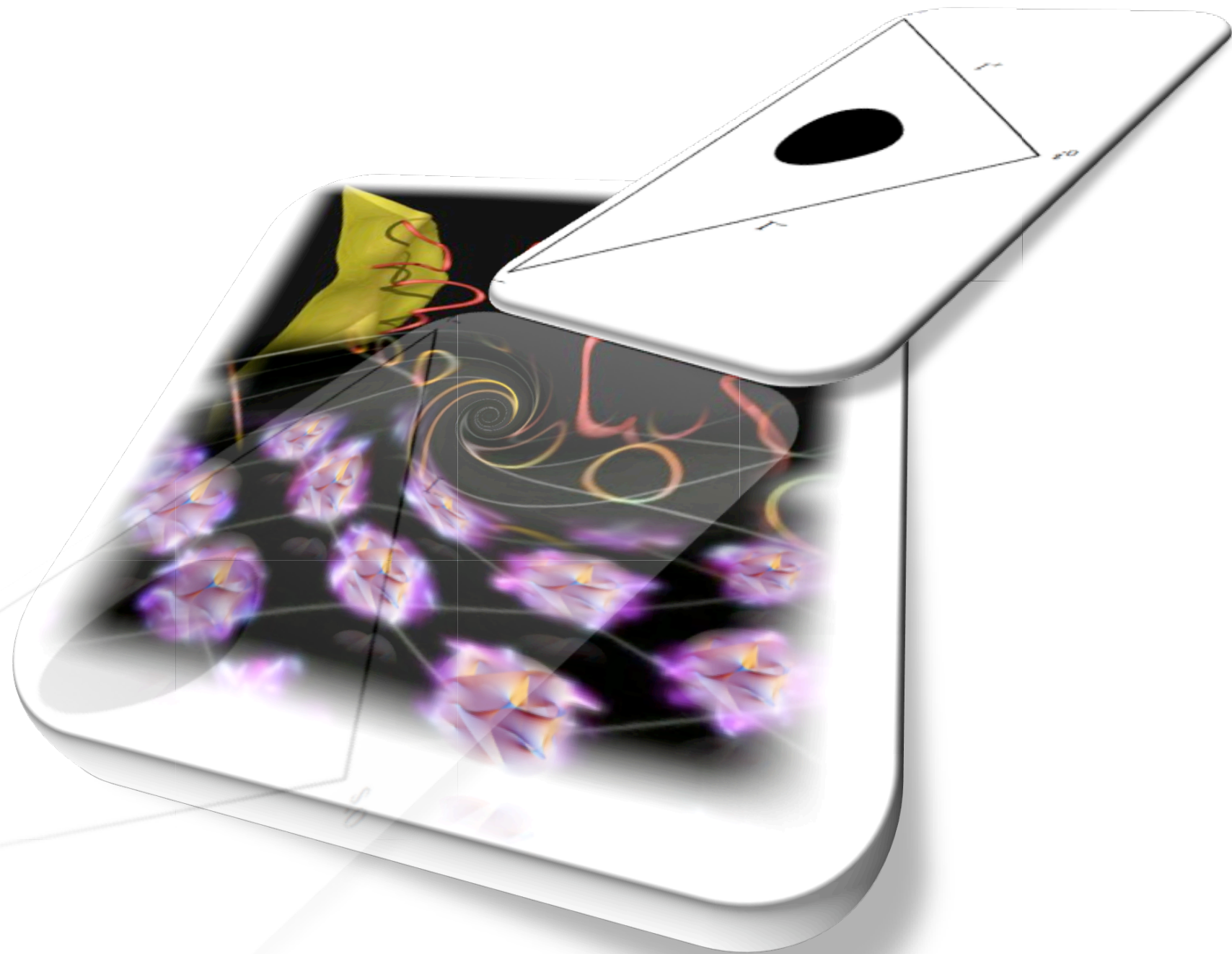


Super-renormalizable Quantum Gravity



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Outline

- Super-renormalizable Gravitational Theories.
 - Unitarity (no ghosts).
 - Renormalizability (Dimensional Regularization)
- Finite gravitational theories,
- Finite gauge theories,
- Towards a finite theory of particle physics.

- Exact solutions and spacetime singularities :
 - Ricci flat, (A)dS, FRW spacetimes.
 - Singularity theorem in nonlocal gravity.

- Scattering amplitudes.

- Conformal invariant quantum gravity.
- Nonsingular spacetimes in Conformal gravity.



Gravity

Einstein Gravity

$$S_{\text{EH}} = 2\kappa_4^{-2} \int d^D x \sqrt{-g} R(g) .$$



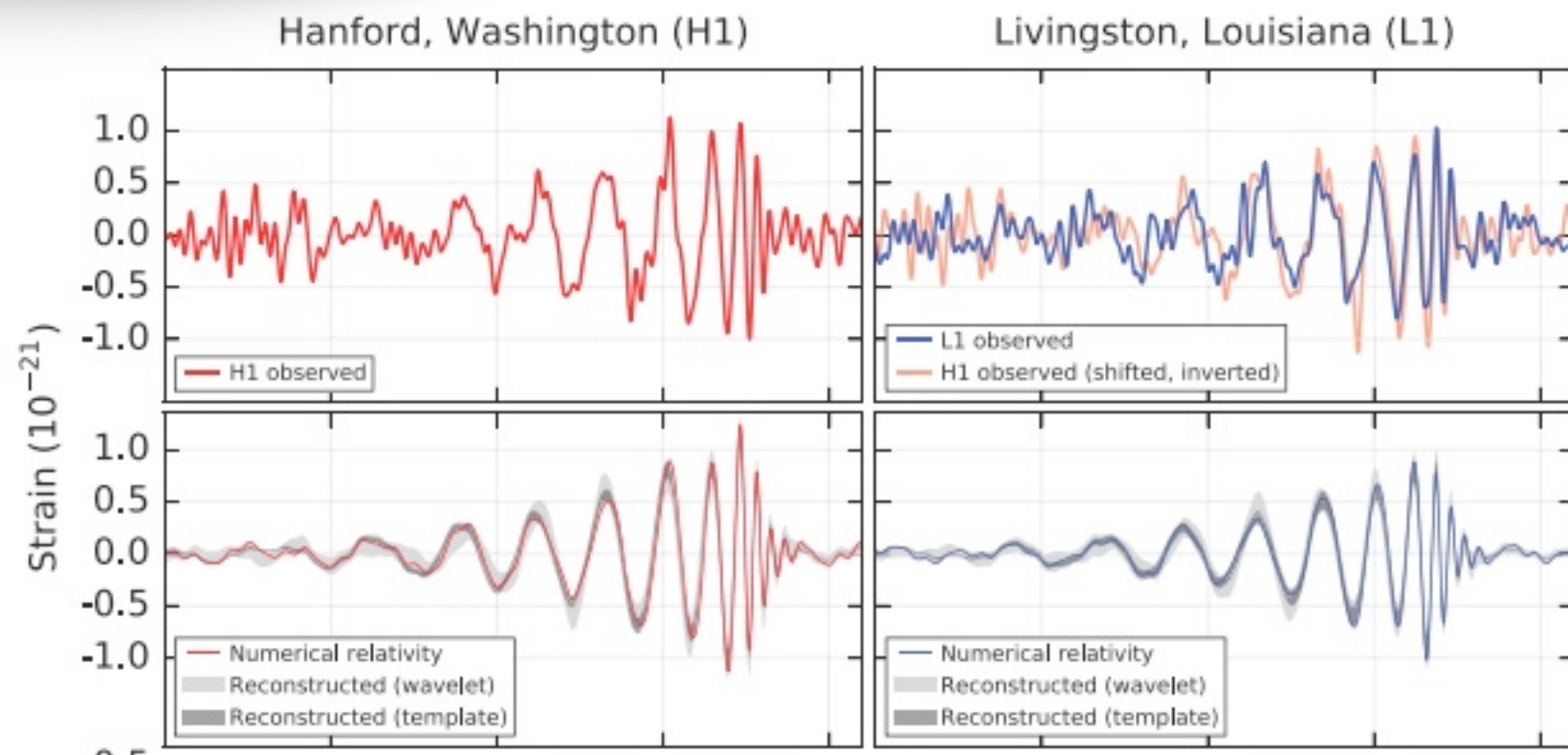
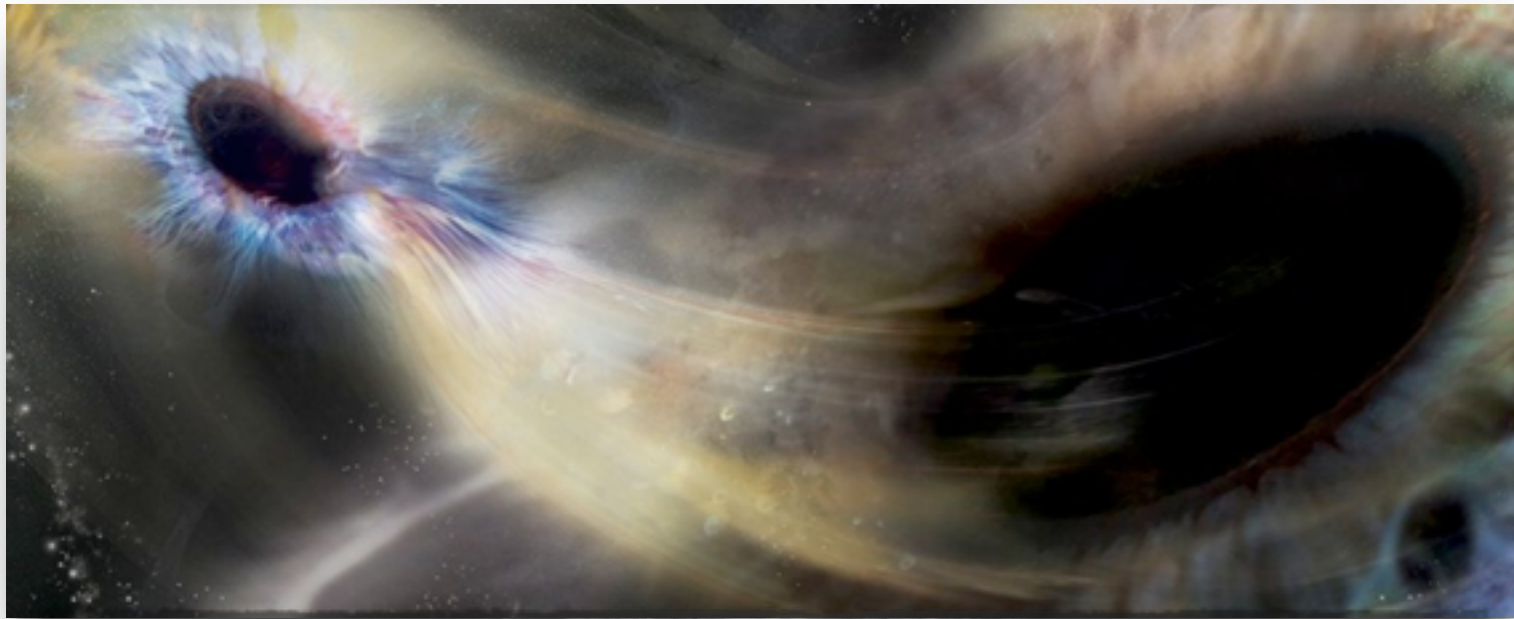
Quadratic Starobinsky Gravity

