Teleparallelism: A New Insight into Gravity

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Plan of the Seminar

1. Preliminaries
   - Tangent bundle
   - Lorentz connections
   - Inertial Lorentz connection

2. General relativity revisited
   - Christoffel (or Levi-Civita) connection
   - Interaction and geometry

3. Teleparallel gravity
   - Teleparallel spin connection
   - Equivalence with general relativity
   - Achievements of teleparallel gravity

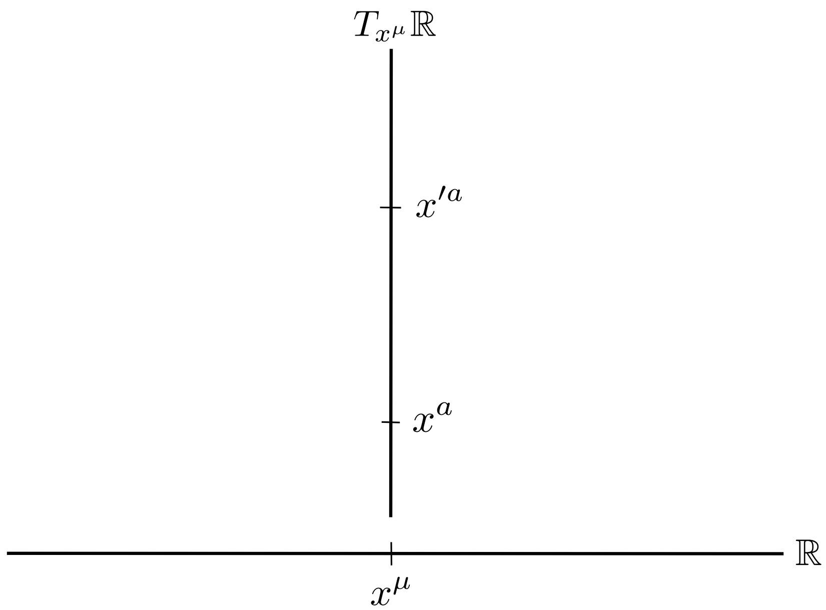
The Tangent Bundle

The geometrical setting of any theory for gravitation

is the tangent bundle

$$\Downarrow$$

At each point of spacetime — the base space of the bundle — there is a tangent space attached to it — the fiber of the bundle



- The Latin alphabet $a, b, c, \ldots$ denote Minkowski tangent spaces indices
- The Greek alphabet $\mu, \nu, \rho, \ldots$ denote spacetime indices

The Tangent Bundle is Soldered

Spacetime (the base space of the bundle) and the tangent space (the fiber of the bundle) are connected to each other

$$\Downarrow$$

This connection is made by the ...

... **Tetrad Field** $h^a{}_\mu$

$$\Downarrow$$

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \, \eta_{ab}$$

Absence of gravitation: Trivial Tetrad $e^a{}_\mu = \partial_\mu x^a$

$$\Downarrow$$

$$\eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu \, \eta_{ab}$$

Lorentz Connections

A Lorentz connection A_μ is a 1-form assuming values in the Lie algebra of the Lorentz group:

$$A_\mu = \tfrac{1}{2}\, A^{ab}{}_\mu\, S_{ab}$$

$A^{ab}{}_\mu \to$ connection components

$S_{ab} \to$ representation of the Lorentz generators

$$\Downarrow$$

Connection $A^{ab}{}_\mu$ can be written with spacetime indices:

$$A^a{}_{b\mu} \longleftrightarrow \Gamma^\rho{}_{\nu\mu}$$

Relation Between $A^{ab}{}_{\mu}$ and $\Gamma^{\rho}{}_{\nu\mu}$

$$\Gamma^{\rho}{}_{\nu\mu} = h_a{}^{\rho} A^{a}{}_{b\mu} h^{b}{}_{\nu} + h_a{}^{\rho} \partial_{\mu} h^{a}{}_{\nu}$$

