Gravitational Waves from Coalescing Binaries

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Outline

1 Introduction to GW: sources and observations
- General Relativity
- Astrophysics signals

2 GW detector and sensitivity
- The observatories
- Data Analysis

3 The physics of GW production
- How to compute waveforms
- Bonus slides: How to test gravity

4 Bonus slides: Cosmology with GW's

Gravitational Wave astronomy

Gravitational waves are produced by the coherent motion of large astrophysical masses $> M_\odot$

- neutron stars (e.g. pulsars)
- stellar mass black holes (e.g. binary x-ray sources)
- supermassive black holes (e.g. in galaxy centers)

or by cosmological production mechanisms

- cosmic strings
- inflation
- phase transitions
- ...

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Einstein equations and the TT-gauge

Weak field approximation, approximatly Cartesian coor.:

$$g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}, \qquad ||h_{\mu\nu}|| \ll 1$$

A gravity wave is the radiative, high frequency part of $h_{\mu\nu} \supset$

1. 4 gauge degrees of freedom
2. 2 physical, radiative degrees of freedom
3. 4 physical, non-radiative degrees of freedom

1&3 propagate with "the speed of thought" (Eddington 1922):
After fixing the diffeomorphism invariance:

$$h_{\mu\nu} = \begin{pmatrix} -2\Phi & \Xi_i \\ \Xi_i & h^{TT}_{ij} + \theta\delta_{ij} \end{pmatrix}$$

$\partial_i \Xi^i = h^{TT}_{ij}\delta^{ij} = \partial^i h_{ij} = 0$: 4 d.o.f.'s eaten by gauge fixing, 6 left

$$\begin{aligned} \text{Einstein eq.'s}: \quad \nabla^2\Phi = \nabla^2\Xi_i = \nabla^2\Theta &= 0 \\ \Box h^{TT}_{ij} &= 0 \end{aligned}$$

Wave generation: localized sources

Einstein formula relates h_{ij} to the source quadrupole moment Q_{ij}

$$\begin{aligned} Q_{ij} &= \int d^3x\rho\left(x_i x_j - \frac{1}{3}\delta_{ij}x^2\right) \qquad v^2 \simeq G_N M/r \\ h_{ij} &= \frac{2G_N}{r}\frac{d^2Q_{ij}}{dt^2} \simeq \frac{2G_N\mu v^2}{r}\cos(2\phi(t)) \end{aligned}$$

$$\begin{aligned} f &= 1\mathrm{kHz}\left(\frac{r}{14\mathrm{Km}}\right)^{-3/2}\left(\frac{M}{m_\odot}\right)^{1/2} \\ v &= 0.3\left(\frac{f}{1\mathrm{kHz}}\right)^{1/3}\left(\frac{m}{m_\odot}\right)^{1/3} < \frac{1}{\sqrt{6}} \end{aligned}$$

$$\frac{dE}{dt} = \frac{r^2}{16\pi G_N}\int d\Omega\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle = \frac{32\eta^2}{5G_N}v^{10} \qquad \eta = \mu/M$$

Do GWs exist? The Hulse-Taylor binary pulsar

GW's have been observed in the NS-NS binary system:

PSR B1913+16

Observation of orbital parameters $(a_p \sin\iota,\ e,\ P,\ \dot{\theta},\ \gamma,\ \dot{P})$

$\downarrow$

determination of m_p, m_c (1PN physics, GR)

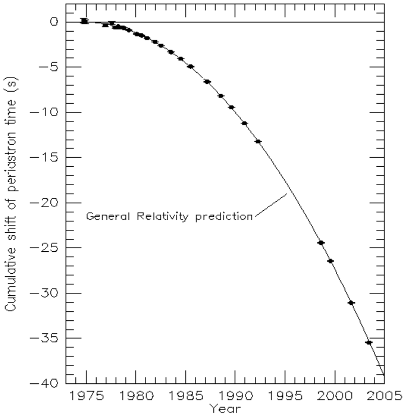
Energy dissipation in GW's $\rightarrow \dot{P}^{(GR)}(m_p, m_c, P, e)$, compared with $\dot{P}^{(obs)}$

$$\frac{1}{2\pi}\phi = \int_0^T \frac{1}{P(t)}\, dt \simeq \frac{T}{P_0} - \frac{\dot{P}_0}{P_0^2}\frac{T^2}{2}$$

