Post-Newtonian analysis of analytic Palatini $f(R)$ theories and Solar System tests

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Table of contents

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This because the metric contains gravitational potentials which are not covered by the PPN formalism and, consequently, the PPN parameters could not maintain its original physical meaning.

One should then look to the PN equations of motion for light rays and massive bodies in order to infer the PPN parameters. This is our approach.

PPN formalism review

The parametrized post-Newtonian (PPN) formalism is a practical way to test gravitational theories in confrontation with Solar System tests.

It is considered a perfect fluid as a source producing a weak gravitational field, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \, , \tag{1}$$

$$T^{\mu\nu} = \left(\rho + \rho\Pi/c^2 + p/c^2\right) u^\mu u^\nu + p g^{\mu\nu} \, . \tag{2}$$

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The $h_{\mu\nu}$'s should depend on gravitational potentials whose coefficients, the PPN parameters, can be constrained by observational data.

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The PPN metric is,

$$
\begin{aligned}
g_{00} &= -1 + 2U - 2\beta U^2 + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 + \\
&\quad + 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 \; + 2(1 + \zeta_3)\Phi_3 + \\
&\quad + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - (\zeta_1 - 2\xi)\Phi_6 + \\
&\quad - 2\xi\Phi_W + (1 + \alpha_2 - \zeta_1 - 2\xi)\partial_{tt}\chi \\
g_{0i} &= -(2\gamma + 2 + \alpha_1/2)V_i \qquad (3) \\
g_{ij} &= \; (1 + 2\gamma\, U)\, \delta_{ij} \, .
\end{aligned}
$$

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The PPN potentials are,

$$
\begin{aligned}
&U=\int\frac{\rho^*(t,\boldsymbol{x}')\,d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|},\quad \chi=\int\rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x',\quad \Phi_1=\int\frac{\rho^{*\prime}v'^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',\\
&\Phi_2=\int\frac{\rho^{*\prime}U'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',\quad \Phi_3=\int\frac{\rho^{*\prime}\Pi'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',\quad \Phi_4=\int\frac{p'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',\\
&\mathcal{A}=\int\frac{\rho^{*\prime}[\boldsymbol{v}'\cdot(\boldsymbol{x}-\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d^3x',\quad \mathcal{B}=\int\frac{\rho^{*\prime}(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\cdot\frac{d\boldsymbol{v}'}{dt}\,d^3x',\\
&\Phi_w=\int\rho^{*\prime}\rho^{*\prime\prime}\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\cdot\left(\frac{\boldsymbol{x}'-\boldsymbol{x}''}{|\boldsymbol{x}-\boldsymbol{x}''|^3}-\frac{\boldsymbol{x}-\boldsymbol{x}''}{|\boldsymbol{x}'-\boldsymbol{x}''|^3}\right)\,d^3x'd^3x''.\\
&V^i=\int\frac{\rho^*(t,\boldsymbol{x}')v^i}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',\quad W^j=\int\frac{\rho^{*\prime}\boldsymbol{v}'\cdot(\boldsymbol{x}-\boldsymbol{x}')(x-x')^j}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d^3x'.
\end{aligned}
\tag{4}
$$

PPN formalism review

Table: The PPN parameters, their relation with physical phenomena and respective GR values.

Parameter	Meaning	Value in GR
γ	Light deflection and time delay	1
β	Perihelion shift	1
ξ	Preferred-location effects	0
α_1	Preferred-frame effects	0
α_2		0
α_3		0
ζ_1	Violation in momentum conservation	0
ζ_2		0
ζ_3		0
ζ_4		0

PPN formalism review

A theory whose PN solution does contain distinct potentials is not covered by the PPN formalism.

It is necessary to obtain the equations of motion of massive bodies and photons in order to determine how the new potentials influence in the dynamics of planets and light rays.

Through the equations of motion is then expected to obtain the PPN parameters.

We proceed in these lines with Palatini gravity.

Palatiny $f(R)$

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The field equations read,

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu}\,, \tag{5}$$

$$\nabla_{\alpha}\left(\sqrt{-g}f'(R)g^{\mu\nu}\right) = 0\,, \tag{6}$$

where κ is the coupling constant, the prime indicates a derivative with respect to R and ∇ represents a covariant derivative constructed with the afinne connection Γ.

Palatiny $f(R)$

Equation $\nabla_\alpha \left(\sqrt{-g} f'(R) g^{\mu\nu}\right) = 0$, can be rewritten as a metricity condition of a conformal metric,

$$\overline{g}_{\mu\nu} = f'(R)\, g_{\mu\nu}. \tag{7}$$

Once that $\nabla_\alpha \overline{g}_{\mu\nu} = 0$, it follows that $\Gamma^\lambda_{\mu\nu}$ is related with $\overline{g}_{\mu\nu}$ as a Levi-Civita connection, namely

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, \overline{g}^{\lambda\alpha} \left(\partial_\nu \overline{g}_{\alpha\mu} + \partial_\mu \overline{g}_{\alpha\nu} - \partial_\alpha \overline{g}_{\mu\nu}\right), \tag{8}$$

with $\partial_\mu = \partial/\partial x^\mu$.