$$A^{a}{}_{b\mu} = h^{a}{}_{\nu} \Gamma^{\nu}{}_{\rho\mu} h_b{}^{\rho} + h^{a}{}_{\nu} \partial_{\mu} h_b{}^{\nu}$$

$$\Downarrow$$

$A^{a}{}_{b\mu}$ and $\Gamma^{\rho}{}_{\nu\mu}$ are two ways of writing the same connection

$$\Downarrow$$

These connections define two covariant derivatives

$$\mathcal{D}_{\mu} = \partial_{\mu} + A_{\mu} \quad \leftrightarrow \quad \nabla_{\mu} = \partial_{\mu} + \Gamma_{\mu}$$

$$\Downarrow$$

Whereas the derivative $\mathcal{D}_{\mu}$ can be defined for bosonic and fermonic fields, the covariant derivative ∇_{μ} can be defined for bosonic fields only

Curvature and Torsion

Given a general Lorentz connection $A^a{}_{b\mu}$, its curvature and torsion are defined by

$$R^a{}_{b\nu\mu} = \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu}$$

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + A^a{}_{e\nu} h^e{}_\mu - A^a{}_{e\mu} h^e{}_\nu$$

$$\Downarrow$$

Through contraction with tetrads, these tensors can be written in spacetime–indexed forms:

$$R^\rho{}_{\lambda\nu\mu} \equiv h_a{}^\rho \, h^b{}_\lambda \, R^a{}_{b\nu\mu}$$

$$T^\rho{}_{\nu\mu} \equiv h_a{}^\rho \, T^a{}_{\nu\mu}$$

Inertial Lorentz Connection

To see how an inertial Lorentz connection
shows up in special relativity ...

$$\Downarrow$$

... let us consider the equation of motion of a
free particle in Minkowski spacetime

$$\frac{du'^a}{d\sigma} = 0$$

$u'^a \;\rightarrow\;$ Denotes the four-velocity in an inertial frame

$d\sigma^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu \;\rightarrow\;$ Minkowski invariant interval

Inertial Lorentz Connection

The corresponding equation in a non-inertial frame is obtained by performing a local Lorentz transformation

$$u^a = \Lambda^a{}_b(x)\, u'^b$$

$$\Downarrow$$

In this frame, the equation of motion assumes the form

$$\frac{du^a}{d\sigma} + \overset{\bullet}{A}{}^a{}_{b\mu}\, u^b\, u^\mu = 0$$

$$\Downarrow$$

$$\overset{\bullet}{A}{}^a{}_{b\mu} = \Lambda^a{}_e(x)\, \partial_\mu \Lambda_b{}^e(x)$$

is a purely inertial Lorentz connection.

It represents inertial effects present in the new frame

GENERAL RELATIVITY

Gravitation is Universal

Particles with different masses and compositions feel gravitation in such a way that all of them acquire the same acceleration

$$\Updownarrow$$

Provided the initial conditions are the same, all of them will follow the same trajectory

$$\boxed{\begin{array}{c}\textit{Universality of Free Fall}\\ \Updownarrow \\ \textit{Weak Equivalence Principle}\end{array}}$$

Strong Equivalence Principle

Since the inertial forces are also universal,

it then follows the

$$\boxed{\textit{Strong Equivalence Principle}}$$

$\Downarrow$

It establishes the **local equivalence between**

inertial effects and gravitation

$\Downarrow$

General relativity was constructed to comply with these principles

Lorentz Connection of General Relativity

The fundamental connection of general relativity is the Levi-Civita (or Christoffel) connection

$$\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \tfrac{1}{2}\, g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$$\Downarrow$$

The corresponding spin connection is

$$\boxed{\overset{\circ}{A}{}^{a}{}_{b\mu} = h^{a}{}_{\nu} \overset{\circ}{\Gamma}{}^{\nu}{}_{\rho\mu} h_{b}{}^{\rho} + h^{a}{}_{\nu} \partial_\mu h_{b}{}^{\nu}}$$

$$\Downarrow$$

It is a connection with vanishing torsion, but non-vanishing curvature:

$$\overset{\circ}{T}{}^{a}{}_{\nu\mu} = 0 \qquad \text{and} \qquad \overset{\circ}{R}{}^{a}{}_{b\nu\mu} \neq 0$$

