations to Third order in the Comoving gauge:

Newtonian

$$\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} + \frac{1}{a} \nabla \cdot (\delta \mathbf{u}) = \frac{1}{a} (\nabla \delta) \cdot [2\varphi \mathbf{u} - \Delta^{-1} (\nabla X + \mathbf{Y})],$$

$$\frac{1}{a} \nabla \cdot \left(\dot{\mathbf{u}} + \frac{\dot{a}}{a} \mathbf{u} \right) + 4\pi G \varrho \delta + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = \frac{1}{a^2} \left\{ -\frac{2}{3} \varphi \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) + 4 \nabla \cdot \left[\varphi \left(\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u} \right) \right] \right. \\ \left. + \frac{2}{3} X \nabla \cdot \mathbf{u} + \mathbf{u} \cdot (\nabla X + \mathbf{Y}) - \Delta [\mathbf{u} \cdot \Delta^{-1} (\nabla X + \mathbf{Y})] \right\} + 2 \frac{\dot{a}}{a} \frac{1}{a} u^{,ij} \Delta^{-1} (a^2 Z_{ij}).$$

Vector

Tensor

$$X \equiv 2\varphi \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \varphi + \frac{3}{2} \Delta^{-1} \nabla \cdot (\mathbf{u} \cdot \nabla \nabla \varphi + \mathbf{u} \Delta \varphi),$$

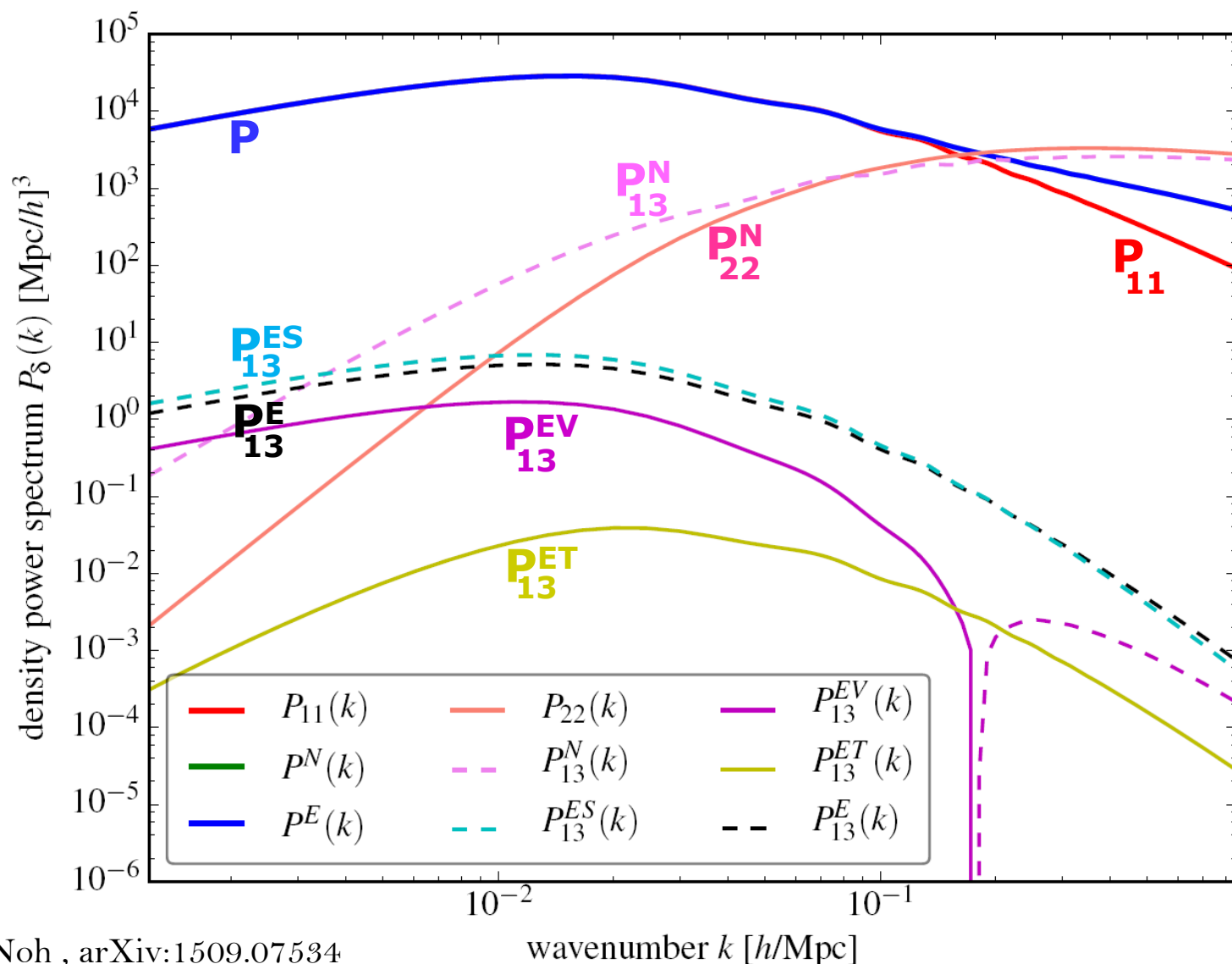
$$\mathbf{Y} \equiv 2 [\mathbf{u} \cdot \nabla \nabla \varphi + \mathbf{u} \Delta \varphi - \nabla \Delta^{-1} \nabla \cdot (\mathbf{u} \cdot \nabla \nabla \varphi + \mathbf{u} \Delta \varphi)]. \quad \leftarrow \text{Vector}$$