Test of the

- 1PN conservative
- leading order dissipative dynamics

Weisberg and Taylor (2004)



$$\frac{\dot{P}_{GR}-\dot{P}_{exp}}{\dot{P}} \sim 10^{-3}$$

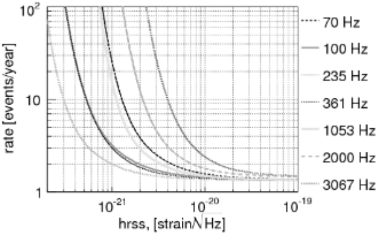
10 pulsars in NS-NS, still ~ 100Myr for coalescence

Burst signals

- Supernovae and collapsing stars, typical amplitude

$$h \sim 6 \cdot 10^{-21} \left(\frac{E}{10^{-7} m_{\odot}} \right) \left(\frac{1\mathrm{msec}}{T} \right)^{1/2} \left(\frac{1\mathrm{kHz}}{f} \right) \left(\frac{10\mathrm{kpc}}{r} \right)$$

Limits for signals centered at different freq's



1.7 yr of data, LIGO/Virgo PRD85 (2012)

Asymmetric rotating neutron star

- Pulsar asymmetry $\epsilon \to h \sim \epsilon (Rf)^2 M/r$

$$L_{GW} = f^6 M^2 R^4 \epsilon^2 \to t_{SD} \sim \frac{M(Rf)^2}{L_{GW}}$$

  If all the spin-down is due to GW emission

$$\begin{array}{lll} \epsilon \simeq 7 \cdot 10^{-3} & \epsilon_{GW} < 1.4 \cdot 10^{-4} & \text{Crab} \\ \epsilon \simeq 1.2 \cdot 10^{-3} & \epsilon_{GW} < 5 \cdot 10^{-4} & \text{Vela} \end{array}$$

  Beating the spin-down limit!

  LIGO/Virgo APJ 2010

- Best upper limit $h_{ul} \sim 10^{-24}$ @ 150Hz

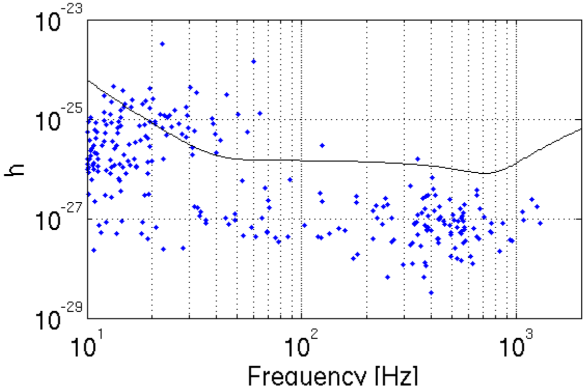
  LIGO/Virgo APJ 2010, 2011, PRD 2012

- Upper limit on GW's emitted by Vela pulsar glitch
  $h_{ul} < 10^{-20}$ $E_{GW} < 10^{45}$ erg

  LIGO PRD 2011

Future detectors (2017+)

Integrated sensitivity (T_{obs}=1yr) for known pulsars vs. spin-down limit



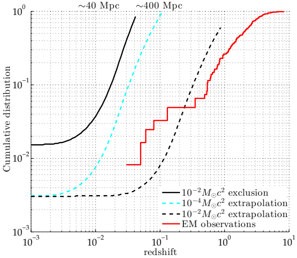
C. Palomba

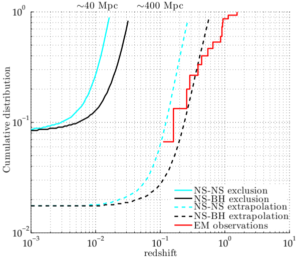
Coincidence with Gamma Ray Bursts

Search in a $-600 \div +60$ sec window around the GRB event

- 150 analyzed via an un-modeled burst search excluded within $D_{burst} \sim 17$ Mpc$(E_{GW}/10^{-2} M_\odot)^{1/2}$ for signals at 150 Hz
- 24 analyzed via matched filtering with coalescing binary signals
  $D_{NS-NS} \sim 17$ Mpc $\quad M_{NS} \sim 1.4 \pm 0.2$
  $D_{BH-NS} \sim 29$ Mpc $\quad M_{NS} \sim 1.4 \pm 0.4 \quad M_{BH} \sim 10 \pm 6$

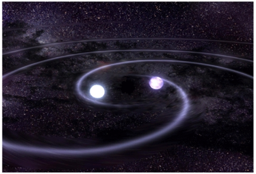
Distance cumulative distibutions:

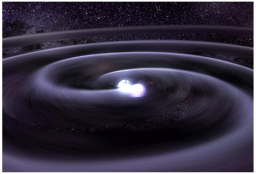


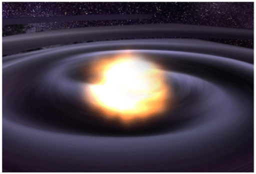


Looks promising for Adv-LIGO/Virgo (LIGO/Virgo et al. APJ (2012))