$$S_{\text{S}} = 2\kappa_4^{-2} \int d^D x \sqrt{-g} (R + \epsilon R^2) .$$



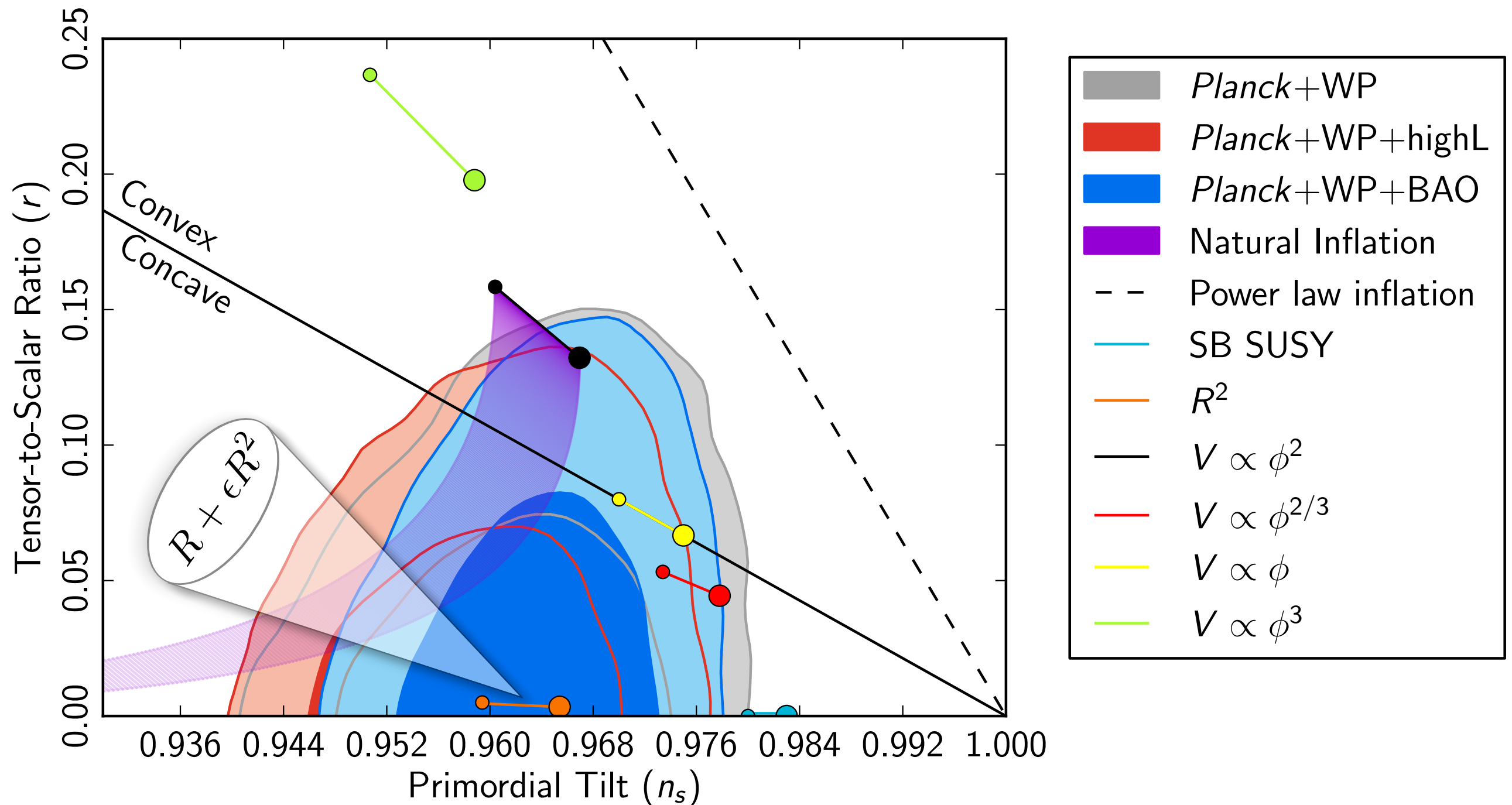
Gravitational Waves

$$\square \mathbf{h} = 8\pi \mathbf{T} \quad (\square \mathbf{A} = \mathbf{J}).$$

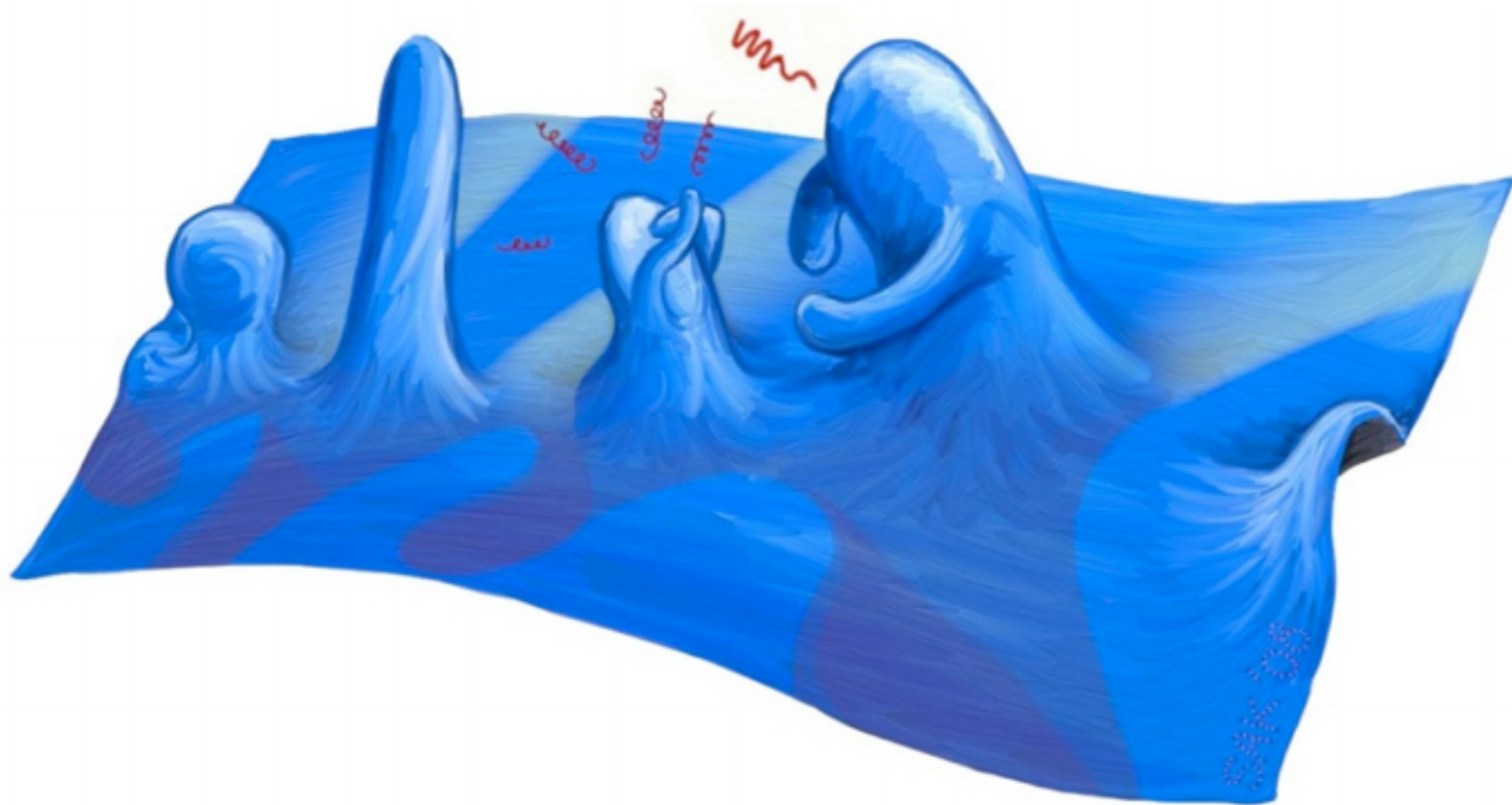


Planck Results

$$S_S = 2\kappa_4^{-2} \int d^D x \sqrt{-g} (R + \epsilon R^2) .$$



Quantum Field Theory of Gravity



Empirical Evidences

- General Covariance,
- Perturbative Quantum Field Theory (gravitons observed),
- Higher Derivative Theory (Starobinsky Inflation more reliable).

+

(Super-)renormalizability in Quantum Field Theory.

Nonlocal or Lee-Wick Gravity.

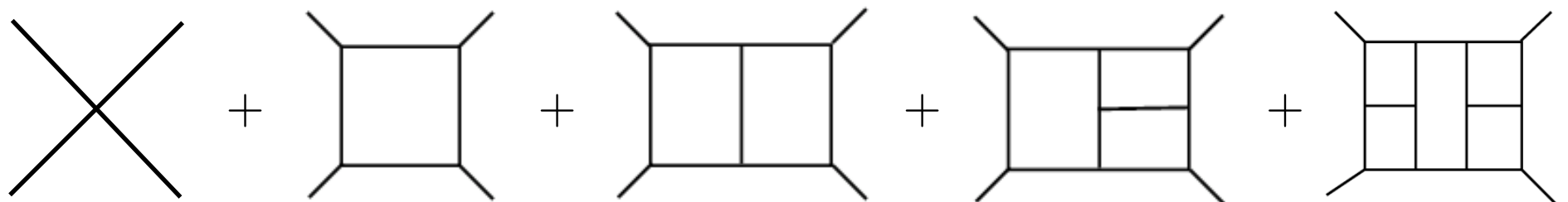
Einstein Gravity

$$\kappa^{-2} R$$

Superficial degree of divergence of a Feynman diagram

$$S_{\text{E-H}} = -2\kappa_D^{-2} \int d^4x \sqrt{|g|} R(g), \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa_D h_{\mu\nu}.$$

Expansion in gravitons : $L_{EH} = \partial h \partial h + \kappa h \partial h \partial h + \kappa^2 h^2 \partial h \partial h + \dots + \kappa^n h^n \partial h \partial h + \dots$

$$S_{fi}^{(4\text{-gravitons})} =$$


L : number of loops,

V : number of vertices,

I : number of internal lines in the graph.

Topological relation : $L = 1 + I - V$,

$$\text{DIVERGENCE} \propto \int (d^D p)^L \left(\frac{1}{p^2} \right)^I p^{2V},$$

$$\delta = L D + 2V - 2I = 2 + (D - 2)L,$$

$$D = 4 \rightarrow \delta = 2 + 2L.$$

$$\text{Regularization : } S = \int d^D x \sqrt{|g|} \left[2\kappa^{-2} R + \underbrace{\frac{\alpha}{\epsilon} \sum_{m,n} \nabla^n R^m}_{n+2m=2+(D-2)L} \right].$$

Quantum Gravity & Quantum Electrodynamics

Peturbative Quantum Gravity

$$L_G = -\frac{1}{16\pi G_N} \sqrt{-g} R(g)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} ,$$

$$L_G \approx \partial h \partial h + \sum_n \kappa^n h^n \partial h \partial h$$

Quantization of $h_{\mu\nu}$

$$L_{QG} = -\sqrt{-g} \left[\kappa^{-2} R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^3 + \alpha_4 R_{\mu\nu} R^{\mu\rho} R^\nu_\rho + \underbrace{\dots}_{\infty \alpha_i} \right].$$

Peturbative Quantum Electrodynamics

$$L_{ED} = -\frac{1}{4} F^2(A) + L_m(\psi) + e j_\mu(\psi) A^\mu ,$$

$$A_\mu \rightarrow \underbrace{A_\mu^{(0)}}_{=0} + A_\mu ,$$

Quantization of A_μ

$$L_{QED} = -\frac{1}{4} F^2(A') + L_m(\psi') + e' j_\mu(\psi') A'^\mu .$$

Quadratic Gravity

Stelle, Fradkin, Tseytlin, Barvinsky, Vilkovisky, Tomboulis, Antoniadis, de Berredo-Peixoto, Shapiro.

$$\kappa^{-2} R + \alpha R^2 + \beta R_{\mu\nu}^2$$

Higher Derivative Quantum Gravity (Stelle 1977)

$$S = \int d^4x \sqrt{-g} \left[-\frac{R}{\kappa^2} + \frac{2\lambda}{\kappa^2} + \frac{1}{f^2} \left(R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) + \frac{\omega}{3f^2} R^2 \right].$$

Propagator

$$S = \int d^4x \sqrt{-g} \left[-\underbrace{\frac{R}{\kappa^2}}_{\sim p^2} + \underbrace{\frac{1}{f^2} \left(R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) + \frac{\omega}{3f^2} R^2}_{\sim p^4} \right].$$

$$G(p) = \frac{P^{(2)}}{p^2 \left(1 + \frac{\kappa^2}{4f^2} p^2 \right)} - \frac{P^{(0)}}{2p^2 \left(1 - \frac{\omega\kappa^2}{2f^2} p^2 \right)} + \text{gauge terms.}$$

Power Counting Renormalizability

$$\text{DIVERGENCE} \propto \int (d^4p)^L \left(\frac{1}{p^4} \right)^I (p^4)^V = \int (dp)^{4L} \left(\frac{1}{p^4} \right)^{I-V} = \int (dp)^{4L} \left(\frac{1}{p^4} \right)^{L-1},$$

$$\delta \leq 4L - 4(L - 1) = 4. \quad .$$

Regularization & Renormalization

$$\begin{aligned}
 S &\rightarrow \int d^4x \sqrt{-g} \left[-\frac{R}{\kappa^2} + \frac{1}{f^2} \left(R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) + \frac{\omega}{3f^2} R^2 \right] + \frac{1}{32\pi^2 \epsilon} \int d^4x \sqrt{g} \left[\beta_2 \left(R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) + \beta_3 R^2 + \beta_4 R + \beta_5 \right] = \\
 &\equiv \int d^4x \sqrt{-g} \left[-\frac{Z_\kappa}{\kappa^2} R + \frac{Z_f}{f^2} \left(R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) + \frac{Z_f Z_\omega \omega}{3f^2} R^2 \right] \\
 &\equiv \int d^4x \sqrt{-g} \left[-\frac{1}{\kappa_B^2} R + \frac{1}{f_B^2} \left(R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) + \frac{\omega_B}{3f_B^2} R^2 \right].
 \end{aligned}$$

Asymptotic Freedom

$$f(t), \quad \omega(t), \quad \lambda(t), \quad \text{where } t := \frac{1}{(4\pi)^2} \log \frac{\mu}{\mu_0},$$

$$f(t)^2 = \frac{f(0)^2}{1 + f(0)^2 \beta_2 t},$$

$$h_{\mu\nu} \rightarrow f(t) h_{\mu\nu},$$

$$S_{\text{UV}} \approx \frac{1}{\kappa^2} (f^2 h \square h + \dots) + (h \square^2 h + O(f)) - \omega (h \square^2 h + O(f)).$$