$$Z_{ij} \equiv N_{ij} - 2\Delta^{-1} \nabla_{(i} N_{j),k}^k + \frac{1}{2} \Delta^{-2} (\nabla_i \nabla_j + \delta_{ij} \Delta) N^{k\ell}_{,k\ell}, \quad \leftarrow \text{Tensor}$$

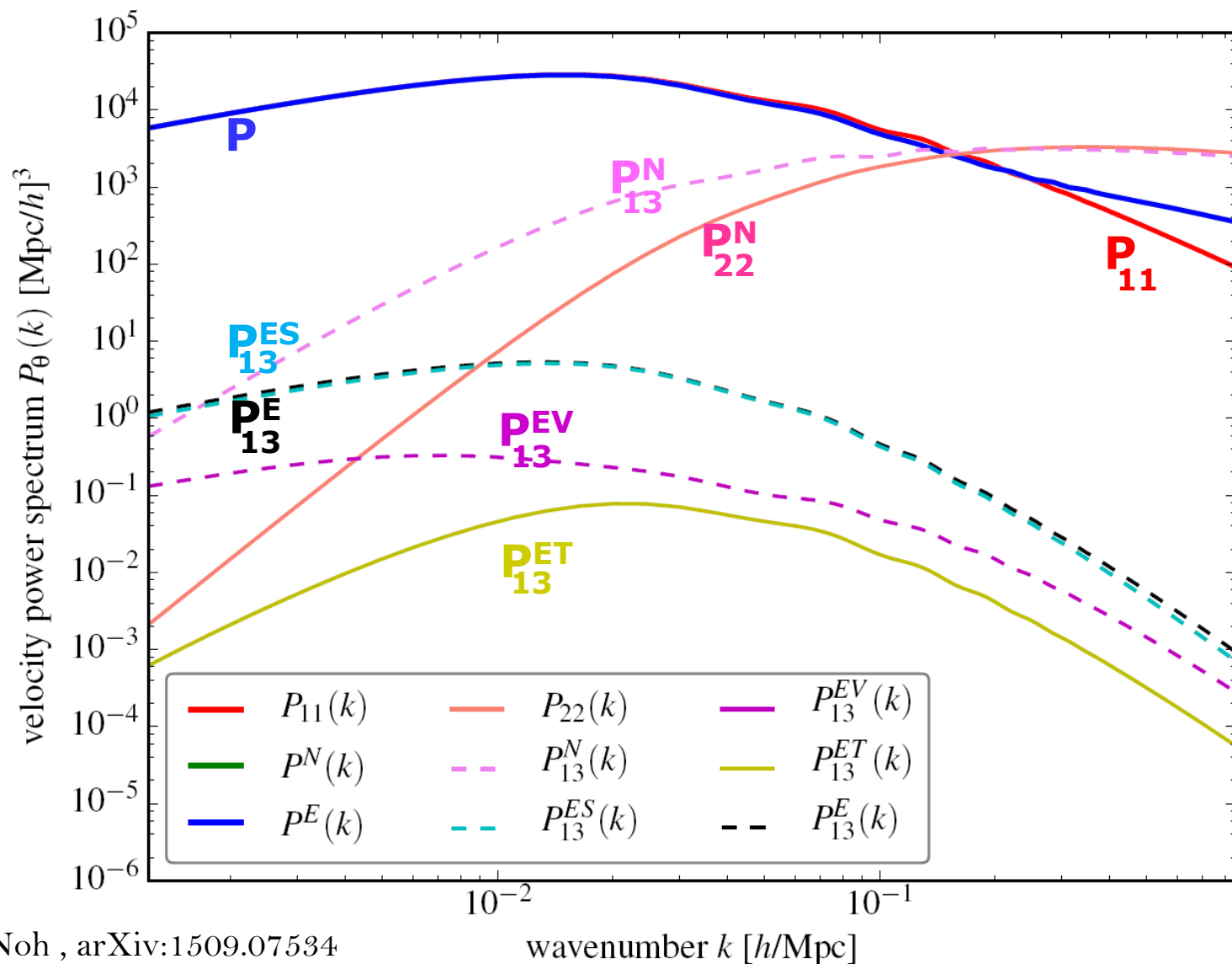
$$a^2 N_{ij} \equiv -\frac{1}{c^2} \left\{ u_{,ij} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}_{,ij} - \frac{1}{3} \delta_{ij} [(\nabla \cdot \mathbf{u})^2 + \mathbf{u} \cdot (\Delta \mathbf{u})] \right\},$$