Particle Equation of Motion

In general relativity, the equation of motion of a (spinless) particle is the geodesic equation

$$\frac{du^a}{ds} + \overset{\circ}{A}{}^a{}_{b\mu}\, u^b\, u^\mu = 0$$

The left-hand side is the particle four-acceleration

$\Downarrow$

The right-hand side represents then the gravitational force acting on the particle

$\Downarrow$

Its vanishing means that in general relativity there is no the concept of **gravitational force**

How does general relativity describe the gravitational interaction?

When restricted to general relativity, only the torsionless Levi–Civita connection is present

$$\Downarrow$$

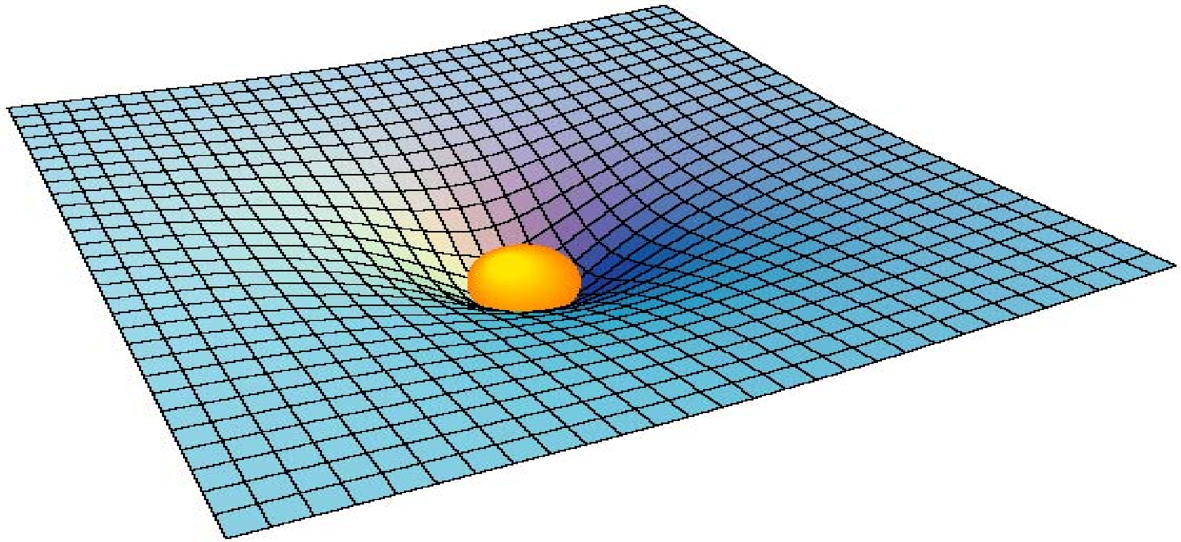
In this case, its curvature can be interpreted, together with the metric, as part of the spacetime definition

$$\Downarrow$$

One can then talk about "curved spacetime"