Binary coalescence: a tale made of three stories



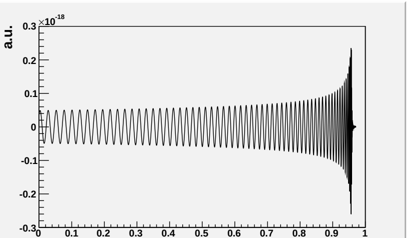


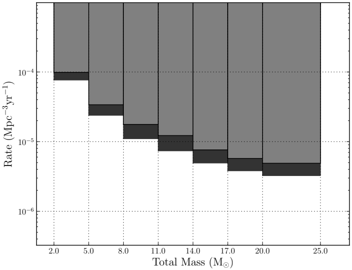


Inspiral phase post-Newtonian approximation: v/c

Merger: fully non-perturbative

Ring-down: Perturbed Kerr Black Hole

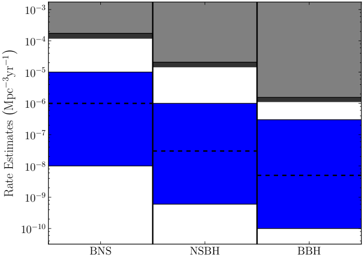




Low mass ($< 25M_\odot$) binary (inspiral)
Upper limit from previous runs
improved by the 7/2009-10/2010 run

Combined upper limit from old and new runs
compared with
astrophysical estimates

LIGO/Virgo PRD 2012



Upper limits for high mass systems

- For high masses $25 < M/M_\odot < 100$ also merger and ring-down are in band and necessary to have complete analytic description of coalescence waveform
  Observative bound on coalescence rate of equal mass system with $19 < (m_1, m_2)/M_\odot < 28$
  $R_c < 0.3$ Mpc^{-3} Myr^{-1}

  LIGO/Virgo PRD 2011

- For higher masses $100 < M/M_\odot < 500$ signals are burst-like:
  Best upper limit for equal mass system with $M \sim 170 M_\odot$:
  Merger Rate $R_m < 0.13$ Mpc^{-3} Myr^{-1}

  LIGO/Virgo PRD 2012

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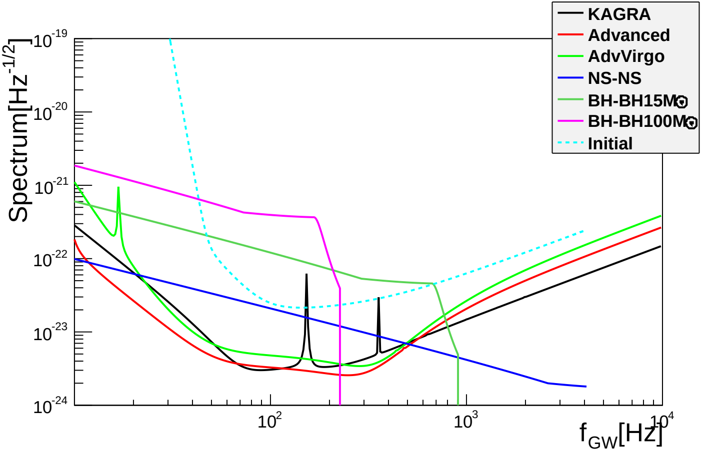
Detector locations

All are now being upgraded to their Advanced version due to to start data taking in 2015

2017+ for design sensitivity

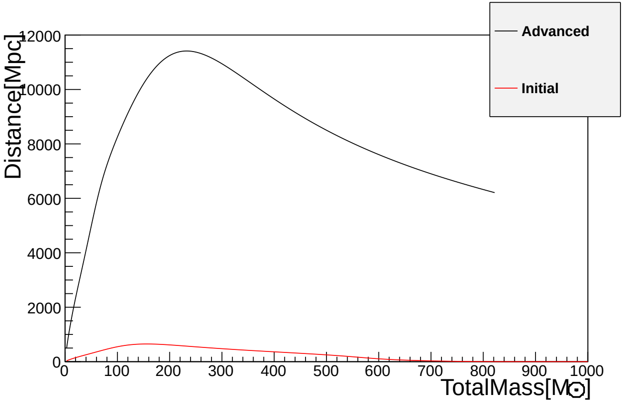
Old coincident runs terminated in October 2010

Advanced detectors



Sensitivity vs. signals @ 200Mpc w. optimal orientation

Distance reach for compact binary coalescence



Observational rate estimates

LIGO/Virgo Advanced Observatories will detect

($SNR = 8$, optimal orientation)