$$c^2 \frac{\Delta}{a^2} \varphi = -4\pi G \varrho \delta + \frac{\dot{a}}{a} \frac{1}{a} \nabla \cdot \mathbf{u}.$$

Nonlinear **Density** Power-spectrum with vector and tensor contributions:



Nonlinear **Velocity** Power-spectrum with vector and tensor contributions:



Newtonian Limit

Chandrasekhar, ApJ (1965): **OPN, Minkowski**

JH, Noh & Puetzfeld, JCAP (2008): **cosmological**

Here: **as a limit of FNL&E PT**

JH & Noh, JCAP **04** (2013) 035

Infinite speed-of-light Limit in ZSG & UEG:

$$\alpha \ll 1, \quad \varphi \ll 1, \quad \frac{\widehat{v^k v_k}}{c^2} \ll 1, \quad \widetilde{p} \ll \widetilde{\varrho} c^2, \quad \frac{1}{c^2} \widetilde{\Pi} \ll 1, \quad \frac{c^2 k^2}{a^2 H^2} \gg 1,$$

Covariant energy-conservation:

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N} \widehat{v^k} + \frac{c}{a} \chi^k \right) \nabla_k \right] \widetilde{\varrho} + \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2} \right) \left\{ \mathcal{N} \left(3 \frac{\dot{a}}{a} - \kappa \right) + \frac{(\mathcal{N} \widehat{v^k})_{,k}}{a(1+2\varphi)} + \frac{\mathcal{N} \widehat{v^k} \varphi_{,k}}{a(1+2\varphi)^2} + \frac{1}{\widehat{\gamma}} \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)} \left(\mathcal{N} \widehat{v^k} + \frac{c}{a} \chi^k \right) \nabla_k \right] \widehat{\gamma} \right\} = 0.$$

$$\alpha = -\frac{1}{c^2} U, \quad \varphi = \frac{1}{c^2} V, \quad \widehat{v^k} = \mathbf{v},$$

ADM momentum-constraint:

$$\kappa = -\frac{12\pi G \varrho a}{c^2 \Delta} \nabla \cdot [(1 + \delta) \mathbf{v}],$$

Tracefree ADM propagation:

$$\varphi = -\alpha.$$

$$\mathcal{N} \equiv \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{a^2 (1 + 2\varphi)}},$$