Unitarity Violation

$$\begin{aligned} G(p) &= \frac{P^{(2)}}{p^2 \left(1 + \frac{\kappa^2}{4f^2} p^2\right)} - \frac{P^{(0)}}{2p^2 \left(1 - \frac{\omega \kappa^2}{2f^2} p^2\right)} + \text{gauge terms} = \\ &= P^{(2)} \left[\frac{1}{p^2} - \frac{1}{p^2 + \frac{4f^2}{\kappa^2}} \right] - \frac{P^{(0)}}{2} \left[\frac{1}{p^2} - \frac{1}{p^2 + \frac{2f^2 \kappa^2}{\omega^2}} \right]. \end{aligned}$$

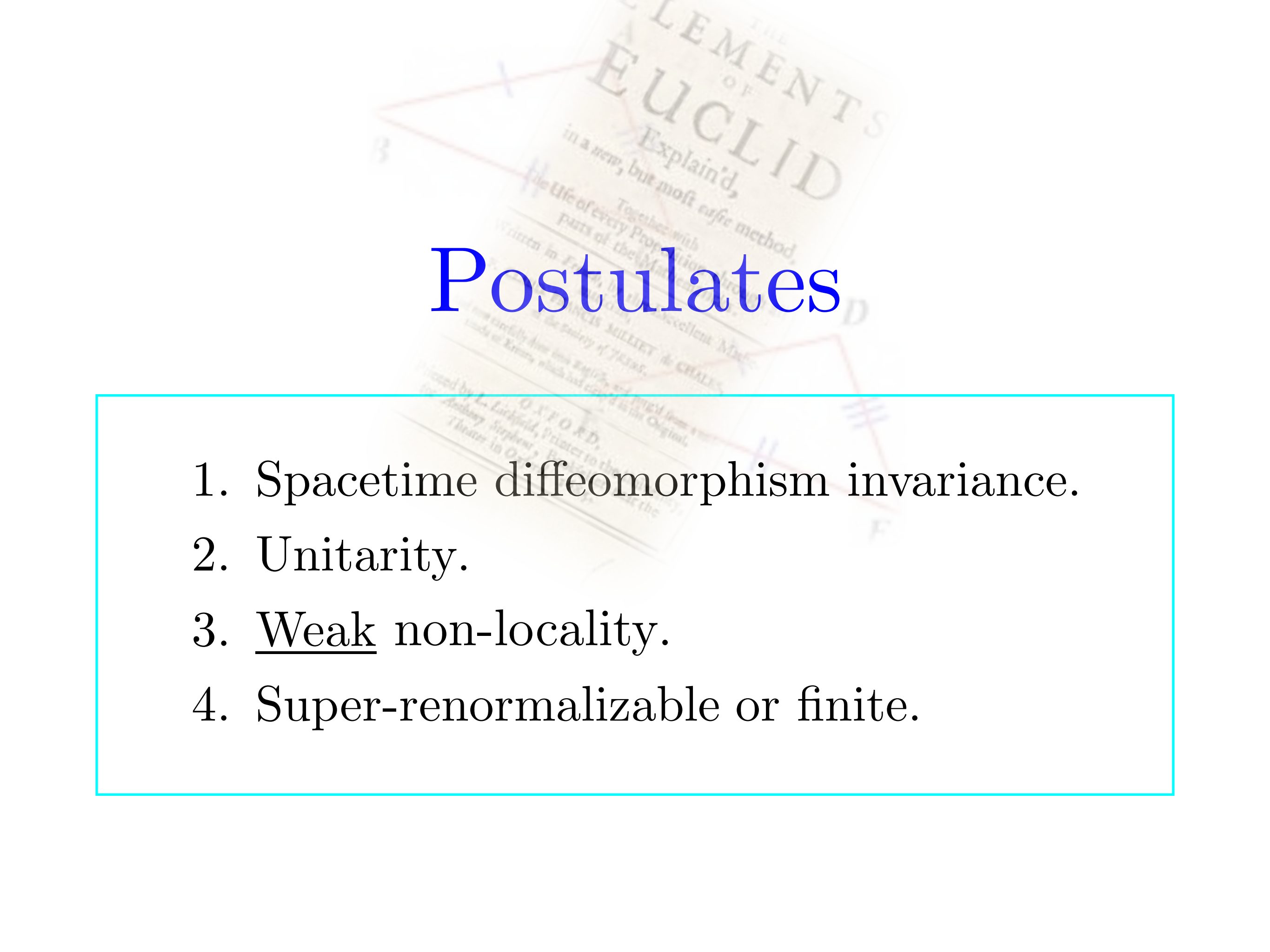
Can we do more? Can we have finite Stelle's quantum gravity?

Yes!

Conformal Supergravity N=4 coupled
to N=4 SYM with gauge group $SU(2) \times U(1)$ or $U(1)^4$.

ES. Fradkin and A.A. Tseytlin,

Conformal anomaly in Weyl theory and anomaly free superconformal theories,
Phys. Lett. 134B (1984) 187—193.

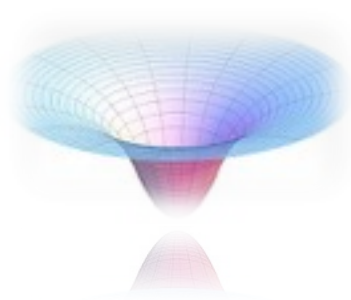


Postulates

1. Spacetime diffeomorphism invariance.
2. Unitarity.
3. Weak non-locality.
4. Super-renormalizable or finite.

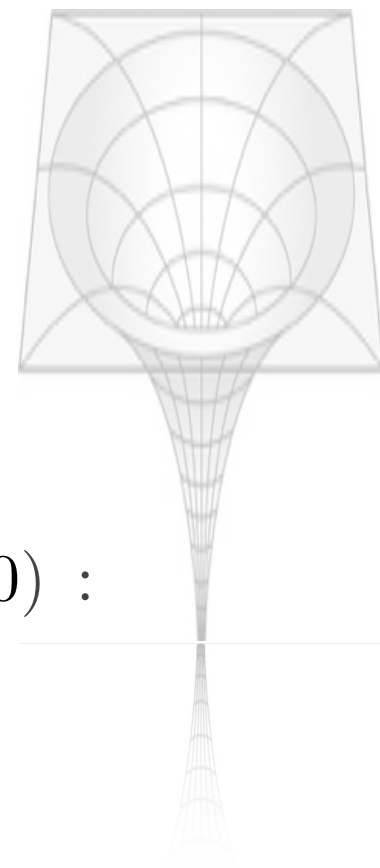
Classical Guiding Principle

Desingularization



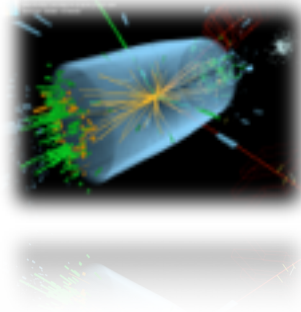
- General covariance ,
- $\nabla^a G_{ab} = 0$
- , • GRAVITY = CURVATURE ,
- low energy limit ($M_P \rightarrow +\infty$) $\rightarrow$ Einstein's gravity,
- non singular classical solutions. Ex. Black holes ($r \sim 0$) :

$$ds^2 \approx -(1 - r^\alpha)dt^2 + \frac{dr^2}{1 - r^\alpha} + r^2 d\Omega^{(2)} , \quad \alpha \geq 2 .$$



Quantum Guiding Principle

Super-renormalizability or Finiteness



- Renormalizability and/or super-renormalizability,

$$G(p) \rightarrow \frac{1}{p^n}, \quad n \geq 4 \text{ in the UV,}$$

- Unitary quantum theory (ghosts free),

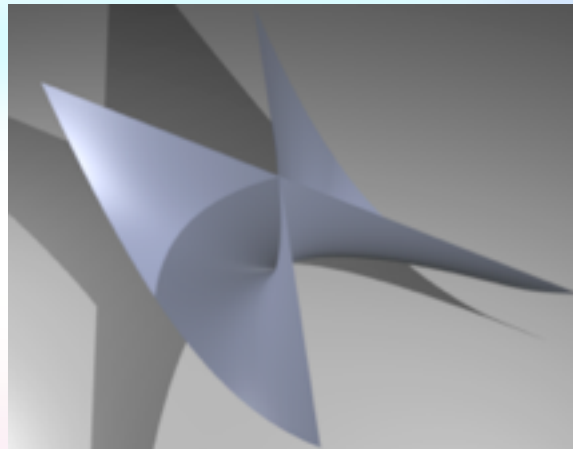
$$G(p) \propto \frac{f(p)}{p^2}, \quad f(p) \text{ does not have poles.}$$

- Decreasing of the “spacetime dimension” at small distances :

$$d_s \leq 2.$$

Classical Guiding Principle

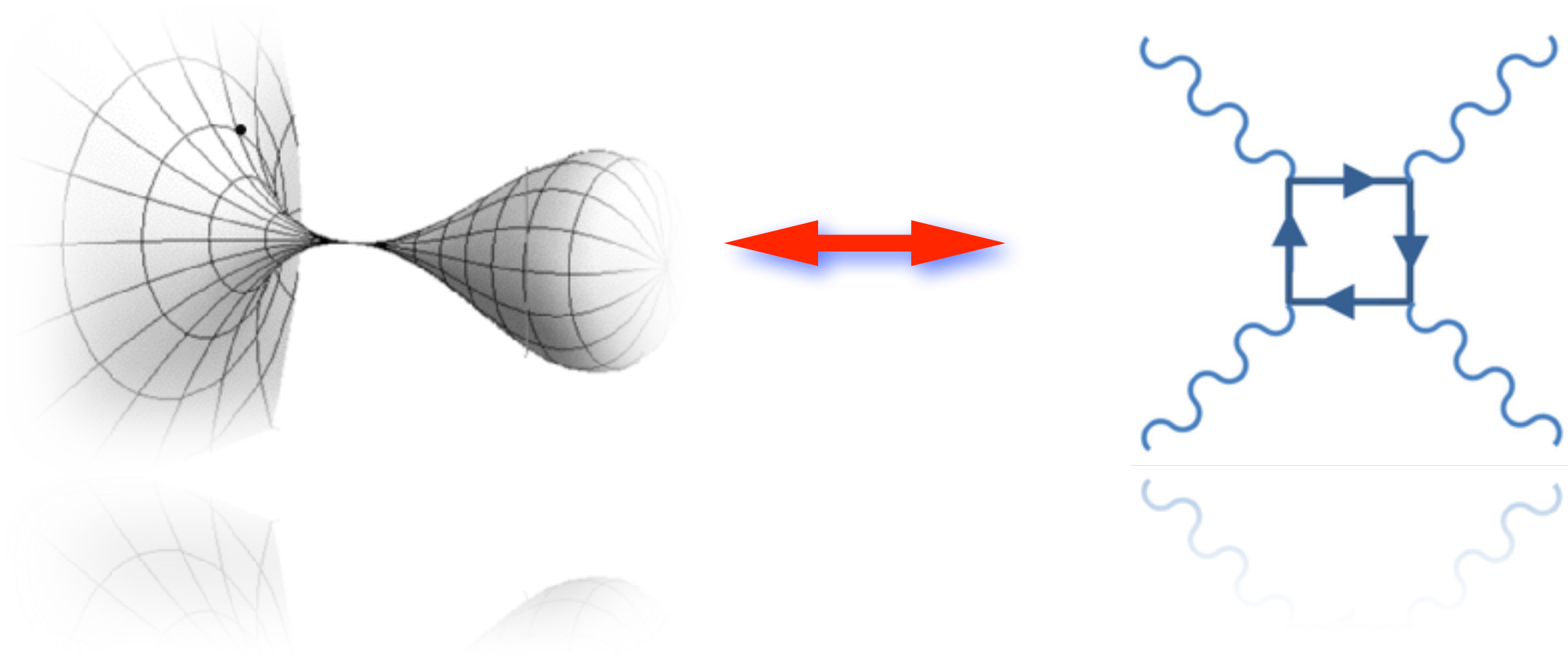
Desingularization



Quantum Guiding Principle

Super-renormalizability or Finiteness

Conjecture



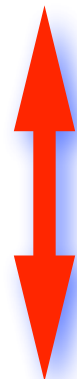
Singularities
in Classical field Theory



Ultraviolet Divergences
in Quantum field Theory

Conjecture

Singularity-Free Classical Gravity



Finita Quantum Gravity

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Int. J. Mod. Phys. A **12**, 5711 (1997) [hep-th/9610006].

NONLOCAL FINITE GRAVITY

L. Modesto and L. Rachwal,

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A. S. Koshelev and S. Y. Vernov, I. Dimitrijevic, B. Dragovich, J. Grujic and Z. Rakic, J. Stankovic.