Palatiny $f(R)$

However, due to the diffeomorphism invariance, one has

$$\nabla_C^{\mu} T_{\mu\nu} = 0, \tag{9}$$

where ∇_C is constructed from the Christoffel symbol,

$$C^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right). \tag{10}$$

This implies that test particles will follow geodesics determined from the connection C, not the fundamental Γ connection.

PN limit of Palatini $f(R)$

To determine the PN corrections in the equations of motion it is necessary to know $g_{00} \sim O(4)$, $g_{0i} \sim O(3)$ and $g_{ij} \sim O(2)$.

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In order to solve the field equations we consider an analytic class of $f(R)$ and expand it in a polynomial series around $R = 0$,

$$f(R) = R + \sum_{n=2}^{\infty} a_n R^n, \tag{11}$$

where a_n are constants coefficients. We are considering

$$a_0 = 0 \quad \rightarrow \quad \text{Neglecting cosmological constant effects}$$

$$a_1 = 1 \quad \rightarrow \quad \text{not physical since } \kappa \text{ is arbitrary}$$

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The field equation is then simplified to,

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + 2\kappa^2 a_2 T \left(T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T \right) + O(6).$$

PN metric of $f(R)$ Palatini gravity

The PN metric (not the PPN original gauge) reads,

$$g_{00} = -1 + 2\left(U + \lambda\rho^*\right) + 2\left(\psi + \frac{1}{2}\partial_{tt}\chi - U^2 - 2\lambda\Phi_P\right) - \\ - \lambda\rho^*\left[10U + v^2 - 2\Pi + \frac{6p}{\rho^*} - \left(2\lambda + \frac{3\sigma}{\lambda}\right)\rho^*\right] + O(6), \quad (12)$$

$$g_{0i} = -4V^i + O(5), \quad (13)$$

$$g_{ij} = \delta_{ij} + 2\left(U - \lambda\rho^*\right)\delta_{ij} + O(4). \quad (14)$$

with

$$\psi = \frac{3}{2}\Phi_1 - \Phi_2 + \Phi_3 + 3\Phi_4, \quad \Phi_P = \int \frac{\rho^{*\prime 2}}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x', \quad (15)$$

$$\rho^* = \rho u^0\sqrt{-g}, \quad \lambda = 8\pi a_2 \quad \text{and} \quad \sigma = 64\pi^2 a_3. \quad (16)$$

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$$g_{0i} = -4V^i + O(5), \quad (18)$$

$$g_{ij} = \delta_{ij} + 2\left(U - \lambda\rho^*\right)\delta_{ij} + O(4). \quad (19)$$

in black $\longrightarrow$ same terms as the PN metric of GR

in red $\longrightarrow$ new terms brought by Palatini gravity,

Motion of light

Post-Newtonian effects in the equations of motion for photons are entirely described by knowing $g_{\mu\nu}$ up to order $O(2)$, which reads

$$g_{00} \approx -1 + 2\left(U + \lambda\rho^*\right), \quad g_{0i} \approx 0, \quad g_{ij} \approx \delta_{ij} + 2\left(U - \lambda\rho^*\right)\delta_{ij}.$$

All the experiments used to constraint the trajectories and time delays of electromagnetic waves do not depend on a given medium, they only depend on the vacuum spacetime.

Thus, Palatini version to the equations of motion of light rays is the same as in Einstein's theory, leading to an agreement with bending of light and Shapiro time delay tests.

In other words, $\gamma = 1$ for Palatini $f(R)$ theories.

Motion of light

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From light equation of motion, one gets

$$\gamma = \frac{h_{00} + H}{2U} - 1, \quad \text{only if} \quad \left(h_{00} + H\right) \propto U. \tag{20}$$

This formula can be applied to our case and results $\gamma = 1$.

Note that the gravitational slip is different,

$$\gamma_s = \frac{H}{h_{00}} = \frac{U + \lambda\rho^*}{U - \lambda\rho^*}. \tag{21}$$

Conserved quantities

Integrating equation $\nabla_\nu T^{\mu\nu} = 0$, is possible to obtain integrals statements that represent conserved quantities.

For $\mu = 0$, the leading terms return a continuity equation,

$$\partial_t(\rho^*) + \partial_i(\rho^* v^i) = 0, \quad \text{with} \quad \rho^* = u^0\sqrt{-g}\,\rho. \tag{22}$$

Together with the PN terms, one has a total mass-energy conservation law,

$$\frac{dM}{dt} = 0, \quad M = m + E, \quad m = \int \rho^* d^3x, \tag{23}$$

$$E \equiv \int \left(\frac{1}{2}\rho^* v^2 - \frac{1}{2}\rho^* U + \rho^* \Pi - \frac{\lambda}{2}\rho^{*2}\right) d^3x. \tag{24}$$

Conserved quantities

Integrating equation $\nabla_\nu T^{\mu\nu} = 0$, is possible to obtain integrals statements that represent conserved quantities.