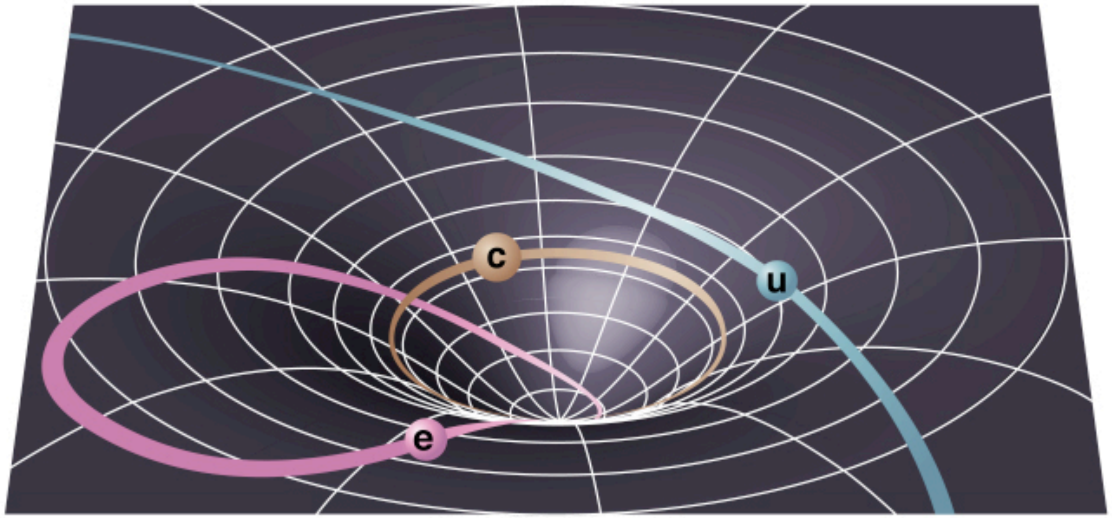
Geometrizing the Interaction

The presence of a gravitational field was supposed by Einstein to produce a *curvature* in spacetime



… Geometrizing the Interaction

A particle in a gravitational field simply follows the geodesics of the curved spacetime



In general relativity, the responsibility of describing the gravitational interaction is transferred to spacetime

$$\Downarrow$$

Geometry replaces the concept of force

How does matter curve spacetime?

The answer is given by Einstein equation

$$\boxed{\overset{\circ}{R}{}^{a}{}_{\mu} - \frac{1}{2}\, h^{a}{}_{\mu}\, \overset{\circ}{R} = k\, T^{a}{}_{\mu}}$$

$$\Downarrow$$

Given a matter distribution represented by the energy-momentum tensor $T^{a}{}_{\mu}$

$$\Downarrow$$

one solves the equation and obtain the Riemann curvature tensor $R^{a}{}_{b\lambda\mu}$

Gravitational Interaction in General Relativity

What About Torsion?

A general Lorentz connection has both curvature and torsion

$$\Downarrow$$

Why should energy and momentum

produce only curvature?

$$\overset{\circ}{R}{}^{a}{}_{\mu} - \tfrac{1}{2}\, h^{a}{}_{\mu}\, \overset{\circ}{R} = k\, T^{a}{}_{\mu}$$

Was Einstein wrong when he made this assumption by choosing the **torsionless** Levi-Civita connection?

Does torsion play any role in gravitation?

TELEPARALLEL GRAVITY

The fundamental connection of teleparallel gravity is the purely inertial connection

$$\boxed{\overset{\bullet}{A}{}^{a}{}_{b\mu} = \Lambda^{a}{}_{e}(x)\,\partial_{\mu}\Lambda_{b}{}^{e}(x)}$$

$$\Downarrow$$

In teleparallel gravity, therefore, Lorentz connections keep their special relativity role of representing inertial effects only, not gravitation

$$\Downarrow$$

The gravitational theory that follows from this choice is a

gauge theory for the translation group

Why the translation group?

The answer follows from playing the gauge game

$\Downarrow$

The source of gravitation is energy and momentum

$\Downarrow$

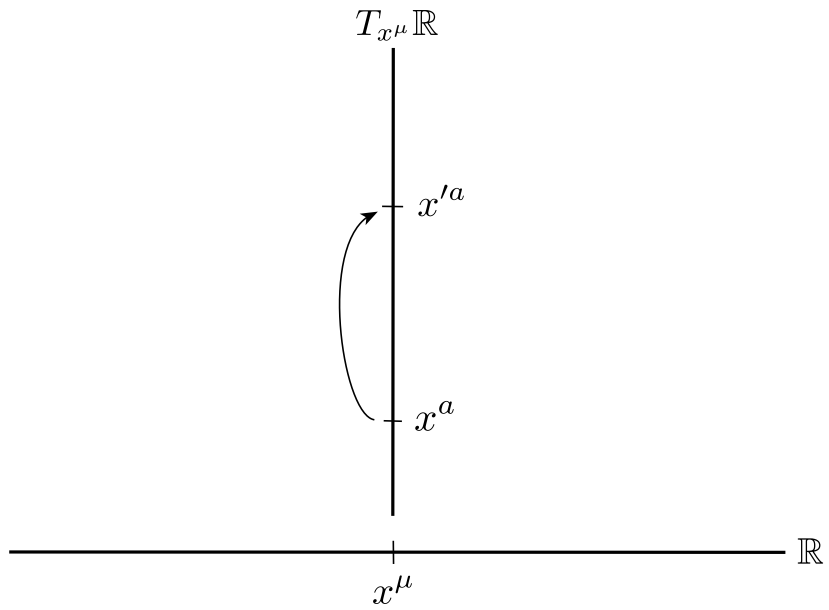
From Noether's theorem, the energy–momentum tensor is conserved
provided the source lagrangian is invariant under
spacetime translations