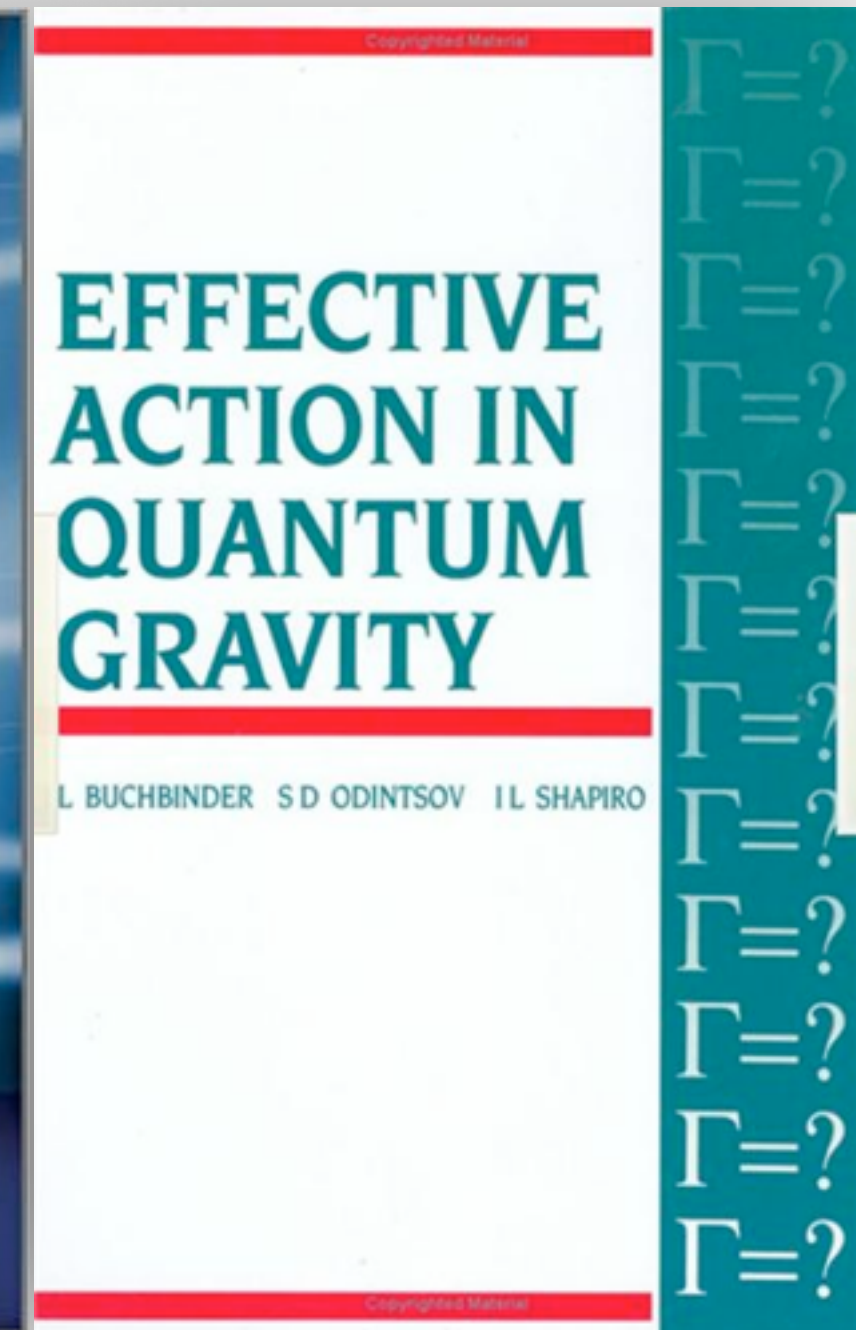
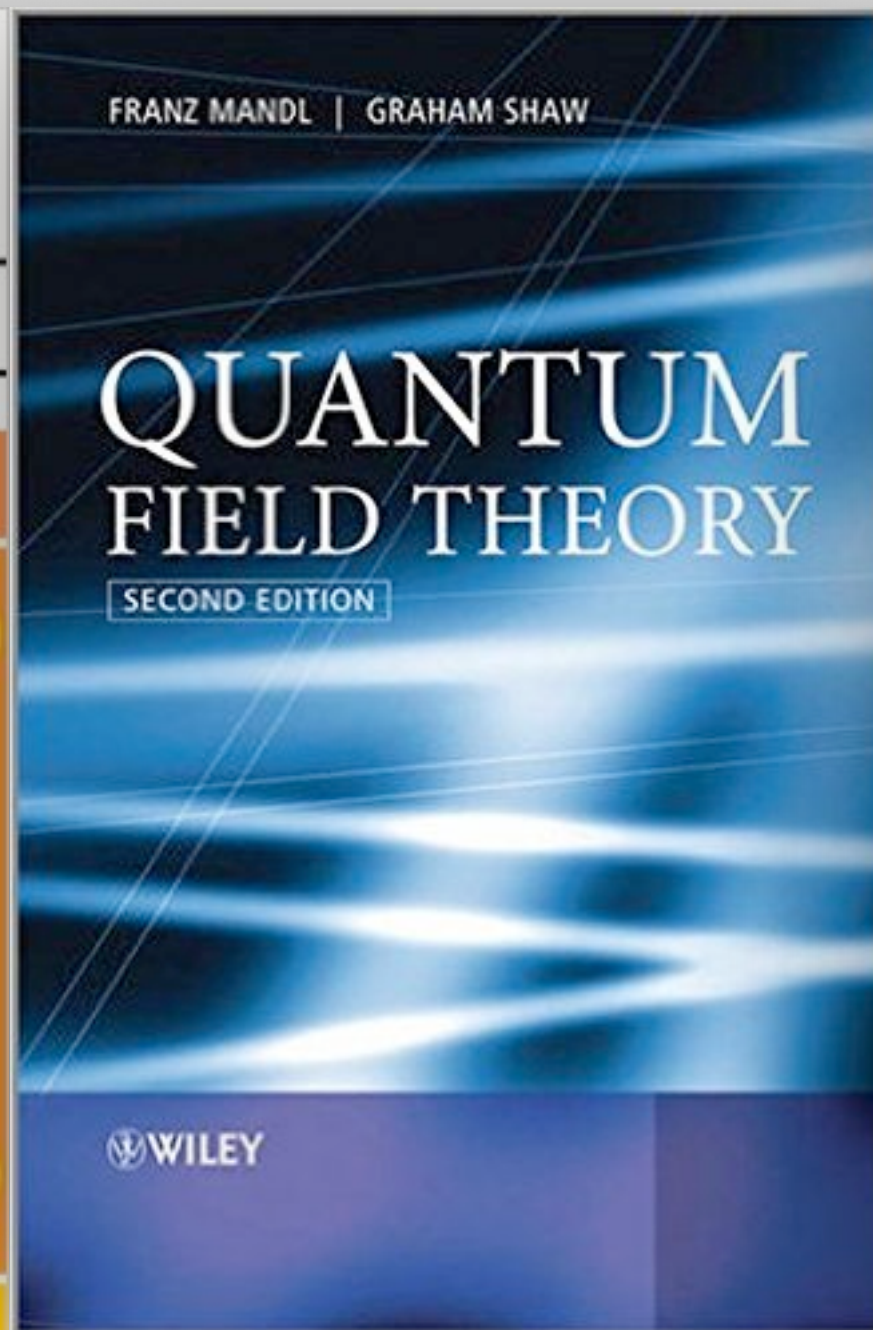
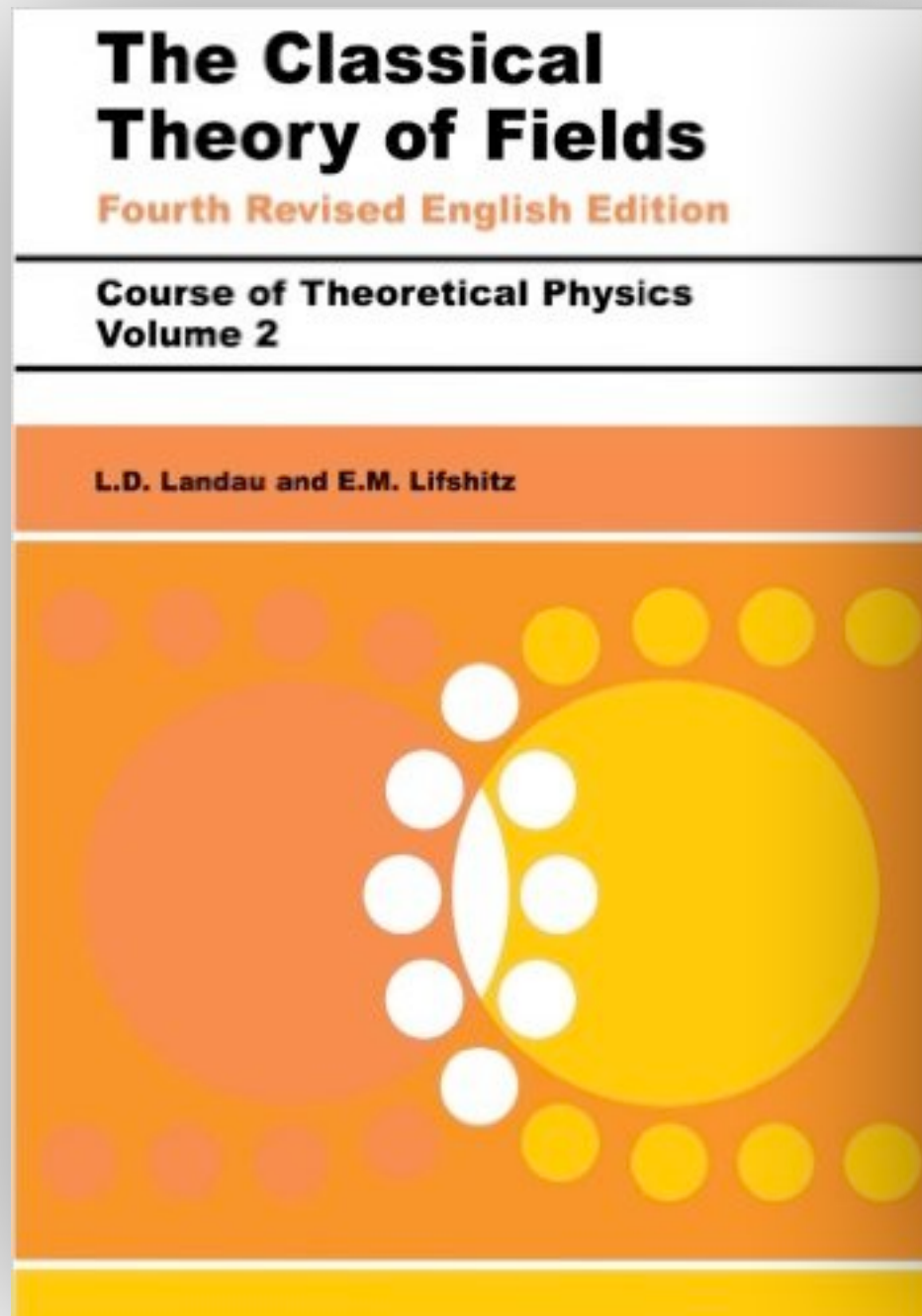
IMPLICATIONS in STRING M-THEORY

Calcagni, L. Modesto, and G. Nardelli.

NONLOCAL GAUGE THEORY

L. Modesto, M. Piva, L. Rachwal.

BOOKS



N. V. Krasnikov

Nonlocal Quantum Gravity

All theories of gravitation based on the use of local Lagrangians are... In this section, we construct a superrenormalizable nonlocal Lagrangian for a purely gravitational field. The generalization to the supersymmetric case...

We choose the nonlocal Lagrangian for the gravitational field in the

$$L = [-1/2 \gamma R + R f_1(\Box^{\text{cov}}) R + R_{\mu\nu} f_2(\Box^{\text{cov}}) R_{\mu\nu} - 1/3 R f_2(\Box^{\text{cov}}) R] \sqrt{-g}.$$

re, $\Box^{\text{cov}}$ is the covariant d'Alembertian operator.

Using the background-field method, we show that the nonlocal theory is renormalizable.

ere

can always

contains only

DFTUZ 96-15, hep-th/9610006
September, 1996

Some remarks on high derivative quantum gravity

M. Asorey^a, J.L. López^a and I.L. Shapiro^{a,b,1}

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UCLA/97/TEP/2
hep-th/9702146

Superrenormalizable Gauge and Gravitational Theories¹

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Nuclear Physics B 889 (2014) 228–248

www.elsevier.com/locate/nuclphysb

NUCLEAR PHYSICS B

Super-renormalizable and finite gravitational theories

Leonardo Modesto^{*}, Lesław Rachwał

Department of Physics & Center for Field Theory and Particle Physics, Fudan University, 200433 Shanghai, China

Finite nonlocal gravity

Yu. V. Kuz'min

Institute of Nuclear Physics, Moscow State University

(Submitted 23 March 1989)

Yad. Fiz. 50, 1630–1635 (December 1989)

A version of a nonlocal, superrenormalizable theory of gravity is considered. For certain values of the Lagrangian parameters the ultraviolet divergence is absent.