For $\mu = i$, the leading terms return the Euler equation,

$$\rho^* \frac{dv^j}{dt} = \rho^* \partial^j U - \partial^j p + \frac{\lambda}{2} \partial^j (\rho^{*2}). \tag{25}$$

With the PN contributions we have a law of momentum conservation,

$$\frac{d}{dt} P^j = 0, \tag{26}$$

with

$$P^j = \int \rho^* v^j \left(1 + \frac{v^2}{2} - \frac{U}{2} + \Pi + \frac{3p}{\rho^*} - \lambda \rho^* \right) d^3x - \frac{1}{2} \int \rho^* W^j d^3x. \tag{27}$$

Conserved quantities

The previous results show that Palatini $f(R)$ gravity does not violate total conservation of energy and momentum in the PN regime.

This is an expected outcome since any Palatini gravity model is a Lagrangian based metric theory with matter action being independent from the affine connection (Lee theorem).

In the context of the PPN formalism, the results obtained here directly show that the PPN parameters $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and α_3 are all zero in Palatini $f(R)$ gravity.

Equations of motion

Now, we split the fluid description of the source into N separated bodies.

$$m_A = \int_A \rho^* d^3x. \tag{28}$$

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The center-of-mass of a body A, its velocity and acceleration is then defined as

$$\boldsymbol{r}_A(t) \equiv \frac{1}{m_A}\int \rho^* \boldsymbol{x}\, d^3x, \quad \boldsymbol{v}_A(t) \equiv \frac{d\boldsymbol{r}_A}{dt} = \frac{1}{m_A}\int \rho^* \boldsymbol{v}\, d^3x, \tag{29}$$

$$\boldsymbol{a}_A(t) \equiv \frac{d\boldsymbol{v}_A}{dt} = \frac{1}{m_A}\int \rho^* \frac{d\boldsymbol{v}}{dt}\, d^3x. \tag{30}$$

In the last equation above we use the PN-Euler equation inside the integral.

Equations of motion

Skipping the mathematical details, the final result is

$$\boldsymbol{a}_A = -\sum_{B\neq A}\frac{M_B}{r_{AB}^2}\boldsymbol{n}_{AB} - \sum_{B\neq A}\frac{M_B}{r_{AB}^2}\Big\{\Big[v_A^2 - 4(\boldsymbol{v}_A\cdot\boldsymbol{v}_B) + 2v_B^2 - \frac{3}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)^2 -$$

$$-\frac{5M_A}{r_{AB}} - \frac{4M_B}{r_{AB}}\Big]\boldsymbol{n}_{AB} - \big[\boldsymbol{n}_{AB}\cdot(4\boldsymbol{v}_A - 3\boldsymbol{v}_B)\big](\boldsymbol{v}_A - \boldsymbol{v}_B) +$$

$$+\frac{7}{2}\sum_{C\neq A,B} M_C\frac{r_{AB}}{r_{BC}^2}\boldsymbol{n}_{BC} - \sum_{C\neq A,B} M_C\left[\frac{4}{r_{AC}} + \frac{1}{r_{BC}} - \frac{r_{AB}}{2r_{BC}^2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC})\right]\boldsymbol{n}_{AB}\Big\},$$

These equations have no explicit dependence on either a_2 or a_3 and they are identical, in form, to the corresponding GR expression.

The above equation of motion can be directly compared to the correspondent equation in the PPN formalism to conclude that $\beta = 1$ and $\alpha_1 = \alpha_2 = \xi = 0$.

Equations of motion

There is a single implicit difference, it is inside the constant M_B, which is the mass-energy associated with the planet indexed with B and contains a correction dependent on a_2

$$M = m + E, \quad E \equiv \int \left(\frac{1}{2}\rho^* v^2 - \frac{1}{2}\rho^* U + \rho^* \Pi - \frac{\lambda}{2}\rho^{*2} \right) d^3x.$$

Thus, GR and Palatini $f(R)$ are not identical, but their differences are insensitive to solar system tests described by the PPN formalism.

Final comments

Although the PN limit of Palatini $f(R)$ gravity has corrections in the metric it does not present corrections in the PN equations of motions.

The only correction is present in the way masses are calculated and this is not covered by Solar System tests.

From the point of view of PPN formalism, the new parameter introduced by Palatini $f(R)$ gravity has no influence on the traditional ones and cannot be constraint by the Solar System tests. Thus, the PPN parameters of Palatini gravity are the same as in GR.

More details and laboratory tests in arXiv: 1912.12234/gr-qc

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