$\Downarrow$

If gravity is to be described by a gauge theory with
energy-momentum as a source, it must be a
gauge theory for the translation group

Gauge Transformations ...

... are local translations in the Minkowski tangent space, the fiber of the tangent bundle



$$x^a \to x'^a = x^a + \epsilon^a(x^\mu)$$

Fundamental Field

The fundamental field of teleparallel gravity is the translational gauge potential ${B^a}_\mu$, a 1-form assuming values in the Lie algebra of the translation group

$$B_\mu = {B^a}_\mu \, P_a$$

$P_a = \partial/\partial x^a \rightarrow$ generators of infinitesimal translations

$$\Downarrow$$

It appears as the nontrivial part of the tetrad field ${h^a}_\mu$

$${h^a}_\mu = \partial_\mu x^a + {B^a}_\mu$$

$$\Downarrow$$

By non-trivial part we mean

$${B^a}_\mu \neq \partial_\mu \epsilon^a$$

Curvature and Torsion

The inertial connections has vanishing curvature:

$$\overset{\bullet}{R}{}^{a}{}_{b\nu\mu} \equiv \partial_\nu \overset{\bullet}{A}{}^{a}{}_{b\mu} - \partial_\mu \overset{\bullet}{A}{}^{a}{}_{b\nu} + \overset{\bullet}{A}{}^{a}{}_{e\nu}\, \overset{\bullet}{A}{}^{e}{}_{b\mu} - \overset{\bullet}{A}{}^{a}{}_{e\mu}\, \overset{\bullet}{A}{}^{e}{}_{b\nu} = 0$$

$$\Downarrow$$

However, for a non-trivial tetrad:

$$\overset{\bullet}{T}{}^{a}{}_{\nu\mu} \equiv \partial_\nu h^{a}{}_{\mu} - \partial_\mu h^{a}{}_{\nu} + \overset{\bullet}{A}{}^{a}{}_{e\nu}\, h^{e}{}_{\mu} - \overset{\bullet}{A}{}^{a}{}_{e\mu}\, h^{e}{}_{\nu} \neq o$$

$$\Downarrow$$

Torsion is the field-strength of teleparallel gravity

$$\Downarrow$$

In teleparallel gravity the gravitational field is fully represented by the translational gauge potential $B^{a}{}_{\mu}$

Teleparallel Lagrangian

The Lagrangian of teleparallel gravity, like any gauge Lagrangian, is quadratic in the field strength, or torsion:

$$\boxed{\overset{\bullet}{\mathcal{L}} = \frac{h}{4k}\left[T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + 2\,T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - 4\,T_{\rho\mu}{}^{\rho}T^{\nu\mu}{}_{\nu}\right]}$$

$$h = \det(h^{a}{}_{\mu}) = \sqrt{-g}$$

Teleparallel Field Equations

Variation of the lagrangian in relation to the gauge field $B^a{}_\rho$ yields the sourceless teleparallel field equations

$$\boxed{\partial_\sigma(h\overset{\bullet}{S}_a{}^{\rho\sigma}) - k\,(h\,\overset{\bullet}{\jmath}_a{}^{\rho}) = 0}$$

$\overset{\bullet}{S}_a{}^{\rho\sigma} = -\overset{\bullet}{S}_a{}^{\sigma\rho} \quad \rightarrow \quad$ Superpotential