Subhorizon limit



$$\dot{\widetilde{\varrho}} + 3 \frac{\dot{a}}{a} \widetilde{\varrho} + \frac{1}{a} \nabla \cdot (\widetilde{\varrho} \mathbf{v}) = 0$$

Equations in Newtonian limit:

Covariant energy-conservation: $\dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\widetilde{\varrho} = -\frac{1}{a}\nabla \cdot (\widetilde{\varrho}\mathbf{v}),$

Covariant momentum-conservation: $\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{a}\nabla U - \frac{1}{a\widetilde{\varrho}}\nabla \widetilde{p},$

Trace of ADM propagation: $\frac{\Delta}{a^2}U = -4\pi G (\widetilde{\varrho} - \varrho).$

With Relativistic Pressure

**Infinite speed-of-light limit,
except for pressure**

JH & Noh, JCAP 10 (2013) 054

Case with Relativistic Pressure:

$$\alpha \ll 1, \quad \varphi \ll 1, \quad \frac{\widehat{v}^k \widehat{v}_k}{c^2} \ll 1, \quad \widetilde{p} \not\ll \widetilde{\varrho} c^2, \quad \frac{1}{c^2} \widetilde{\Pi} \ll 1, \quad \frac{c^2 k^2}{a^2 H^2} \gg 1,$$



Covariant energy-conservation:

$$\dot{\widetilde{\varrho}} + 3 \frac{\dot{a}}{a} \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2} \right) + \frac{1}{a} \nabla \cdot \left[\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2} \right) \mathbf{v} \right] = \frac{1}{c^2} \frac{2}{a} \mathbf{v} \cdot \nabla \widetilde{p},$$

Covariant momentum-conservation:

$$\dot{\mathbf{v}} + \frac{\dot{a}}{a} \mathbf{v} + \frac{1}{a} \mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{a} \nabla U = - \frac{1}{\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}} \left(\frac{1}{a} \nabla \widetilde{p} + \frac{\dot{\widetilde{p}}}{c^2} \mathbf{v} \right),$$

Trace of ADM propagation:

$$\frac{\Delta}{a^2} U = \underline{-4\pi G (\widetilde{\varrho} - \varrho)},$$

No pressure!

Previous works were not successful in guessing the correct forms.

(Whittaker 1935; McCrea 1951; Harrison 1965; Coles & Lucchin 1995; Lima et al. 1997; ... ; Harko 2011)

Post-Newtonian Approximation

Chandrasekhar, ApJ (1965): **1PN, Minkowski**
JH, Noh & Puetzfeld, JCAP (2008): **cosmological**
Here: **as a limit of FNL&E PT**
Noh & JH, JCAP 08 (2013) 040

1PN convention: (Chandrasekhar 1965)

$$ds^2 = - \left[1 - \frac{1}{c^2} 2U + \frac{1}{c^4} (2U^2 - 4\Phi) \right] c^2 dt^2 - \frac{1}{c^3} 2a P_i c dt dx^i + a^2 \left(1 + \frac{1}{c^2} 2V \right) \gamma_{ij} dx^i dx^j,$$

$$\tilde{\mu} \equiv \mu \equiv \varrho c^2 \left(1 + \frac{1}{c^2} \Pi \right), \quad \tilde{p} = p, \quad \tilde{u}^i \equiv \frac{1}{c} \frac{1}{a} \bar{v}^i \tilde{u}^0, \quad \sim \text{Shear}$$

Identification:

$$\alpha = -\frac{1}{c^2} \left[U - \frac{1}{c^2} (U^2 - 2\Phi) \right], \quad \overset{\text{PT}}{\downarrow} \varphi = \frac{1}{c^2} V, \quad \overset{\text{1PN}}{\downarrow} \kappa = -\frac{1}{c^2} \left(3 \frac{\dot{a}}{a} U + 3\dot{V} + \frac{1}{a} P^k{}_{,k} \right),$$

$$\chi_i = \frac{1}{c^3} a P_i, \quad v_i = \frac{1}{c} \left\{ \bar{v}_i + \frac{1}{c^2} \left[\bar{v}_i \left(\frac{1}{2} \bar{v}^2 + U + 2V \right) - P_i \right] \right\},$$

