# Degeneracy in physical systems: Dynamical dimensional reduction

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# Multivalued Hamiltonians

Hamiltonians that are multivalued functions of the momenta are generated from Lagrangians with time derivatives of order higher than two.

Such cases arise in several interesting physical situations:

- Modifications of GR.
- Nonlinear Electrodynamics
- K-essence models.
- Classical time crystals.

To investigate such configurations we start with the most simple non-canonical kinetic term [3, 4]

$$L = \frac{\beta}{4}\dot{\phi}^4 - \frac{\kappa}{2}\dot{\phi}^2 - V(\phi), \qquad (1)$$

where $\beta$ and $\kappa$ are positive.
The conjugated momenta is

$$p_\phi = \beta\dot{\phi}^3 - \kappa\dot{\phi}, \qquad (2)$$

which is a multivalued function of $\dot{\phi}$.

The equation of motion is

$$\ddot{q}^j \frac{\partial p_i}{\partial \dot{q}^j} + \dot{q}^j \frac{\partial p_i}{\partial q^j} - \frac{\partial L}{\partial q^i} = 0, \tag{3}$$

Thence, in such extreme points, the equations of motion become singular. The system becomes degenerate. In the system given above

$$(3\beta\dot{\phi}^2 - \kappa)\ddot{\phi} = -V'(\phi), \tag{4}$$

dot and slash denote derivatives with respect to time and $\phi$, respectively.

The system becomes degenerate for $\dot{\phi}^2 = \kappa/3\beta$, as the coefficient multiplying the acceleration goes to zero.

We are able to describe the system through a first order Lagrangian in the space spanned by $\vec{z} = (\phi, \rho)$

$$L = (\beta\rho^3 - \kappa\rho)\dot{\phi} - H \tag{5}$$

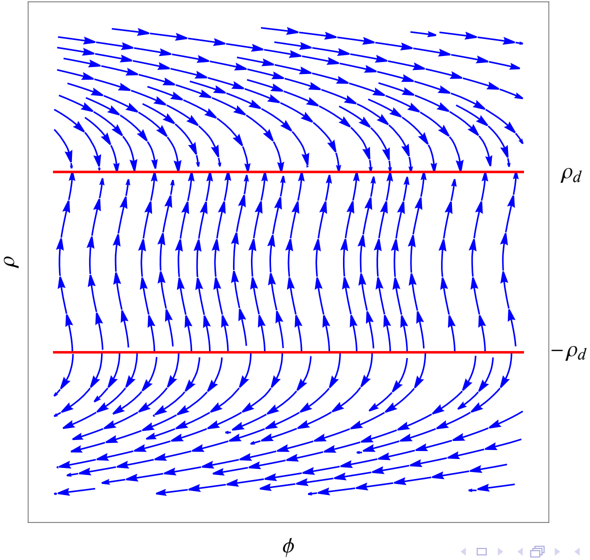
The eq. of motion

$$\Delta\dot{\rho} = -V'(\phi), \qquad \Delta(\dot{\phi} - \rho) = 0, \tag{6}$$

being $\Delta = (3\beta\rho^2 - \kappa)$. Henceforth, as expected, the degeneracy surfaces are $\rho = \pm\rho_d = \pm\sqrt{\kappa/3\beta}$, for which $\Delta = 0$. The flux $\Phi = j^i n_i$ into the surface is

$$\Phi = -6\beta\rho V'(\phi_d), \tag{7}$$

# Linear growing potential



# Sombrero potential

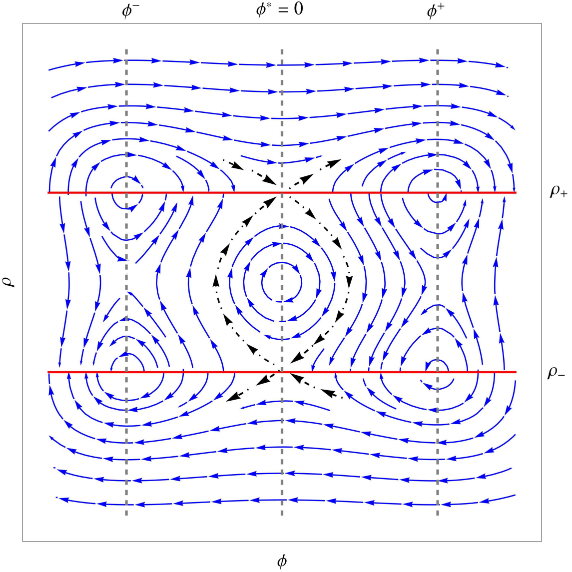
Let now the potential be

$$V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{\omega}{2}\phi^2, \tag{8}$$

being $\lambda$ and $\omega$ positive. The potential has two global minima $\phi^{\pm} = \pm\omega/\lambda$, and one local maximum $\phi^{*} = 0$.
Further, the flux is

$$\Phi = -6\beta\rho\phi(\lambda\phi^2 - \omega). \tag{9}$$

# Sombrero potential



To understand what happens in the degeneracy surface we examine the constraints

$$G_1 = p_\phi - \beta\rho^3 + \kappa\rho \approx 0, \qquad G_2 = p_\rho \approx 0. \tag{10}$$

The Poisson bracket between them

$$\{G_1, G_2\} = -\Delta, \tag{11}$$

which vanishes in the degeneracy surface $\Delta = 0$.