It is well known that the quantization of general relativity using the background-field formalism leads to a nonrenormalizable theory.¹ The addition to the Lagrangian of terms proportional to the square of the curvature tensor makes the theory renormalizable, but leads to a loss of unitarity at the tree level.² Here we abandon the locality of the Lagrangian in order to reconcile the properties of unitarity and renormalizability.

We write the Lagrangian as

$$\mathcal{L} = -\sqrt{-g} \left[\frac{1}{2\kappa} R + \alpha R f_1\left(\frac{\Box_c}{\mu^2}\right) R + \beta R^{ik} f_2\left(\frac{\Box_c}{\mu^2}\right) R_{ik} + \gamma R^{iklm} f_3\left(\frac{\Box_c}{\mu^2}\right) R_{iklm} \right]. \quad (1)$$

Here $\Box_c$ is the D'Alembertian constructed from covariant derivatives and μ is a constant with the dimension of

PHYSICAL REVIEW D 86, 044005 (2012)

Super-renormalizable quantum gravity

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revised manuscript received 27 May 2012; published 3 August 2012

Super-renormalizable quantum gravity is an extension of the Stelle higher derivative gravity into the ultraviolet. We know that the usual quadratic action is renormalizable but not finite because of the presence of a ghost (state of negative norm) in the theory. The ghost is first introduced by Tomboulis in 1997, but we expand and extend the theory to the quantum level. This theory is ghost-free, since the introduction of the ghost in the model with the property does not introduce new poles. The higher derivative theory is recovered expanding the entire functions to the infinite series. Any truncation of the entire functions gives rise to the usual quadratic theory. All the infinite series, we do not fall into these troubles. The theory is renormalizable and finite from two loops on. Since only one-loop Feynman diagrams are super-renormalizable. We analyze the fractal properties of the theory. The spacetime dimension at short scales. Black hole spherical symmetry. The high curvature corrections in the equation of motions. The

The Lagrangian

L.M., L. Rachwal

$$\mathcal{L}_g = -2\kappa_D^{-2} \sqrt{-g} \left(R + R\gamma_0(\square)R + R_{\mu\nu}\gamma_2(\square)R^{\mu\nu} + R_{\mu\nu\rho\sigma}\gamma_4(\square)R^{\mu\nu\rho\sigma} + \mathcal{V}(\mathcal{R}) \right),$$

$$\gamma_0(\square) = -\frac{(D-2)(e^{H_0} - 1) + D(e^{H_2} - 1)}{4(D-1)\square} + \gamma_4(\square),$$

$$\gamma_2(\square) = \frac{e^{H_2} - 1}{\square} - 4\gamma_4(\square).$$

$$\mathcal{V}(R) = s_1 \mathcal{R}^3 + \cdots + s_k \mathcal{R}^{\frac{D}{2}} + \cdots + s_n \mathcal{R}^{\frac{D}{2}} \nabla^{2\gamma-4} \mathcal{R}^2.$$

$$D = 4$$

N. V. Krasnikov (1988), A.T. Tomboulis (1997), L.M. (2011)

$$\mathcal{L}_{4D} = \frac{2}{\kappa_4^2} \left(R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} \right) .$$

Nonlocal Gravity in Einstein (Ricci) Basis

L.M., L. Rachwal

$$\mathcal{L}_E = -2\kappa_D^{-2} \sqrt{|g|} \left[\mathbf{R} + \mathbf{G} \gamma_G (-\square_\Lambda) \mathbf{Ric} + \mathbf{R} \gamma_{S'} (-\square_\Lambda) \mathbf{R} + \mathbf{V}_g \right],$$

$$\gamma_G = \gamma_2 = \frac{e^{H_2} - 1}{\square}, \quad \gamma_{S'} = \frac{1}{2} \gamma_2 + \gamma_0 = \frac{D-2}{4(D-1)} \frac{e^{H_2} - e^{H_0}}{\square}.$$

Unitarity: Anselmi, Piva.
F. Briscese, LM, S. Tsujikawa

Nonlocal Gravity in Weyl Basis

Y. D. Li, M., Rachwal, Koshelev, Starobinsky.

$$\mathcal{L}_C = -2\kappa_D^{-2} \sqrt{|g|} \left[\mathbf{R} + \mathbf{C} \gamma_C(\square) \mathbf{C} + \mathbf{R} \gamma_S(\square) \mathbf{R} + \mathbf{V}(\mathbf{C}) \right],$$

$$\gamma_C = \frac{D-2}{4(D-3)} \frac{e^{H_2} - 1}{\square}, \quad \gamma_S = -\frac{D-2}{4(D-1)} \frac{e^{H_0} - 1}{\square}.$$

Nonlocal Gravity in Bach Basis

L.M., L. Rachwal

$$\mathcal{L}_B = -2\kappa_D^{-2} \sqrt{|g|} \left[R + B_{ab} \gamma_B(\square) B^{ab} + R \gamma_S(\square) R + \mathbf{V}(\mathbf{B}) \right],$$

$$\gamma_B = \left(\frac{D-2}{D-3} \right)^2 \frac{e^{H_2} - 1}{\square^3},$$

$$\gamma_S = - \frac{(D-2)(D-3)(e^{H_0} - 1) + D(D-4)(e^{H_2} - 1)}{4(D-1)(D-3)\square}.$$

$$\begin{aligned} B_{ac} &= \nabla^b \nabla^d C_{abcd} - R^{bd} C_{abcd} = \\ &= \frac{D-3}{D-2} \square R_{ac} - \frac{D-3}{2(D-1)} R_{;ac} + \frac{D(D-3)}{(D-2)^2} R_{ab} R_c^b - \frac{D-3}{D-2} R^{bd} C_{abcd} - \frac{D-3}{(D-2)^2} g_{ac} R_{bd} R^{bd} \\ &\quad - \frac{D(D-3)}{(D-1)(D-2)^2} R R_{ac} - \frac{D-3}{2(D-1)(D-2)} g_{ac} \square R + \frac{D-3}{(D-1)(D-2)^2} g_{ac} R^2 - R^{bd} C_{abcd}. \end{aligned}$$

The Form Factor (general properties)

$$V^{-1}(z) := e^{H(z)}.$$

- $V^{-1}(z)$ has no zeros on the whole complex plane $|z| < +\infty$.

Therefore there are no gauge-invariant poles other than the transverse massless physical graviton pole.

- $|V^{-1}(z)|$ has the same asymptotic behavior along the real axis at $\pm \infty$.
- There exists $\Theta > 0$ and $\Theta < \pi/2$, such that asymptotically :

$$|V^{-1}(z)| \rightarrow |z|^{\gamma+N+1}, \text{ when } |z| \rightarrow +\infty$$

($\gamma \geq D_{\text{even}}/2$ & $\gamma \geq (D_{\text{odd}} - 1)/2$ respectively),

for z in the conical regions C defined by :

$$C = \{z \mid -\Theta < \arg z < +\Theta, \pi - \Theta < \arg z < \pi + \Theta\}.$$

$$N = \frac{D-4}{2}.$$

$$V^{-1}(z) := e^{H(z)}.$$

$$H(z) = \int_0^{p_{\gamma+1}(z)} \frac{1 - f(\omega)}{\omega} d\omega.$$

1. $p_{\gamma+1}(z)$: real polynomial of degree $\gamma + 1$, $p_{\gamma+1}(0) = 0$,
2. $f(z)$: entire and real function on the real axis with $f(0) = 1$,
3. $|f(z)| \rightarrow 0$ for $|z| \rightarrow \infty$ in the conical region around the real axis.

$$f(z) = e^{-q_{\text{poly}}(z)} g_{\text{poly}}(z) \text{ (entire function of finite order } |f(z)| \leq e^{|z|^\rho}).$$

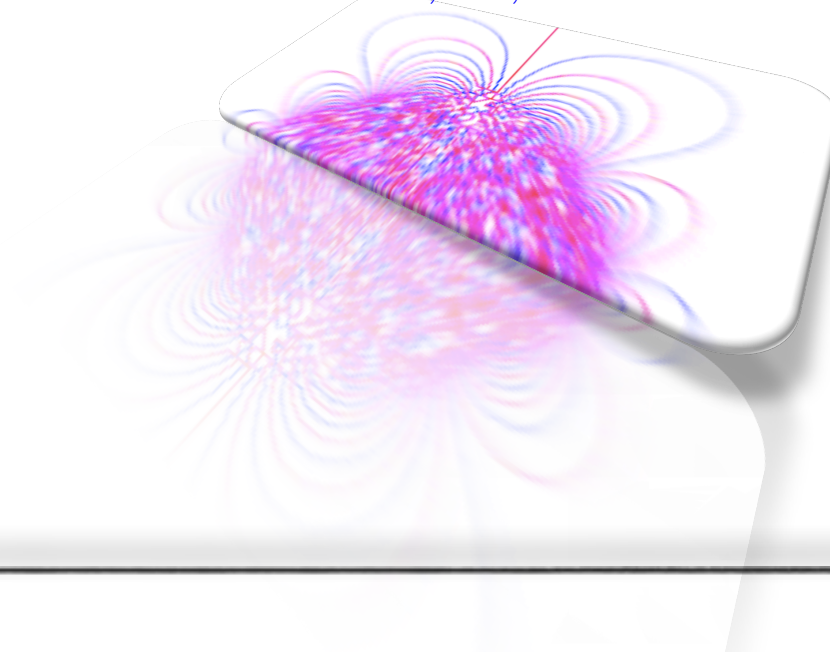
$$a. \quad g_{\text{poly}}(z) : \text{ polynomial such that } g_{\text{poly}}(0) = 1.$$

$$b. \quad \deg(q_{\text{poly}}(z)) < \rho, \quad q_{\text{poly}} = c_n z^{2n} + \cdots + c_1 z^2, \quad c_n > 0.$$

$$\text{No zeros} \implies g_{\text{poly}}(z) = 1.$$

The Form Factor (explicit example)

Tomboulis, M., Rachwal.



$$V^{-1}(z) \equiv e^{H(z)} = e^{\frac{1}{2}[\Gamma(0, p^2(z)) + \gamma_E + \log(p^2(z))]}$$

$$\stackrel{\mathbb{R}}{=} e^{\frac{1}{2}[\Gamma(0, p^2(z)) + \gamma_E]} |p(z)|$$

$$\approx e^{-p^2(z)} \left(\frac{1}{2p^2(z)} - \frac{1}{2p^4(z)} + O\left(\frac{1}{p^6(z)}\right) + \dots \right) e^{\frac{\gamma_E}{2}} |p(z)| + \boxed{e^{\frac{\gamma_E}{2}} |p(z)|},$$

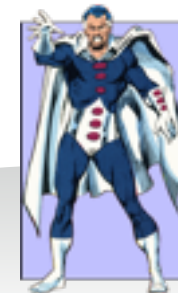
$$p_{\gamma+N+1}(z) = a_{\gamma+N+1} z^{\gamma+N+1} + \dots + a_{\gamma+N+1-\frac{D}{2}} z^{\gamma+N+1-\frac{D}{2}} + \dots$$

Unitarity

Propagator and Unitarity

2 d.o.f.

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$



For $H_2(\square) = H_0(\square)$:

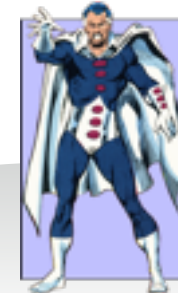
$$\mathcal{O}^{-1}(k) = \frac{V(k^2/\Lambda^2)}{k^2} \left(P^{(2)} - \frac{P^{(0)}}{D-2} \right).$$

Propagator and Unitarity

3 d.o.f.