	NS-NS	10 $M_\odot$ BH-BH
Distance (Mpc)	$450\mathrm{Mpc}$	$1\mathrm{GPc}$
Rates $\mathrm{MWEG}^{-1}\mathrm{Myear}^{-1}$	$1 \div 10^3$	$4 \cdot 10^{-2} \div 100$

$$N = 0.011 \times \frac{4}{3}\pi \left(\frac{D_H/\mathrm{Mpc}}{2.26} \right)^3 \mathrm{MWEG}$$

Realistic case:

$$R_{NS-NS} \sim 10\mathrm{yr}^{-1} \qquad R_{BH-BH} \sim 10^2\mathrm{yr}^{-1}$$

for LIGO/Virgo at design sensitivity

LIGO/Virgo CQG 2010

Advanced LIGO/Virgo goals

- Make first "direct" detection of GWs from neutron stars and/or black holes
- Probe intermediate mass black hole range: $M \sim 100 M_{\odot}$
- Measure rate of binary coalescences
- Measure pulsar parameters
- Possible probe of neutron star interior/nuclear matter at high density
- Verify association between short GRB's and GW's
- Combine EM and GW detection
- Make strong tests of GR
- Use coalescing binaries as standard sirens for cosmology

Data analysis technique: Matched filtering

An experimental apparatus output: time series

$$O(t) = h(t) + n(t) \qquad h(t) = D^{ij} h_{ij}(t)$$

Noise is conveniently characterized by its spectral function

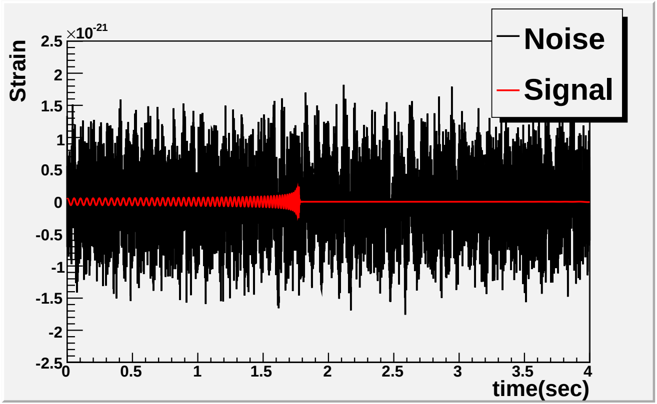
$$\langle \tilde{n}(f) \tilde{n}^*(f') \rangle = \delta(f - f') S_n(f) \qquad [Hz^{-1}]$$

Matched filter enhances the sensitivity

$$\frac{1}{T}\int_0^T O(t)h(t)\,dt = \frac{1}{T}\int_0^T h^2(t)\,dt + \frac{1}{T}\int_0^T n(t)h(t)\,dt \sim h_0^2 + \sqrt{\frac{\tau_0}{T}} n_0 h_0$$

Hunting for tiny signals

Detector's output is flooded with noise:

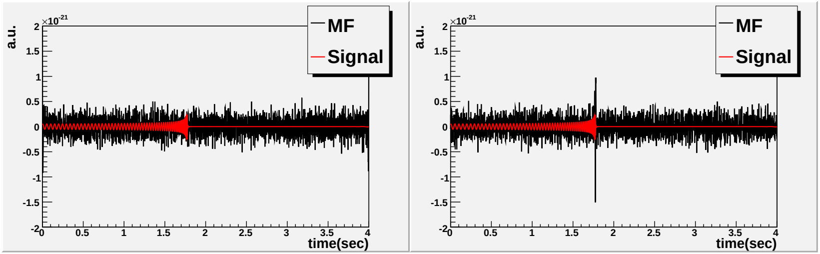


Noise + GW signal from 2+12 $M_{\odot}$ system at 50 Mpc distance

Matched filtering

Matched filtering enhances sensitivity:

$$O(t) \rightarrow MF(t) \propto \int \frac{O(f)h^*(f)}{S_n(f)} e^{2\pi i f t} df$$



MF with $h' \neq h$ MF with h

but requires good model of the signal h

or a complete bank of $h's$

GW detection

Inspiral $h = A\cos(\phi(t))$ $\qquad \frac{\dot{A}}{A} \ll \dot{\phi}$
Virial relation:

$$v \equiv (G_N M \pi f_{GW})^{1/3} \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$\begin{aligned} E(v) &= -\frac{1}{2}\nu M v^2 \left(1 + \#(\nu)v^2 + \#(\nu)v^4 + \ldots\right) \\ P(v) \equiv -\frac{dE}{dt} &= \frac{32}{5G_N} v^{10} \left(1 + \#(\nu)v^2 + \#(\nu)v^3 + \ldots\right) \end{aligned}$$