Following the Dirac-Bergmann procedure, we write the total Hamiltonian as

$$
\begin{aligned}
H_T &= p_\phi \dot{\phi} + p_\rho \dot{\rho} - L + \mu^i G_i \qquad (12) \\
&= (p_\phi - A_1)\dot{\phi} + p_\rho \dot{\rho} + H + \mu^i G_i = H + \bar{\mu}^i G_i . \qquad (13)
\end{aligned}
$$

Thence, the consistency conditions $\dot{G}_i = \{G_i, H_T\} \approx 0$ gives

$$
\dot{G}_1 = -V'(\phi) - \bar{\mu}^2 \Delta \approx 0, \qquad (14)
$$

$$
\dot{G}_2 = \Delta(\bar{\mu}^1 - \rho) \approx 0, \qquad (15)
$$

which determine the Lagrange multipliers. The equations of motion are

$$
\dot{\phi} = \{\phi, H_T\} = \bar{\mu}^1, \qquad \dot{\rho} = \{\rho, H_T\} = \bar{\mu}^2. \qquad (16)
$$

So, the evolution of the system is given by the Lagrange multipliers.

## A degenerated Universe

The model is a FLRW Universe filled with a k-essence field. The Lagrangian for the field can be cast as

$$\mathcal{L} = f(\phi)k(X) - V(\phi), \tag{17}$$

with $X = (\nabla\phi)^2/2 = \dot{\phi}^2/2$ and $f(\phi) > 0$. Important quantities in the model are the pressure $p = \mathcal{L}$ and the energy density

$$\varepsilon = 2X\mathcal{L}_{,X} - \mathcal{L} = f(2Xk_{,X} - k) + V. \tag{18}$$

Moreover, the dynamics of the model is found through Einstein's equations

$$H^2 = \frac{8\pi}{3}\varepsilon, \quad \varepsilon_{,X}\ddot{\phi} = -3Hp_{,X}\dot{\phi} - \varepsilon_{,\phi}. \tag{19}$$

# A degenerated Universe

The dynamical equation of state for the scalar field is

$$w = \frac{fk - V}{f(2Xk_{,X} - k) + V}, \tag{20}$$

so that $w = -1$ implies that $Xk_{,X} = 0$.
Another important constraint is due to the stability of the model against perturbations: the squared sound speed $c_s^2$ shall be positive. That is

$$c_s^2 = \frac{p_{,X}}{\varepsilon_{,X}} = \frac{k_{,X}}{2Xk_{,XX} + k_{,X}} \geq 0. \tag{21}$$

# A degenerated Universe

We require that both functions $p_{,X}$ and $\varepsilon_{,X}$ approach zero for $X = X_d$, i.e.: $k_{,X}(X_d) = 0$ and $k_{,XX}(X_d) = 0$. Under these conditions, expanding the function $k(X)$ around $X_d$, we find the Lagrangian for the model

$$\mathcal{L} = f(\phi)(X - X_d)^3 - V(\phi). \tag{22}$$

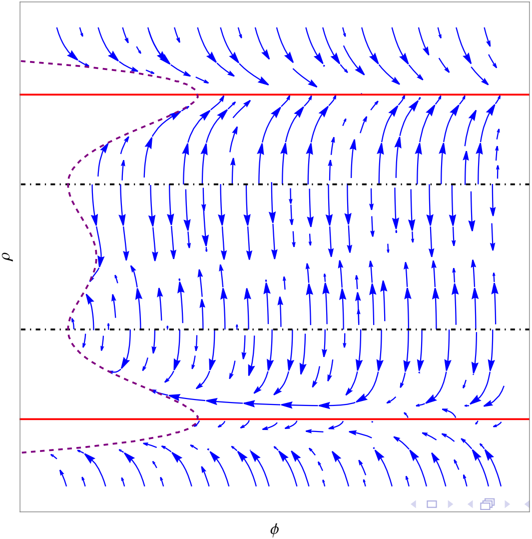
So that

$$\varepsilon_{,X} = 3f(X - X_d)(5X - X_d), \tag{23}$$

and

$$c_s^2 = \frac{X - X_d}{5X - X_d}. \tag{24}$$

# Dynamical evolution of the system for $f = cte$ and a linear growing potential.



# Final remarks

- We have shown the appearence of degenerated surfaces in multivalued Hamiltonians, in which the system looses degrees of freedom.
- We have also presented an interesting possible cosmological consequence of this process: A dynamical dark energy which degenerates into a cosmological constant.
- Now we are looking for possible observables to answer the question: Is it possible that we live in a degenerated Universe?

# References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# Thank you!