F. Brisce, LM, S. Tsujikawa

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$



For $H_2(\square) \neq H_0(\square)$:

$$\begin{aligned} \mathcal{O}^{-1}(k) &= \frac{P^{(2)}}{k^2 e^{H_2(-k^2)}} - \frac{P^{(0)}}{(D-2) k^2 e^{H_0(k^2)} \left(1 + \frac{k^2}{m^2}\right)} \cdot \\ &= \frac{P^{(2)}}{k^2 e^{H_2}} - \frac{P^{(0)}}{(D-2) k^2 e^{H_0}} + \frac{P^{(0)}}{(D-2) e^{H_0} (k^2 + m^2)} \cdot \end{aligned}$$

A. koshelev, LM, L. Rachwal, A. Starobinsky

Perturbative Unitarity

$$S^\dagger S = 1, \quad S = 1 + iT \quad \Longrightarrow \quad T - T^\dagger = iT^\dagger T.$$

$$T_{fi} - T_{if}^* = i \sum_k T_{kf}^* T_{ki}.$$

Def. $\text{Disc}(T_{fi}) = T_{fi}(s + i\epsilon) - T_{fi}(s - i\epsilon).$

If $(T_{if}(s + i\epsilon))^* = T_{fi}(s - i\epsilon) \quad \Longrightarrow \quad \boxed{\text{Disc}(T_{fi}) = i \sum_k T_{kf}^* T_{ki}},$

$$\boxed{2\text{Im}T_{if} > 0}.$$

$$\mathcal{L}_{h\Theta} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu,\rho\sigma} h^{\rho\sigma} - g h_{\mu\nu} \Theta^{\mu\nu}.$$

Amplitude :

$$S^{(2)} = iT = (-i)^2 g^2 \Theta^{\mu\nu} i\Delta_{F\mu\nu,\rho\sigma} \Theta^{\mu\nu}, \quad \langle 0|T_{\text{Wick}}(h_{\mu\nu}(x')h_{\rho\sigma}(x))|0\rangle = i\Delta_{F\mu\nu,\rho\sigma}(k) \equiv i\mathcal{O}_{\mu\nu,\rho\sigma}^{-1}(k).$$

$$\begin{aligned} S_{if}^{(2)} &= iT_{if} = (2\pi)^D \delta(P_i - P_f) iT_{if} = (-i)^2 g^2 \Theta^{\mu\nu} i\Delta_{F\mu\nu,\rho\sigma} \Theta^{\rho\sigma} \\ &= (2\pi)^D \delta(P_i - P_f) i \underbrace{(-i)^2 \Theta^{\mu\nu} \frac{1}{k^2 + i\epsilon} \left(\frac{P^{(2)}}{e^{H_2}} - \frac{P^{(0)}}{(D-2)e^{H_0}} \right)_{\mu\nu,\rho\sigma}}_{T_{if}} \Theta^{\rho\sigma}, \end{aligned}$$

$$T_{if} = (-i)^2 \Theta^{\mu\nu} \frac{1}{k^2 + i\epsilon} \left(\frac{P^{(2)}}{e^{H_2}} - \frac{P^{(0)}}{(D-2)e^{H_0}} \right)_{\mu\nu,\rho\sigma} \Theta^{\rho\sigma}.$$

$$\begin{aligned} 2\text{Im } T_{if} &= 2\text{Im } (-i)^2 g^2 \left\{ \left[\Theta_{\mu\nu} \Theta^{\mu\nu} - \frac{1}{D-1} \Theta_{\mu}^{\mu 2} \right] e^{-H_2(k^2)} - \frac{\Theta_{\mu}^{\mu 2}}{(D-1)(D-2)} e^{-H_0(k^2)} \right\} \left(\frac{k^2 - i\epsilon}{k^4 + \epsilon^2} \right) \\ &= 2g^2 \left\{ \left[\Theta_{\mu\nu} \Theta^{\mu\nu} - \frac{1}{D-1} \Theta_{\mu}^{\mu 2} \right] e^{-H_2(k^2)} - \frac{\Theta_{\mu}^{\mu 2}}{(D-1)(D-2)} e^{-H_0(k^2)} \right\} \left[\frac{\epsilon}{k^4 + \epsilon^2} \right] \\ &\rightarrow 2g^2 \left\{ \Theta_{\mu\nu} \Theta^{\mu\nu} - \frac{\Theta_{\mu}^{\mu 2}}{D-2} \right\} \pi \delta(k^2), \quad \text{for } H_2(0) = H_0(0) = 0. \end{aligned}$$

Perturbative Unitarity

$$T_{ab}^{\mathcal{D}}(s + i\epsilon) = i^{I+V+1} \int \frac{\left(\prod_l^L d^4 k_l\right) B}{\prod_i^I D_i^+}, \quad D^\pm = q^2 - m^2 \pm i\epsilon,$$

$$T_{ab}^{\mathcal{D}}(s - i\epsilon) = (-1)^L i^{I+V+1} \int \frac{\left(\prod_l^L d^4 k_l\right) B}{\prod_i^I D_i^-}.$$

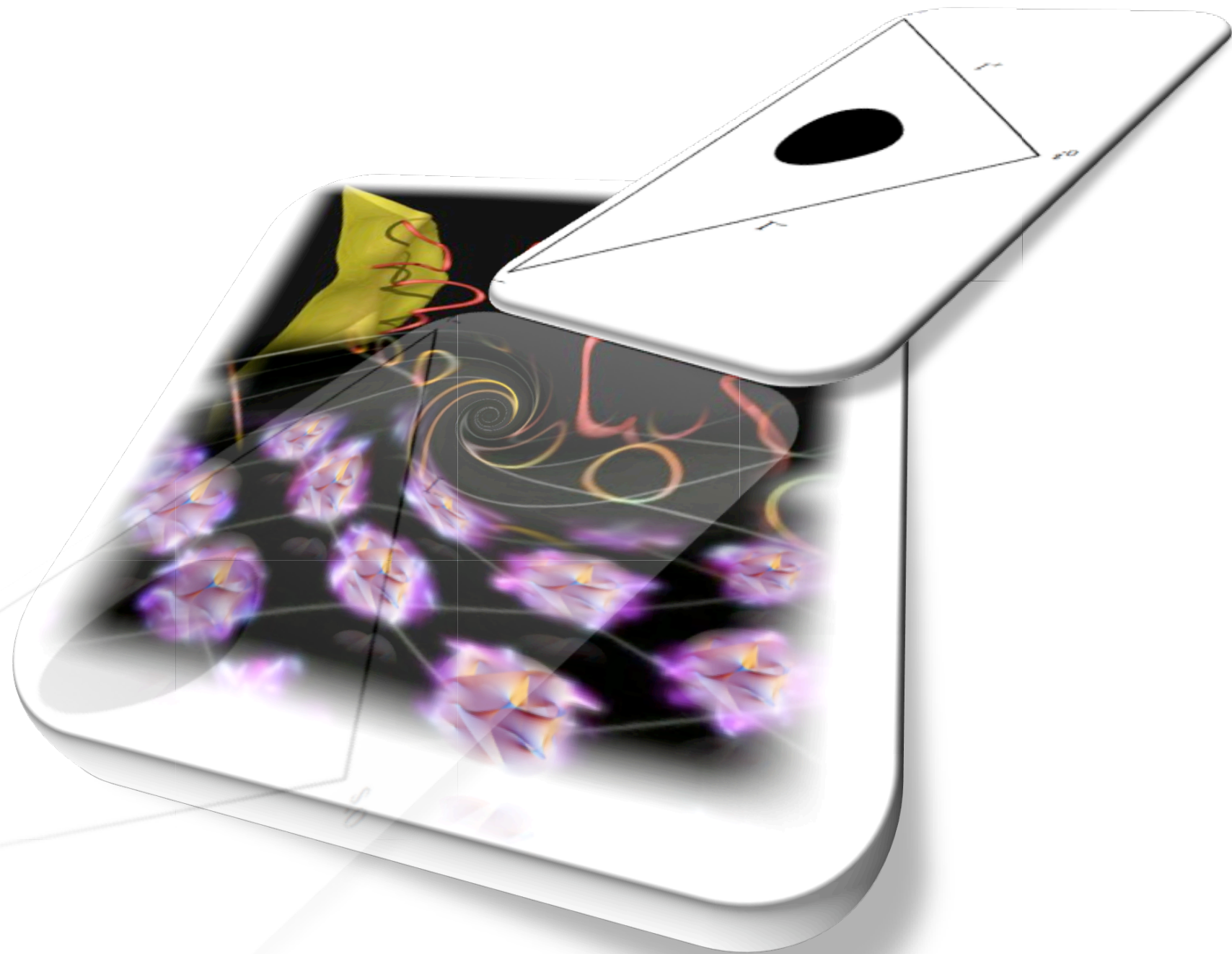
$$\text{Olive :} \quad \left(T_{ba}^{\bar{\mathcal{D}}}(s + i\epsilon)\right)^* = T_{ab}^{\mathcal{D}}(s - i\epsilon) \quad \text{or} \quad \left(T_{ba}^{\bar{\mathcal{D}}}(s + i\epsilon)\right)^* = T_{ab}^{\mathcal{D}}((s + i\epsilon)^*).$$

$$\text{Perturbative Unitarity :} \quad \text{Disc}(T_{ab}) = T_{ab}^{\mathcal{D}}(s + i\epsilon) - T_{ab}^{\mathcal{D}}(s - i\epsilon) = i \sum_k T_{ka}^* T_{kb}.$$

Following Cutkosky and/or Eden-Landshoff-Olive-Polkinghorne (ELOP)
we derive the discontinuity formula across a branch cut:

$$\text{Disc}T_{ab} = (-2\pi i)^r \int B \prod_L d^4 k_L \frac{\delta^{(+)}(q_1^2 - m_1^2) \delta^{(+)}(q_2^2 - m_2^2) \dots \delta^{(+)}(q_r^2 - m_r^2)}{(q_{r+1}^2 - m_{r+1}^2) \dots (q_N^2 - m_N^2)}.$$

Super-renormalizable Quantum Gravity



Leonardo Modesto



SUSTech

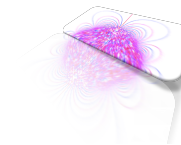
Southern University
of Science and
Technology

Summary

N. V. Krasnikov (1988), A.T. Tomboulis (1997), L.M. (2011), Shapiro (local theory 1996)

$$\mathcal{L} = \frac{2}{\kappa_D^2} \left(R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} \right) + c_1^{(3)} R^3 + \dots + c_1^{(N+2)} R^{N+2} .$$

Entire Function



$$V^{-1}(z) \equiv e^{H(z)} = e^{\frac{1}{2} [\Gamma(0, p^2(z)) + \gamma_E + \log(p^2(z))]} \\ \approx e^{-p^2(z)} \left(\frac{1}{2p^2(z)} - \frac{1}{2p^4(z)} + O\left(\frac{1}{p^6(z)}\right) + \dots \right) e^{\frac{\gamma_E}{2} |p(z)|} + e^{\frac{\gamma_E}{2} |p(z)|} ,$$

$$p(z) = z^{\gamma+N+1} , \quad z = -\square_\Lambda = -\square/\Lambda^2 .$$

Propagator

$$\mathcal{O}^{-1}(k) = \frac{V(k^2/\Lambda^2)}{k^2} \left(P^{(2)} - \frac{P^{(0)}}{D-2} \right) .$$

Super-renormalizability

Super-renormalizability in a nutshell

Asorey, Lopez, Shapiro,M, Rachwal.

Action: $\int d^D x \sqrt{-g} \omega_\gamma \mathcal{R} \square^{\gamma + \frac{D-4}{2}} \mathcal{R} , \quad \omega_\gamma = \frac{\tilde{\omega}_\gamma}{\Lambda^{2\gamma}} , \quad \gamma \in \mathbb{N} .$

Propagator: $G(p) \sim \frac{1}{\omega_\gamma p^{2\gamma+D}} , \quad \text{Vertex: } \mathcal{V}(p) \sim \omega_\gamma p^{2\gamma+D} .$

$$\begin{aligned} \mathcal{A}^{(L)} &= \delta^D(K) \int (d^D p)^L G(p)^I \mathcal{V}(p)^V \\ &= \delta^D(K) \int (d^D p)^L \left(\frac{1}{\omega_\gamma p^{2\gamma+D}} \right)^I (\omega_\gamma p^{2\gamma+D})^V \\ &= \delta^D(K) \Lambda^{2\gamma(L-1)} \int (d^D p)^L \left(\frac{1}{p^{2\gamma+D}} \right)^I (p^{2\gamma+D})^V \\ &= \delta^D(K) \Lambda^{2\gamma(L-1)} \int (d^D p)^L \left(\frac{1}{p^{2\gamma+D}} \right)^{L-1} \\ &= \delta^D(K) \Lambda^{2\gamma(L-1)} (\Lambda_{\text{cut-off}})^{\omega(G)} , \quad \boxed{\omega(G) \equiv D - 2\gamma(L-1)} . \end{aligned}$$

For $\gamma > \frac{D}{2}$ only one loop divergences.