E(v)(P(v)) known up to 3(3.5)PN

$$\begin{aligned} \frac{1}{2\pi}\phi(T) &= \frac{1}{2\pi}\int^T \omega(t)dt = -\int^{v(T)} \frac{\omega(v)dE/dv}{P(v)} dv \\ &\sim \int \left(1 + \#(\nu)v^2 + \ldots + \#(\nu)v^6 + \ldots\right) \frac{dv}{v^6} \end{aligned}$$

GW detection

$$N_{cycles} \simeq 1.6 \cdot 10^4 \left(\frac{10\text{Hz}}{f_{min}}\right)^{5/3} \left(\frac{1.2M_\odot}{M_c}\right)^{5/3}$$

Sensitivity $\propto M_c^{5/3}\sqrt{N_{cycles}} \propto M_c^{5/6}$
$f_{Max} \propto M^{-1}$, $M_c \equiv \eta^{3/5}(m_1 + m_2)$

Important to know the phase at $O(1)$ when taking correlation of detector's output and model waveform

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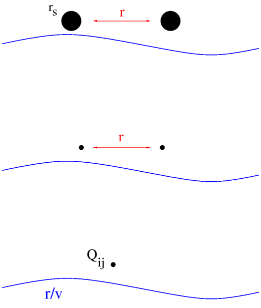
The EFT point of view on the 2-body problem: how to compute waveform

Different scales in EFT:

- Very short distance $\lesssim r_s$ negligible up to 5PN (effacement principle)
- Short distance: orbital scale $r \sim r_s/v^2$, potential gravitons $k_\mu \sim (v/r, 1/r)$
- Long distance: GW's $\lambda \sim r/v \sim r_s/v^3$ $k_\mu \sim (v/r, v/r)$ coupled to point particles with moments

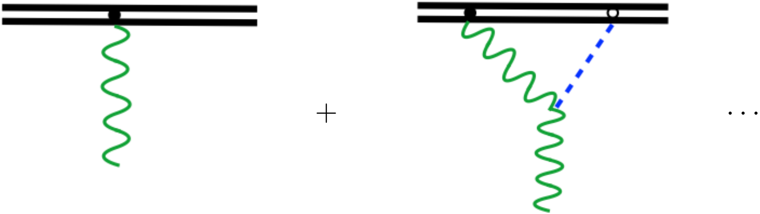


Theory at large distances

Extended object coupled to gravity via multipoles

$$S_{ext} \int dt\, M + L_{ab}\omega_0^{ab} + Q_{ij}E_{ij} + J_{ij}B_{ij} + O_{ijk}\partial_k E_{ij} + \dots$$

Emission amplitude $A_h(k) = \sqrt{G_N} k^2 \epsilon_{ij} Q_{ij} \to P \propto |A_h|^2$



In order to make prediction we need to know what are the multipoles:
Need to match with the theory at the orbital scale:

$$Q_{ij} = \int \ddot{T}_{00} x_i x_j$$

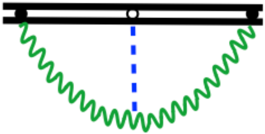
Divergences

Theory of extended object has UV divergencies: assuming source concentrated at $r = 0$ gives divergencies, in the amplitude

$$\left|\frac{A_{h,v^6}}{A_{h,v^0}}\right|^2 = (G_N M\omega)^2 \left(\frac{1}{d-3} + \log\frac{k^2}{\mu^2} + \ldots\right)$$

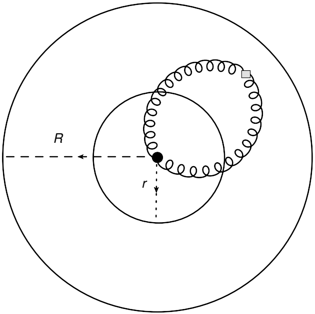
in the radiation-reaction

$$\begin{aligned}\delta\ddot{x}_{ai}(t) &= -\frac{8}{5} x_{aj}(t) G_N^2 M \\ &\int_{-\infty}^{t} dt'\, Q_{ij}^{(7)}(t') \log\frac{t-t'}{\mu}\end{aligned}$$



Log terms give running

$$\mu\frac{dM}{d\mu} = -\frac{214}{105}(G_N m\omega)^2 I_{ij}^{(R)}$$
$$\mu\frac{dI_{ij}}{d\mu} = \frac{2(G_N M\omega)^2}{5}\left(2I_{ij}^{(5)}I_{ij}^{(1)} - 2Q_{ij}^{(4)}I_{ij}^{(2)} + I_{ij}^{(3)}I_{ij}^{(3)}\right)$$