 **1PN equations!** (JH, Noh & Puetzfeld, JCAP 2008)

Covariant E-conservation:

$$\frac{1}{a^3} (a^3 \varrho)^\cdot + \frac{1}{a} (\varrho \bar{v}^i)_{|i} = -\frac{1}{c^2} \left[\varrho \left(\frac{\partial}{\partial t} + \frac{1}{a} \bar{\mathbf{v}} \cdot \nabla \right) \left(\frac{1}{2} \bar{v}^2 + 3U + \Pi \right) + \left(3 \frac{\dot{a}}{a} + \frac{1}{a} \nabla \cdot \bar{\mathbf{v}} \right) p \right],$$

Basic 1PN Equations:

Tracefree ADM propagation: $V = U$.

Covariant energy-conservation:

$$\frac{1}{a^3} (a^3 \bar{\varrho})^\cdot + \frac{1}{a} (\bar{\varrho} \bar{v}^i)_{,i} = -\frac{1}{c^2} \left[\bar{\varrho} \left(\frac{\partial}{\partial t} + \frac{1}{a} \bar{\mathbf{v}} \cdot \nabla \right) \left(\frac{1}{2} \bar{v}^2 + 3U + \bar{\Pi} \right) + \left(3 \frac{\dot{a}}{a} + \frac{1}{a} \nabla \cdot \bar{\mathbf{v}} \right) \bar{p} \right],$$

Covariant momentum-conservation:

$$\frac{1}{a} (a \bar{v}_i)^\cdot + \frac{1}{a} \bar{v}_{i,k} \bar{v}^k - \frac{1}{a} U_{,i} + \frac{1}{a} \frac{\bar{p}_{,i}}{\bar{\varrho}} = \frac{1}{c^2} \left[\frac{1}{a} \bar{v}^2 U_{,i} + \frac{2}{a} (\Phi - U^2)_{,i} + \frac{1}{a} (a P_i)^\cdot + \frac{1}{a} \bar{v}^k (P_{i,k} - P_{k,i}) \right. \\ \left. + \frac{1}{a} \left(\bar{v}^2 + 4U + \bar{\Pi} + \frac{\bar{p}}{\bar{\varrho}} \right) \frac{\bar{p}_{,i}}{\bar{\varrho}} - \bar{v}_i \left(\frac{\partial}{\partial t} + \frac{1}{a} \bar{\mathbf{v}} \cdot \nabla \right) \left(\frac{1}{2} \bar{v}^2 + 3U \right) - \bar{v}_i \frac{1}{\bar{\varrho}} \left(\frac{\partial}{\partial t} + \frac{1}{a} \bar{\mathbf{v}} \cdot \nabla \right) \bar{p} \right],$$

Fourth-order in perturbation!

Trace of ADM propagation:

$$\frac{\Delta}{a^2} U + 4\pi G (\bar{\varrho} - \varrho) = -\frac{1}{c^2} \left\{ \frac{1}{a^2} [2\Delta\Phi - 2U\Delta U + (a P^i_{,i})^\cdot] + 3\ddot{U} + 9\frac{\dot{a}}{a} \dot{U} + 6\frac{\ddot{a}}{a} U \right. \\ \left. + 8\pi G \left[\bar{\varrho} \bar{v}^2 + \frac{1}{2} (\bar{\varrho} \bar{\Pi} - \varrho \Pi) + \frac{3}{2} (\bar{p} - p) \right] \right\},$$

ADM momentum-constraint:

$$0 = \frac{1}{a^2} (P^k_{,ki} - \Delta P_i) - 16\pi G \bar{\varrho} \bar{v}_i + \frac{4}{a} \left(\dot{U} + \frac{\dot{a}}{a} U \right)_{,i},$$

$$\bar{v}_i = \left(1 - \frac{3}{c^2} U \right) \widehat{v}_i + \frac{1}{c^2} P_i, \quad \widehat{v}_i = \left(1 - \frac{v^2}{2c^2} \right) v_i,$$