$\overset{\bullet}{\jmath}_a{}^{\rho} \quad \rightarrow \quad$ Energy-momentum current

Connection Decomposition

According to a theorem by Ricci, given a general Lorentz connection $A^a{}_{b\nu}$, it can always be decomposed according to

$$\boxed{A^a{}_{b\nu} = \overset{\circ}{A}{}^a{}_{b\nu} + K^a{}_{b\nu}}$$

$$K^a{}_{b\nu} = \frac{1}{2}\left(T_b{}^a{}_\nu + T_\nu{}^a{}_b - T^a{}_{b\nu}\right) \;\rightarrow\; \text{Contortion tensor}$$

$$\Downarrow$$

In the specific case of the teleparallel spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$, the decomposition has the form

$$\boxed{\overset{\bullet}{A}{}^a{}_{b\mu} = \overset{\circ}{A}{}^a{}_{b\mu} + \overset{\bullet}{K}{}^a{}_{b\mu}}$$

Equivalence with General Relativity

Using the above relation, the teleparallel lagrangian is found to satisfy

$$\boxed{\overset{\bullet}{\mathcal{L}} = \frac{h}{2k} \overset{\circ}{R} - \partial_\mu \omega^\mu}$$

$$\Downarrow$$

Using the same relation, the teleparallel field equation satisfy

$$\boxed{\partial_\sigma (h \overset{\bullet}{S}{}_a{}^{\rho\sigma}) - k^2 \, (h \, \overset{\bullet}{\jmath}{}_a{}^{\rho}) \equiv h \left(\overset{\circ}{R}{}_a{}^{\rho} - \tfrac{1}{2} \, h_a{}^{\rho} \, \overset{\circ}{R} \right)}$$

$$\Downarrow$$

Teleparallel gravity is equivalent to general relativity

There is no new physics associated to torsion

Why Gravitation Has Two Alternative Descriptions?

Like any other interaction of nature, gravitation has
a description in terms of a gauge theory

$\Downarrow$

Teleparallel gravity corresponds to a
gauge theory for the translation group

$\Downarrow$

On the other hand, **universality of gravitation** allows an
alternative description in terms of a geometrization of spacetime

$\Downarrow$

General Relativity is a geometric theory for gravitation

$\Downarrow$

Universality of the gravitational interaction is responsible
for the existence of the alternative, geometric description

ACHIEVEMENTS OF TELEPARALLEL GRAVITY

Although equivalent theories ...

... there are conceptual differences between GR and TG

$$\Downarrow$$

Perhaps the main difference is that teleparallel gravity

is a gauge theory, whereas general relativity is not

$$\Downarrow$$

As a matter of fact, teleparallel gravity is a

true field theory (in the sense of classical fields),

whereas general relativity is not

Separating Inertia from Gravitation

Let us consider again the connection decomposition

$$\overset{\circ}{A}{}^a{}_{b\rho} = \overset{\bullet}{A}{}^a{}_{b\rho} - \overset{\bullet}{K}{}^a{}_{b\rho}$$

$$\Downarrow$$

Since $\overset{\bullet}{A}{}^a{}_{b\rho}$ represents inertial effects only, this expression corresponds to a separation between inertial effects and gravitation

$$\Downarrow$$

In the local frame in which $\overset{\circ}{A}{}^a{}_{b\rho} = 0$, the above identity becomes

$$\color{red}{\overset{\bullet}{A}{}^a{}_{b\rho}} = \color{blue}{\overset{\bullet}{K}{}^a{}_{b\rho}}$$

$$\Downarrow$$

This expression shows explicitly that, in such a local frame, inertial effects exactly compensate gravitation