Observer at different distance disagree on the value of M, Q_{ij}

Goldberger, Ross PRD 2010
Goldberger, Rothstein, Ross 1211.6095

Foffa & RS PRD 2012

$$P \propto G_N \int \omega^6 \left(Q_{ij}^2 + \frac{16}{45}J_{ij}^2 + \frac{5}{189}\omega^2 I_{ijk}^2 + a(G_N M\omega)Q_{ij}^2 + b(G_N M\omega)^2 Q_{ij}^2 + \ldots\right)$$

Conservative dynamics

To compute emitted flux (and the energy) the theory at orbital scale is needed: heavy, classical sources coupled to gravitons

$$\exp\left[iS_{eff}(x_a)\right] = \int \mathcal{D}h(x) \exp\left[iS_{EH}(h) + iS_{pp}(h, x_a)\right]$$

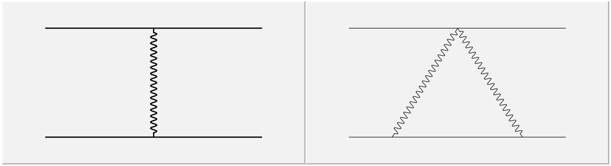
$$\begin{aligned} S_{pp} &= -\sqrt{G_N} m \int dt \left(h_{00}/2 + v_i h_{0i} + v^i v^j h_{ij}/2 + \ldots\right) \\ S_{EH} &= \int d^4x \left[(\partial_i h)^2 - (\partial_t h)^2 + \sqrt{G_N} h (\partial h)^2 + \ldots\right] \end{aligned}$$

Power counting to integrate out potential gravitons in the NR limit

$$\begin{aligned} \int d^{3+1}k \frac{e^{ik^\mu x_\mu}}{k^2} &\sim \int d^3k\, \delta(t) \frac{1}{\mathbf{k}^2}\left(1 - \frac{\partial_t^2}{\mathbf{k}^2} + \ldots\right) \\ &\sim \int d^3k\, \delta(t) \frac{e^{i\mathbf{kx}}}{\mathbf{k}^2}\left(1 + \frac{(\mathbf{kv})^2}{\mathbf{k}^2} + \ldots\right) \end{aligned}$$

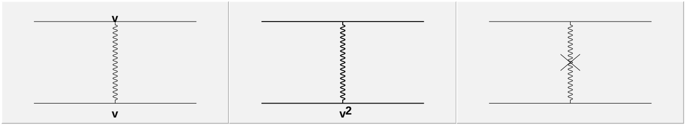
Manifest scaling

The 1PN potential



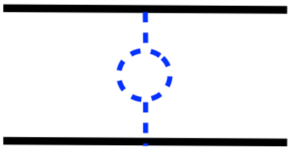
Scaling: L Lv^2

Using virial theorem $v^2 \sim G_N M/r$



$$V = -\frac{Gm_1m_2}{2r}\left[1 - \frac{G_N m_1}{2r} + \frac{3}{2}(v_1^2) - \frac{7}{2}v_1 v_2 - \frac{1}{2}v_1\hat{r}v_2\hat{r}\right] + 1 \leftrightarrow 2$$

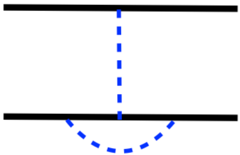
Divergences and quantum effects



Graviton loops are quantum corrections suppressed by $\hbar/L \sim 10^{-76}(M/M_\odot)^{-2}(v/0.1)$

Harmless power law divergence: infinite mass renormalization absorbed in the bare coupling

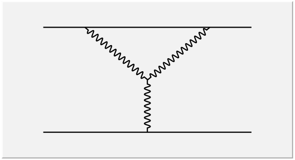
$$\propto \int d^d k \frac{1}{k^2}$$



EFT does not predicting all physical paramaters, some are inputs
log divergencies calls for UV completion of the theory

The 3PN computation automatized

Topologies



Graphs — v and time derivative-insertions

Amplitudes

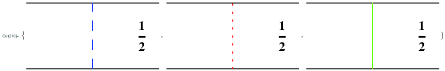
$$A = G_N m_i v_i \int d^d k \, d^d k_1 \frac{1}{k^2 (k-k_1)^2} \dots$$