Examples. Einstein and Stelle

$$\int d^D x \sqrt{-g} \omega_\gamma \mathcal{R} \square^{\gamma + \frac{D-4}{2}} \mathcal{R}, \quad \omega_\gamma = \frac{\tilde{\omega}_\gamma}{\Lambda^{2\gamma}}, \quad \gamma \in \mathbb{N}.$$

Einstein Gravity, $S = M_P^2 \int d^D x \sqrt{-g} R.$

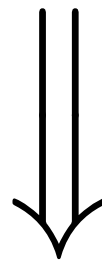
$$\gamma = -1 - \frac{D-4}{2} \implies \omega(G) = (D-2)L + 2.$$

Stelle's Gravity, $S = \int d^D x \sqrt{-g} (M_P^2 R + \alpha R^2 + \beta R_{\mu\nu}^2)$

$$\gamma = -\frac{D-4}{2} \implies \omega(G) = (D-4)L + 4.$$

Counterterms

$$\delta^D(K) \Lambda^{2\gamma(L-1)} (\Lambda_{\text{cut-off}})^{D-2\gamma(L-1)}$$



$$\frac{1}{\epsilon} \int d^D x \sqrt{-g} \mathcal{R}^{\frac{D}{2}} .$$

In general (in DIMREG) :

$$\frac{1}{\epsilon} \mathcal{R}^{\frac{D}{2}} , \quad \frac{1}{\epsilon} \mathcal{R}^{\frac{D}{2}-1} , \quad \dots , \quad \frac{1}{\epsilon} \mathcal{R} , \quad \frac{1}{\epsilon} \mathbf{1} , \dots$$

Rigorous Power-Counting

General Vertex

$$S_{\text{gravity}} = \int d^D x \sqrt{-g} \mathcal{R} \left(\sum_{r=0}^{\infty} a_r (-\tilde{\square})^r \right) \mathcal{R}.$$

$$\mathbf{V}^{(N)} = \sum_{n''+n+n'=N} \mathbf{R}^{(n'')} \cdot v^{(n)} \mathbf{R}^{(n')} = \sum_{n''+n+n'=N} \delta^{n''}(\sqrt{-g} \mathcal{R}) \delta^n \left(\sum_{r=0}^{\infty} a_r (-\tilde{\square})^r \right) \delta^{n'}(\mathcal{R}),$$

$$h(-\tilde{\square}) = \sum_r a_r (-\tilde{\square})^r, \quad -\tilde{\square} = -\square + \mathcal{I}, \quad \mathcal{I} = -\nabla^2 + \square + \mathcal{R}.$$

Typical contribution to $v^{(n)}$:

$$(-\square)^{p_1} \mathcal{I}^{(m_1)} (-\square)^{p_2} \mathcal{I}^{(m_2)} (-\square)^{p_3} \dots (-\square)^{p_\ell} \mathcal{I}^{(m_\ell)} (-\square)^{p_{\ell+1}},$$

where $m_1 + m_2 + \dots m_\ell = n$.

$$S_\ell(Q_1, Q_2, \dots, Q_{\ell+1}) = \sum_{r=\ell}^{\infty} a_r \sum_{\{p_i\}, \sum_{i=1}^{\ell+1} p_i = r-\ell} \prod_{i=1}^{\ell+1} Q_i^{2p_i} = \sum_{i=1}^{\ell+1} \frac{h(Q_i^2)}{\prod_{j \neq i}^{\ell+1} (Q_i^2 - Q_j^2)}.$$

[illegible]

$$\begin{aligned}
v^{(n)} \mathbf{R}^{(n')} &= \delta^n \left(\sum_{r=0}^{\infty} a_r (-\tilde{\square})^r \right) \mathbf{R}^{(n')} = \sum_{\ell=1}^n \sum_{\sigma} \sum_{r=\ell}^{\infty} a_r S_{r,\ell,n}^{\sigma} \left[(-\square)^{r-\ell}, \left(\mathcal{I}^{(m)} \right)^{\ell} \right] \mathbf{R}^{(n')} \\
&= \sum_{\ell=1}^n \sum_{\sigma} \sum_{r=\ell}^{\infty} a_r \sum_{\{p_i\}, \sum_{i=1}^{\ell+1} p_i = r-\ell} (-\square)^{p_1} \mathcal{I}^{(m_1)} (-\square)^{p_2} \mathcal{I}^{(m_2)} (-\square)^{p_3} \dots (-\square)^{p_{\ell}} \mathcal{I}^{(m_{\ell})} (-\square)^{p_{\ell+1}} \mathbf{R}^{(n')} \\
&= \sum_{\ell=1}^n \ell^{(n-1)} \ell! \sum_{r=\ell}^{\infty} a_r \sum_{\{p_i\}, \sum_{i=1}^{\ell+1} p_i = r-\ell} (-\square)^{p_1} \mathcal{I}^{(m_1)} (-\square)^{p_2} \mathcal{I}^{(m_2)} (-\square)^{p_3} \dots (-\square)^{p_{\ell}} \mathcal{I}^{(m_{\ell})} (-\square)^{p_{\ell+1}} \mathbf{R}^{(n')} \\
&= \sum_{\ell=1}^n \ell^{(n-1)} \ell! \left(\prod_{i=1}^{\ell} \mathcal{I}^{(m_i)} \right) S_{\ell}(\{Q_i^2\}) \mathbf{R}^{(n')} = \sum_{\ell=1}^n \ell^{(n-1)} \ell! \left(\prod_{i=1}^{\ell} \mathcal{I}^{(m_i)} (Q_i, Q_{i+1}) \right) S_{\ell}(\{Q_i^2\}) \mathbf{R}^{(n')} \\
&= \sum_{\ell=1}^n \ell^{(n-1)} \ell! \prod_{i=1}^{\ell} \left(a_i Q_{i+1}^2 + b_i Q_i Q_{i+1} \right) \sum_{i=1}^{\ell+1} \frac{h(Q_i^2)}{\prod_{j \neq i}^{\ell+1} (Q_i^2 - Q_j^2)} \mathbf{R}^{(n')} . \\
&= \sum_{\ell=1}^n \ell^{(n-1)} \ell! \prod_{i=1}^{\ell} \left(a_i Q_{i+1}^2 + b_i Q_i Q_{i+1} \right) \times \\
&\quad \times \left[\frac{h(Q_1^2)}{\prod_{j \neq 1}^{\ell+1} (Q_1^2 - Q_j^2)} + \frac{h(Q_2^2)}{\prod_{j \neq 2}^{\ell+1} (Q_2^2 - Q_j^2)} + \frac{h(Q_3^2)}{\prod_{j \neq 3}^{\ell+1} (Q_3^2 - Q_j^2)} + \dots + \frac{h(Q_{\ell+1}^2)}{\prod_{j \neq \ell+1}^{\ell+1} (Q_{\ell+1}^2 - Q_j^2)} \right] \mathbf{R}^{(n')} \\
&\quad \sim \frac{1}{q_1} \qquad \qquad \qquad \sim \frac{1}{q_1}, \quad \frac{1}{q_2} \qquad \qquad \frac{1}{q_1}, \quad \frac{1}{q_2}, \dots, \frac{1}{q_{\ell}} .
\end{aligned}$$

$$Q_{\ell+1} = q_{\ell+1} \in \mathcal{R}.$$

$$\begin{aligned} \mathbf{R}^{(n'')} \cdot v^{(n)} \mathbf{R}^{(n')} &= \mathbf{R}^{(n'')} \sum_{\ell=1}^n \ell^{(n-1)} \ell! \prod_{i=1}^{\ell} \left(a_i Q_{i+1}^2 + b_i Q_i Q_{i+1} \right) \times \\ &\times \left[\frac{h(Q_1^2)}{\prod_{j \neq 1}^{\ell+1} (Q_1^2 - Q_j^2)} + \frac{h(Q_2^2)}{\prod_{j \neq 2}^{\ell+1} (Q_2^2 - Q_j^2)} + \frac{h(Q_3^2)}{\prod_{j \neq 3}^{\ell+1} (Q_3^2 - Q_j^2)} + \cdots + \frac{h(Q_{\ell+1}^2)}{\prod_{j \neq \ell+1}^{\ell+1} (Q_{\ell+1}^2 - Q_j^2)} \right] \mathbf{R}^{(n')} \\ &\sim \frac{1}{q_1} \qquad \qquad \qquad \sim \frac{1}{q_1}, \quad \frac{1}{q_2} \qquad \qquad \frac{1}{q_1}, \quad \frac{1}{q_2}, \dots, \frac{1}{q_{\ell}}. \end{aligned}$$

$$e^{H(z)} = e^{\frac{1}{2}[\gamma_E + \Gamma(0, p(z)^2) + \log(p(z)^2)]} = \sqrt{p(z)^2} \left(1 + \frac{e^{-p(z)^2}}{2p(z)^2} + \dots \right),$$

$$p(z) = q(z)^2.$$

Renormalization & Asymptotic Freedom

LM, L. Rachwal

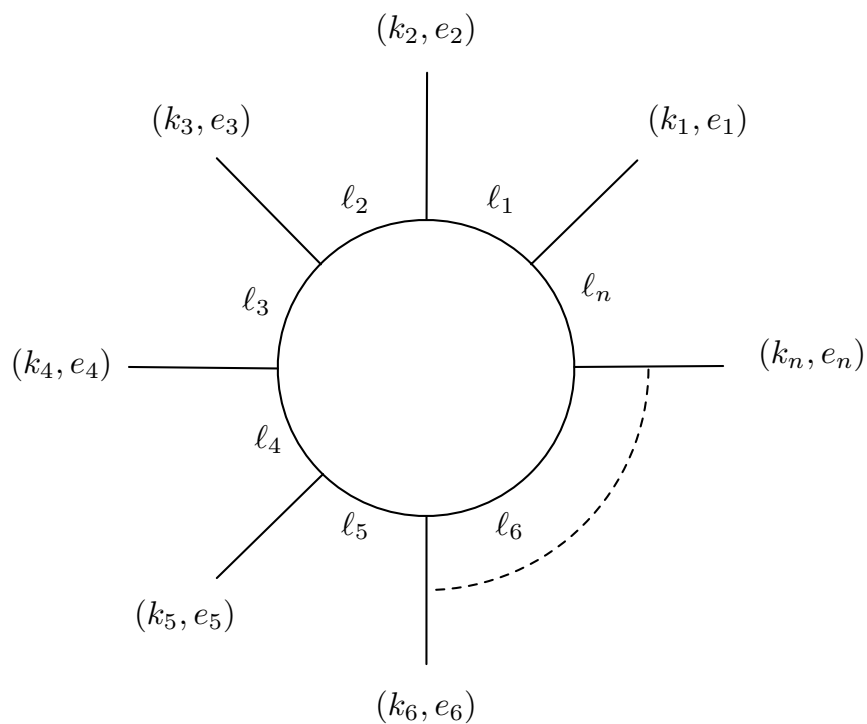
$$\mathcal{L} = \frac{2}{\kappa_D^2} R + 2 G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\tilde{\kappa}_D^2 \square} R^{\mu\nu} + c_1^{(3)} R^3 + \dots + c_1^{(N+2)} R^{N+2} \\ + \sum_{n=0}^N \left[(a_n - \tilde{a}_n) R (-\square_\Lambda)^n R + (b_n - \tilde{b}_n) R_{\mu\nu} (\square_\Lambda)^n R^{\mu\nu} \right].$$

Vertexes

$$\text{set 1 : } \mathcal{R}, \mathcal{R}^2, \mathcal{R}^3, \mathcal{R} \square \mathcal{R}, \dots, \mathcal{R}^{N+2}, \mathcal{R} \square^N \mathcal{R} \\ \Rightarrow h^m (\partial^2 h), h^m (\partial^2 h)^2, h^m (\partial^2 h)^3, \dots, h^m (\partial^2 h)^{N+2},$$

$$\text{set 2 : } \mathcal{R} \frac{\exp H(-\square_\Lambda)}{\square} \mathcal{R} \Rightarrow h^m (\partial^2 h) \frac{p(-\square_\Lambda)}{\square} (\partial^2 h) \sim (\nabla^{2\gamma+2N+4} h^{m+2}).$$

One loop amplitudes



$$A^{(L=1)} \sim \int d^D \ell \frac{\mathcal{V}_1(\ell_1, k_1, e_1) \mathcal{V}_2(\ell_2, k_2, e_2) \dots \mathcal{V}_n(\ell_n, k_n, e_n)}{\ell_1^2 V^{-1}(\ell_1) \ell_2^2 V^{-1}(\ell_2) \dots \ell_n^2 V^{-1}(\ell_n)},$$

divergence : $k^{2N+4} \int d^D \ell \frac{1}{\ell^{N+2} (\ell - k)^{N+2}} \sim \frac{1}{\epsilon} k^{2N+4},$

counterterms : $\frac{1}{\epsilon} R_{\dots}^{N+2}, \frac{1}{\epsilon} R_{\dots} \square^N R_{\dots}, \frac{1}{\epsilon} R_{\dots} R_{\dots} \square^{N-1} R_{\dots}, \dots$

$$\alpha_i \in \{\kappa_D^{-2}, \bar{\lambda}, a_n, b_n, c_1^{(1)}, \dots, c_1^{(N)}\} \equiv \{\kappa_D^{-2}, \bar{\lambda}, \tilde{\alpha}_n\}.$$

$$\begin{aligned}\mathcal{L}_{\text{ren}} &= \mathcal{L} + \mathcal{L}_{\text{ct}} \\ &= \mathcal{L} + 2(Z_\kappa - 1)\kappa^{-2} R + (Z_{\bar{\lambda}} - 1)\bar{\lambda} + (Z_{c_1^{(1)}} - 1)c_1^{(1)} R^3 + \dots + (Z_{c_1^{(N)}} - 1)c_1^{(N)} R^{N+2} \\ &\quad + \sum_{n=0}^N \left((Z_{a_n} - 1)a_n R (-\square_\Lambda)^n R + (Z_{b_n} - 1)b_n R_{MN} (-\square_\Lambda)^n R^{MN} \right),\end{aligned}$$

$$\mathcal{L}_{\text{ct}} = \frac{1}{\epsilon} \left[\beta_\kappa R + \beta_{\bar{\lambda}} + \sum_{n=0}^N \left(\beta_{a_n} R (-\square_\Lambda)^n R + \beta_{b_n} R_{MN} (-\square_\Lambda)^n R^{MN} \right) + \beta_{c_1^{(1)}} R^3 + \dots + \beta_{c_1^{(N)}} R^{N+2} \right],$$

$$\text{Beta functions : } \quad \beta_\kappa, \beta_{\bar{\lambda}}, \beta_{a_n}, \beta_{b_n}, \beta_{c_1^{(1)}}, \dots, \beta_{c_1^{(N)}} ,$$

$$(Z_{\alpha_i} - 1)\alpha_i = \frac{1}{\epsilon}\beta_{\alpha_i} \implies Z_{\alpha_i} = 1 + \frac{1}{\epsilon}\beta_{\alpha_i} \frac{1}{\alpha_i} ,$$

$$\alpha_i(\mu) \sim \alpha_i(\mu_0) + \beta_i \log \left(\frac{\mu}{\mu_0} \right) .$$

$$\beta_{G_N} , \quad \beta_{\Lambda_{cc}}$$

$$N = \gamma \geq 2$$

$$S_N = \int d^4x \sqrt{|g|} \left\{ \omega_{N,R} R \square^N R + \omega_{N,C} C \square^N C + \omega_{N-1,R} R \square^{N-1} R + \omega_{N-1,C} C \square^{N-1} C \right. \\ \left. + \omega_{0,R} R^2 + \omega_{0,C} C^2 + \omega_{EH} R + \omega_{CC} \right\}.$$

$$\Gamma_{\text{EH,div}}^{(1)} = - \frac{\mu^{n-4}}{\epsilon} \frac{1}{6} \int d^4x \sqrt{|g|} \left\{ \frac{5\omega_{N-1,C}}{\omega_{N,C}} + \frac{\omega_{N-1,R}}{\omega_{N,R}} \right\} R ,$$

$$\Gamma_{\text{CC,div}}^{(1)} = \frac{1}{\epsilon} \int d^4x \sqrt{|g|} \left\{ \frac{\omega_{N-1,R}^2}{\omega_{N,R}^2} + \frac{5\omega_{N-1,C}^2}{\omega_{N,C}^2} - 2 \left(\frac{\omega_{N-2,R}}{\omega_{N,R}} + \frac{5\omega_{N-2,C}}{\omega_{N,C}} \right) \right\}.$$

Shapiro, Rachwal, M.