Particle Equation of Motion

In general relativity, the equation of motion of a spinless particle is the geodesic equation

$$\frac{du^a}{ds} + \overset{\circ}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = 0$$

$$\Downarrow$$

Substituting the connection decomposition

$$\overset{\circ}{A}{}^a{}_{b\rho} = \overset{\bullet}{A}{}^a{}_{b\rho} - \overset{\bullet}{K}{}^a{}_{b\rho}$$

$$\Downarrow$$

we obtain the corresponding equation of motion in teleparallel gravity

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = \overset{\bullet}{K}{}^a{}_{b\rho}\, u^b\, u^\rho$$

$$\Downarrow$$

This is the teleparallel version of the equation of motion

… Particle Equation of Motion

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = \overset{\bullet}{K}{}^a{}_{b\rho}\, u^b\, u^\rho$$

$\Downarrow$

This is a force equation, with contortion playing the role of gravitational force

$\Downarrow$

The inertial effects of the frame are represented by the connection of the left-hand side, which is non-covariant by its very nature

$\Downarrow$

The inertial effects remain geometrized in the sense of general relativity.

In general relativity, curvature is used to
geometrize the gravitational interaction

$\Downarrow$

In teleparallel gravity, torsion accounts for gravitation, not by
geometrizing the interaction, but by acting as a **force**

$\Downarrow$

As a consequence, there are no geodesics in teleparallel
gravity, but only force equations

$\Downarrow$

This is quite similar to electromagnetism, a gauge
theory for the unitary group U(1)

$\Downarrow$

This is expected since, like Maxwell theory,
teleparallel gravity is a gauge theory

Although mostly unnoticed ...

... torsion has already been detected

$\Downarrow$

It is responsible for all gravitational effects, including the physics of the solar system, which can be reinterpreted in terms of a force equation, with contortion playing the role of force

$\Downarrow$

According to teleparallel gravity, curvature and torsion are related to the same degrees of freedom of gravity

$\Downarrow$

This means that Einstein did not make any mistake by choosing a torsionless connection

Gravitational Energy Localization

As is well-known, in general relativity the complex defining the energy-momentum density of gravitation is a pseudotensor

$$\Downarrow$$

The basic reason for this is that the spin connection of general relativity ${\stackrel{\circ}{A}}{}^{a}{}_{b\mu}$ includes both gravitation and inertial effects

$$\Downarrow$$

Since inertial effects are non-tensorial by definition, the resulting energy-momentum density is necessarily a pseudotensor

$$\Downarrow$$

This characteristic has been considered, not really a problem, but an inherent property of the gravitational field

... Gravitational Energy Localization

On the other hand, on account of the possibility of separating gravitation from inertial effects, in teleparallel gravity the energy-momentum pseudotensor can be written in the form

$$\stackrel{\bullet}{\jmath}_a{}^\rho \equiv \stackrel{\bullet}{t}_a{}^\rho + \stackrel{\bullet}{\imath}_a{}^\rho$$

$$\Downarrow$$

$\stackrel{\bullet}{t}_a{}^\rho \;\rightarrow\;$ represents gravitation only, and is a tensorial quantity

$\stackrel{\bullet}{\imath}_a{}^\rho \;\rightarrow\;$ represents inertial effects, which is non–tensorial

$$\Downarrow$$

$$\stackrel{\bullet}{t}_a{}^\rho = \frac{1}{k} h_a{}^\lambda \stackrel{\bullet}{S}_c{}^{\nu\rho} \stackrel{\bullet}{T}{}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h} \stackrel{\bullet}{\mathcal{L}}$$

Gravitation and Universality

In the next few years several experiments will test the
weak equivalence principle to an unprecedented precision

$\Downarrow$

If some violation is found, the geometric description
of general relativity breaks down

$\Downarrow$

On the other hand, like Newtonian gravity, teleparallel gravity
remains a consistent theory in the absence of universality

$\Downarrow$

In this sense it can be considered to be
a more fundamental theory

Curved Spacetime Paradigm

The advent of teleparallel gravity breaks dow
the paradigm of the curved spacetime

$$\Downarrow$$

The statement that spacetime is curved depends on the theory
used to describe the gravitational interaction

$$\Downarrow$$

**Whether spacetime is curved or not turns out
to be a matter of convention!**

$$\Downarrow$$

This raises some philosophical question as to
the existence of an objective reality

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