Evaluation — Analytic integral in a database

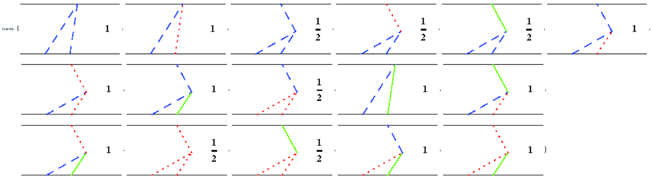
S. Foffa & RS PRD 2011

Feynman diagrams at 3PN order

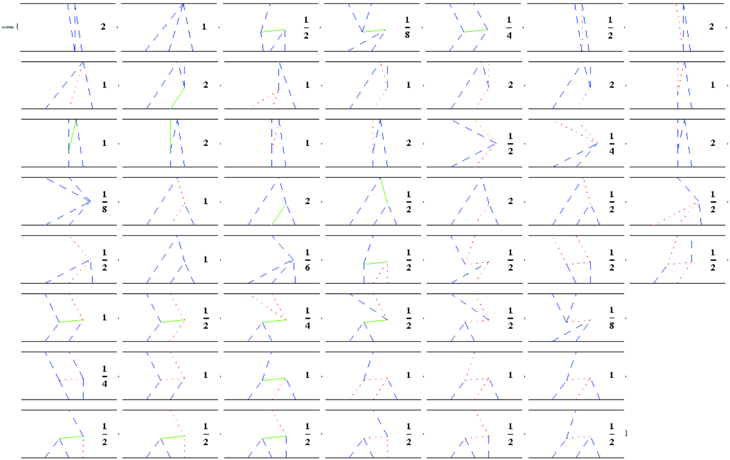
$G_N v^6$



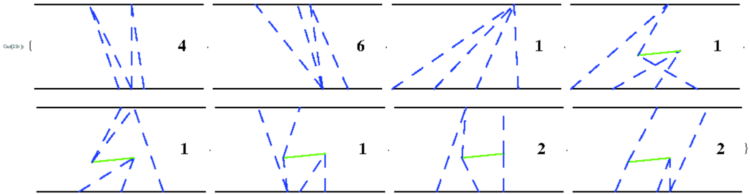
$G_N^2 v^4$



Feynman diagrams at 3PN order: $G_N^3 v^2$



Feynman diagrams at 3PN order: G_N^4



Final result matches previous derivation of 3PN Hamiltonian, see eq. (174) of Blanchet's Living Review on Relativity

The 4 PN status

Recovered the 3PN Hamiltonian

Blanchet et al. PRD '03

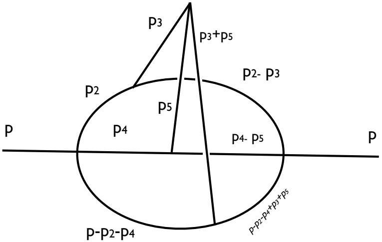
At 4PN we have to compute:

- 3 graphs @ $G_N v^8$ order
- 23 @ $G_N^2 v^6$
- 202 @ $G_N^3 v^4$
- 307 @ $G_N^4 v^2$
- 50 @ G_N^5

Foffa & RS in preparation

The 4 PN on the way

The major obstruction to the computation is one of the G_N^5 graphs equivalent to the 4-loop quantum field theory diagram



in a massless, euclidean theory in $d = 3$
Result at G_N^5 obtained with traditional method (different gauge)

Jaranowski, Schäfer PRD 2013

Testing GR

$$\phi(t) = v^{-5} \sum_{n=0}^{7} \left(\phi_n + \phi_n^{(l)} \log(v) \right) v^n$$

Bayesian model seection between two hypothesis

- $\mathcal{H}_{GR}$
- $\mathcal{H}_{modGR}$: one or more ϕ_i's are not as predicted by GR

Odds ratio:

$$\mathcal{O}_{GR}^{modGR} \equiv \frac{P\left(\mathcal{H}_{modGR} | d, I\right)}{P\left(\mathcal{H}_{GR} | d, I\right)}$$

In absence of noise $\mathcal{O}_{GR}^{modGR} \gtrless 0$ favours $\begin{matrix}\mathbf{modGR} \\ GR\end{matrix}$

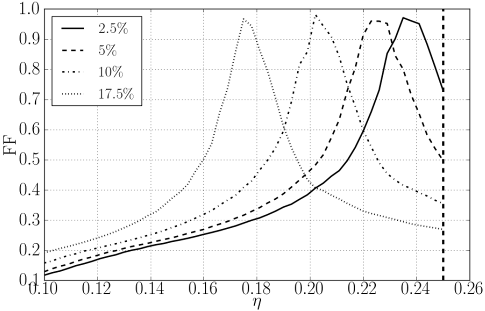
Li, RS et al. JPCS (2012, PRD (2012)

Parameter estimation bias

Waveform match (fitting factor)

$$FF = \int df \frac{h_1^*(f) h_2(f) + h_1(f) h_2^*(f)}{S_n(f)}$$

for $\delta\chi_3$ injections varying η (maximized over other params)

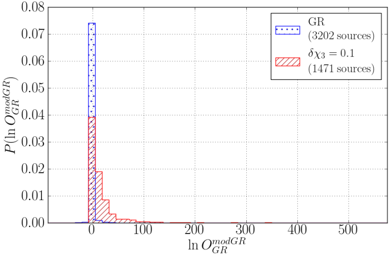


GR deviation do not prevent detection, but considerable bias!