Gauge conditions:

$$\frac{1}{a} P^i{}_{|i} + n\dot{U} + m\frac{\dot{a}}{a}U = 0,$$

Harmonic gauge : $n = 4, m = \text{arbitrary},$

Chandrasekhar's gauge : $n = 3, m = \text{arbitrary},$

Uniform–expansion gauge : $n = 3 = m,$

Transverse–shear gauge : $n = 0 = m.$

Propagation speed issue

Under the general gauge:

Trace of ADM propagation:

$$\begin{aligned}
 & \frac{\Delta}{a^2} U + 4\pi G (\varrho - \varrho_b) \\
 & + \frac{1}{c^2} \left\{ 2 \frac{\Delta}{a^2} \Phi - (n-3) \ddot{U} - (2n+m-9) \frac{\dot{a}}{a} \dot{U} + \left[(6-m) \frac{\ddot{a}}{a} - m \frac{\dot{a}^2}{a^2} \right] U \right. \\
 & \left. + 8\pi G \left[\varrho v^2 + \frac{1}{2} (\varrho \Pi - \varrho_b \Pi_b) + U (\varrho - \varrho_b) + \frac{3}{2} (p - p_b) \right] \right\} = 0.
 \end{aligned}$$

Propagation speed $\frac{c}{\sqrt{n-3}}$ **is gauge dependent!**

of the potential

Resolution using Weyl tensor:

$$E_{ij} = -\frac{1}{c^2} \left[\frac{1}{2} (U + V)_{,i|j} - \frac{\Delta}{6} (U + V) \gamma_{ij} \right] + L^{-2} \mathcal{O}^{-4},$$

$$H_{ij} = \frac{1}{c^3} \frac{1}{2a} \eta_{(i}{}^{kl} \left\{ \left[\frac{1}{2} (P^m{}_{|ml} - \Delta P_l) + \frac{1}{3} v_l \Delta (U + V) \right] \gamma_{kj} + P_{l|kj} - v_l (U + V)_{,k|j} \right\} + L^{-2} \mathcal{O}^{-5},$$

$\dot{E}_{ij}$, $\dot{H}_{ij}$ equations:

$$\begin{aligned} \underline{\ddot{E}_{ij}} + 3 \frac{\dot{a}}{a} \dot{E}_{ij} + \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) E_{ij} \\ = \frac{c^2}{a^2} \left[\underline{\Delta E_{ij}} - E_{(i|j)k}^k - \frac{1}{2} E_{k(i}{}^{|k}{}_{j)} + \frac{1}{2} (E^{kl}{}_{|kl} - \Delta E_k^k) \gamma_{ij} + \frac{1}{2} E_{k|ij}^k \right] \\ + \frac{1}{a^2} \left\{ \left[(V + 2U)_{,i} E_j^k \right]_{|k} - \left[(V + 2U)^{,k} E_{ij} \right]_{|k} + \frac{1}{2} \left[(V + 2U)^{,k} E_{k(i} \right]_{|j)} \\ - \frac{1}{2} \left[(V + 2U)^{,l} E_l^k \right]_{|k} \gamma_{ij} \right\} + \frac{4\pi G}{c^2} (-\Delta \Pi_{ij} + \Pi_{(i|j)k}^k), \end{aligned} \quad (225)$$

Fully NL and exact cosmological pert.:

- | | | |
|------------------------------------|---|---------------------------|
| 1. Formulation | } | MN 433 (2013) 3472 |
| 2. Multi-component fluids , fields | | arXiv: 1511.01360 |
| 3. Minimally coupled scalar field | | JCAP 07 (2014) 037 |
| 4. Newtonian limit | | JCAP 04 (2013) 035 |
| 5. with relativistic pressure | | JCAP 10 (2013) 054 |
| 6. 1PN equations | | JCAP 08 (2013) 040 |

Future extentions:

1. Anisotropic stress $\Rightarrow$ Relativistic magneto-hydrodynamics
2. Light propagation (geodesic, Boltzmann)
3. 2 and higher order PN equations
4. Gauge-invariant combinations
5. TT perturbation should be handled perturbatively

Applications:

1. Relativistic NL perturbations
2. Fitting and Averaging
3. Backreaction
4. Relativistic cosmological numerical simulation