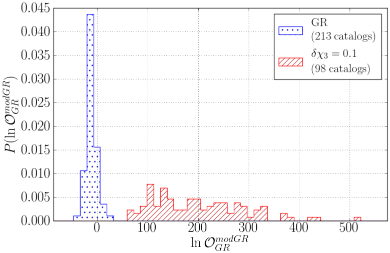
Simulated signals in noise

Constant shift in $\phi_3 \to \phi_3(1 + \delta\chi_3)$ SNR's limited to 8-25

Single sources with noise:
Odds ratio overlaps



15-sources catalogs can disentangle fundamental effects



Future directions for testing GR

- Are waveforms accurate enough? In early inspiral yes
- For neutron stars: are finite size and matter effects important? Mostly after 450Hz (see Hinderer et al. PRD (2010))
- Is the effect of spin important?
  Not for neutron stars, but for BH it has to be considered
- Are internal calibration errors under control? Yes

  Vitale et al. PRD85 (2012)
- What about inclusion of merger and ring-down?
  Work in progress
- Can the computational challenge be satisfied? Maybe yes...

New era is about to start ($\sim$ 4 years)

- GW astronomy will open a new window onto the Universe, both astrophysically and cosmologically
- Observational test of strong gravity field dynamics not available so far: GW detection will give access to strong gravity phenomena
- Field theory method tuned out to be useful in classical gravity

Spare slides

Outline

1 Introduction to GW: sources and observations
- General Relativity
- Astrophysics signals

2 GW detector and sensitivity
- The observatories
- Data Analysis

3 The physics of GW production
- How to compute waveforms
- Bonus slides: How to test gravity

4 Bonus slides: Cosmology with GW's

Measuring H_0

Coalescing binary systems are standard sirens:

$$h(t) = \frac{G_N \eta M^{5/3} f_s^{2/3}}{D} \cos\left[\phi(t)\right]$$

In cosmological settings source and observer clocks tick differently:

$$dt_o = (1+z)dt_s \qquad f_o(1+z) = f_s$$

$$h(t_o) = \frac{G_N \eta f_o^{2/3} M^{5/3} (1+z)^{2/3}}{a(t_o) D} \qquad \cos\left[\phi(t_s(t_o))\right]$$

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$$M\frac{d\phi(t_s/M)}{dt_s} = (1+z)M\frac{d\phi\left((1+z)t_s(t_o)/M(1+z)\right)}{dt_o} \implies$$

$$\phi\left(t_o/\mathcal{M}\right) = \phi\left(t_s/M\right) \qquad \mathcal{M} \equiv M(1+z)$$

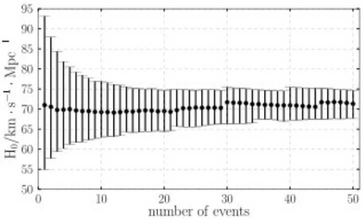
Determining H_0

Hubble law: $z = H_0 d_L$

D_L can be measured, z degenerate with M, however if

- the source in the sky has been localized (α, δ)
- GW sources are in the galaxy catalog with known red-shift

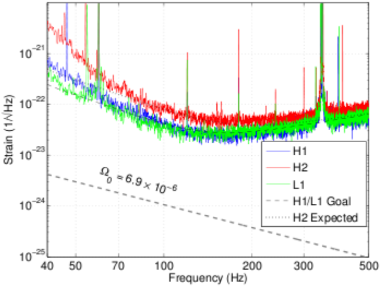
$$P(z, D_L|c_i) = \int d\mathcal{M}\, d\vec{\theta}\, d\alpha\, d\delta\, P(D_L\mathcal{M}, \vec{\theta}, \alpha, \delta|c_i)\pi(z, |\alpha, \delta)$$



Schutz, Nature '86
W. Del Pozzo, arXiv:1108.1317

Stochastic background

$$\Omega_{GW} = \frac{f}{\rho_c}\frac{d\rho_{GW}}{df}$$
$$S_{GW} = \frac{3H_0^2}{10\pi^2}\frac{\Omega_{GW}}{f^3}$$



Between 50 and 150 Hz