Yang-Mills & Gravity

Asymptotic Freedom

$$\mathcal{L}_{\text{YM}} = \frac{1}{2g^2(t)} \text{Tr} F^2 \sim dA dA + g(t) A^2 dA + g^2(t) A^4,$$

$$g(t)^{-2} = g_o^{-2} + \beta_g t,$$

$$\mathcal{L}_{\text{G}} = \frac{2}{\kappa_D^2(t)} \left(R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} \right) \sim \partial h e^{H(-\square_\Lambda)} \partial h + \kappa_D h \partial h \partial h + O(\kappa_D^2),$$

$$\kappa_D^{-2}(t) = \kappa_{D_o}^{-2} + \beta_{\kappa_D} t.$$

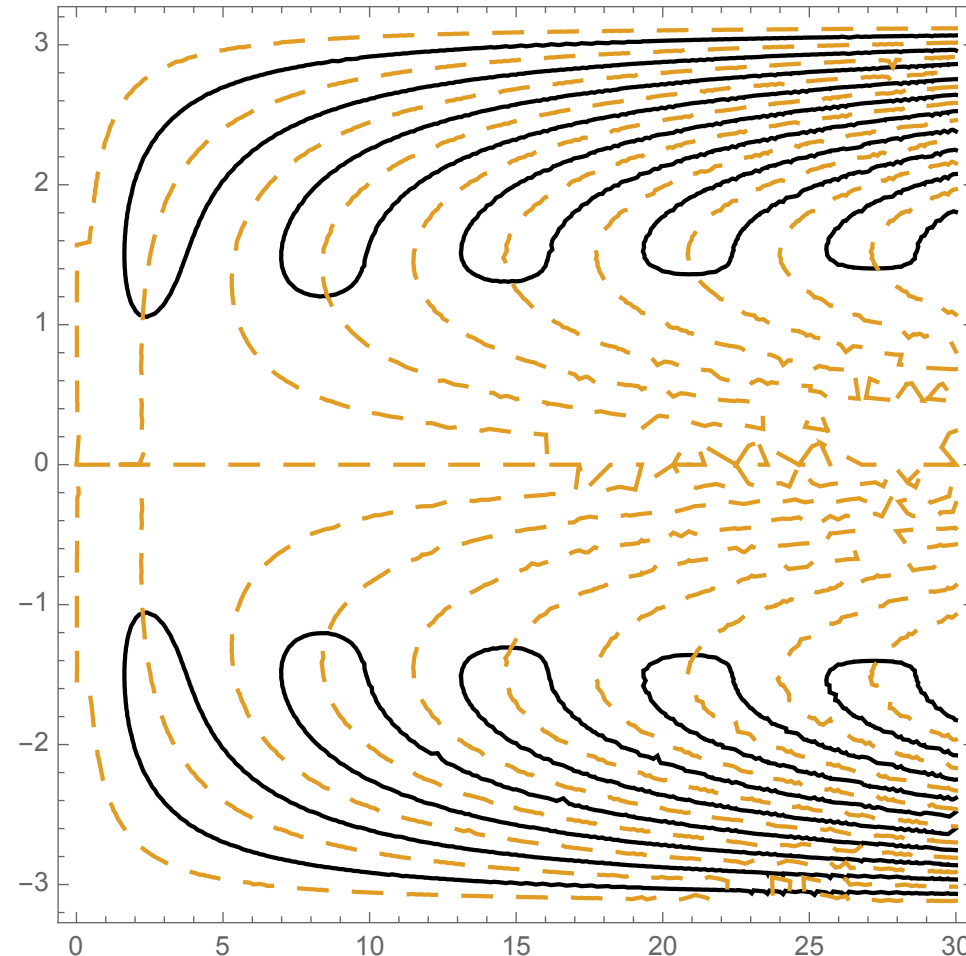
All the beta-functions do not depend on the gauge fixing condition.

Shapiro

Quantum Corrections to the Propagator

$$\mathcal{O}_R^{-1}(k) \sim \frac{V(k^2/\Lambda^2)}{k^2 \left\{ 1 + V(k^2/\Lambda^2) [c_0 k^2 + \cdots + c_N k^{2N+2}] \log\left(\frac{k^2}{\mu^2}\right) \right\}}.$$

LM, I. Shapiro.



$$2\text{Im}(\mathcal{O}^{-1}(k)) = 2\frac{\epsilon}{k^4 + \epsilon^2} \rightarrow 2\pi \delta^4(k^2).$$

X. Calmet, R. Casadio



Lee-Wick Quantum Gravity

LM, I. Shapiro.

PLB, NPB

$$S_{\text{F}} = \int d^4x \sqrt{-g} 2\kappa_4^{-2} \left[-R + 2\Lambda_{\text{cc}} - s_0 G_{\mu\nu} \square^3 R^{\mu\nu} - s_1 R^2 \square R^2 - s_2 R_{\mu\nu}^2 \square R_{\rho\sigma}^2 \right] ,$$

$$S_{\text{SR}} = \int d^4x \sqrt{|g|} 2\kappa_4^{-2} \left[-R + 2\Lambda_{\text{cc}} - s_0 G_{\mu\nu} \square R^{\mu\nu} - \sum_i (\mathbf{Riem}^3)_i \right] .$$

Unitarity $S = 1 + iT$

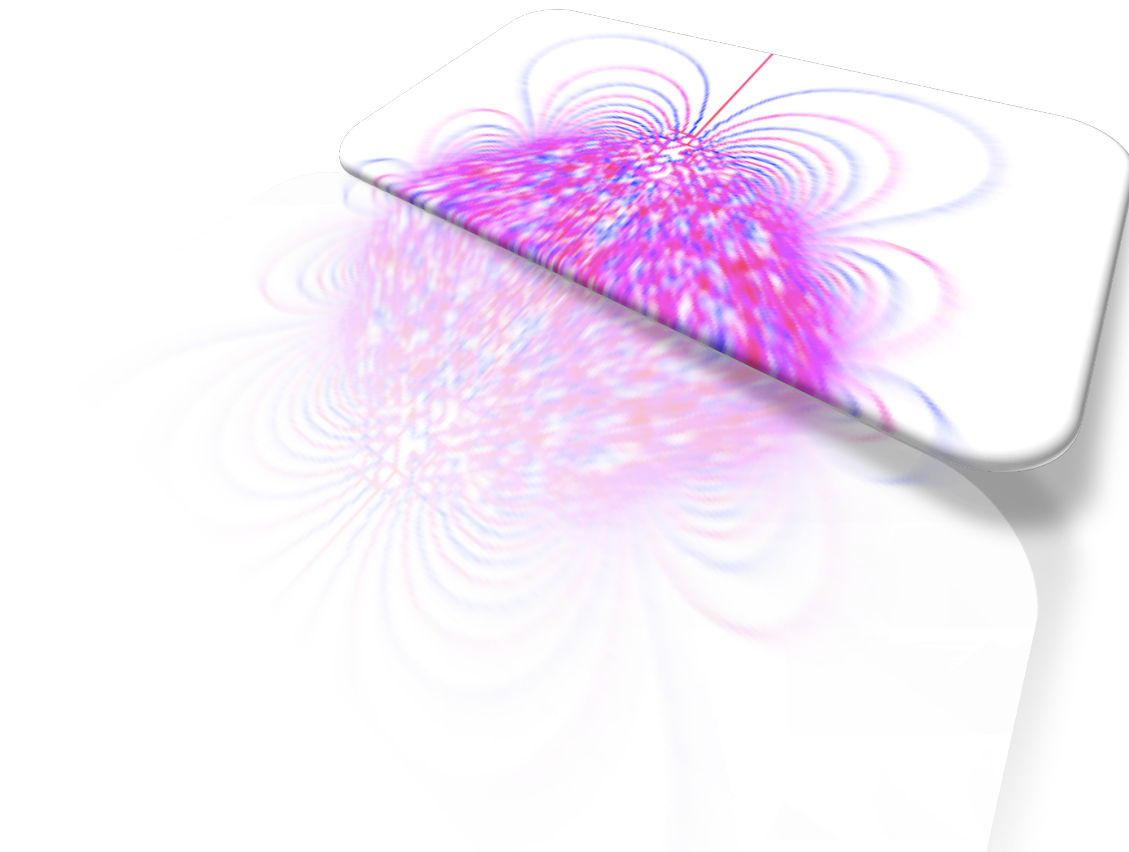
$$T_{if} = (-i)^2 \Theta^{\mu\nu} \left[\frac{1}{k^2 + i\epsilon} + \sum_n \left(\frac{c_n}{k^2 - \eta_n^2} + \frac{c_n^*}{k^2 - (\eta_n^2)^*} \right) \right] \left(P^{(2)} - \frac{P^{(0)}}{D-2} \right)_{\mu\nu, \rho\sigma} \Theta^{\rho\sigma}.$$

$$\begin{aligned} 2\text{Im } T_{if} &= 2\text{Im } (-i)^2 g^2 \left\{ \Theta_{\mu\nu} \Theta^{\mu\nu} - \frac{\Theta_\mu^{\mu 2}}{D-2} \right\} \left[\frac{k^2 - i\epsilon}{k^4 + \epsilon^2} + \sum_n \left(\frac{c_n}{k^2 - \eta_n^2} + \frac{c_n^*}{k^2 - (\eta_n^2)^*} \right) \right] \\ &= 2g^2 \left\{ \Theta_{\mu\nu} \Theta^{\mu\nu} - \frac{\Theta_\mu^{\mu 2}}{D-2} \right\} \left[\frac{\epsilon}{k^4 + \epsilon^2} \right] \rightarrow 2g^2 \left\{ \Theta_{\mu\nu} \Theta^{\mu\nu} - \frac{\Theta_\mu^{\mu 2}}{D-2} \right\} \pi \delta(k^2) \end{aligned}$$

Perturbative Unitarity

CUTKOSKY, LAHDSHOFF, OLIVE, POLKINGHORNE (CLOP).

Super-renormalizable Quantum Gravity



Summary

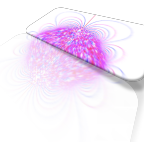
The Lagrangian

N. V. Krasnikov (1988), A.T. Tomboulis (1997), L.M. (2011), Shapiro (local theory 1996)

$$\mathcal{L} = \frac{2}{\kappa_D^2} \left(R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} \right) + c_1^{(3)} R^3 + \dots + c_1^{(N+2)} R^{N+2}.$$

Coupling constants : $\alpha_i \equiv \{\bar{\lambda}, \kappa_D^{-2}, a_n, b_n, c_i^{(3)}, \dots, c_i^{(N+2)}\}.$

Entire Function



$$V^{-1}(z) \equiv e^{H(z)} = e^{\frac{1}{2}[\Gamma(0, p^2(z)) + \gamma_E + \log(p^2(z))]} \\ \approx e^{-p^2(z)} \left(\frac{1}{2p^2(z)} - \frac{1}{2p^4(z)} + O\left(\frac{1}{p^6(z)}\right) + \dots \right) e^{\frac{\gamma_E}{2}} |p(z)| + e^{\frac{\gamma_E}{2}} |p(z)|,$$

$$p(z) = z^{\gamma+N+1}, \quad z = -\square_\Lambda = -\square/\Lambda^2.$$

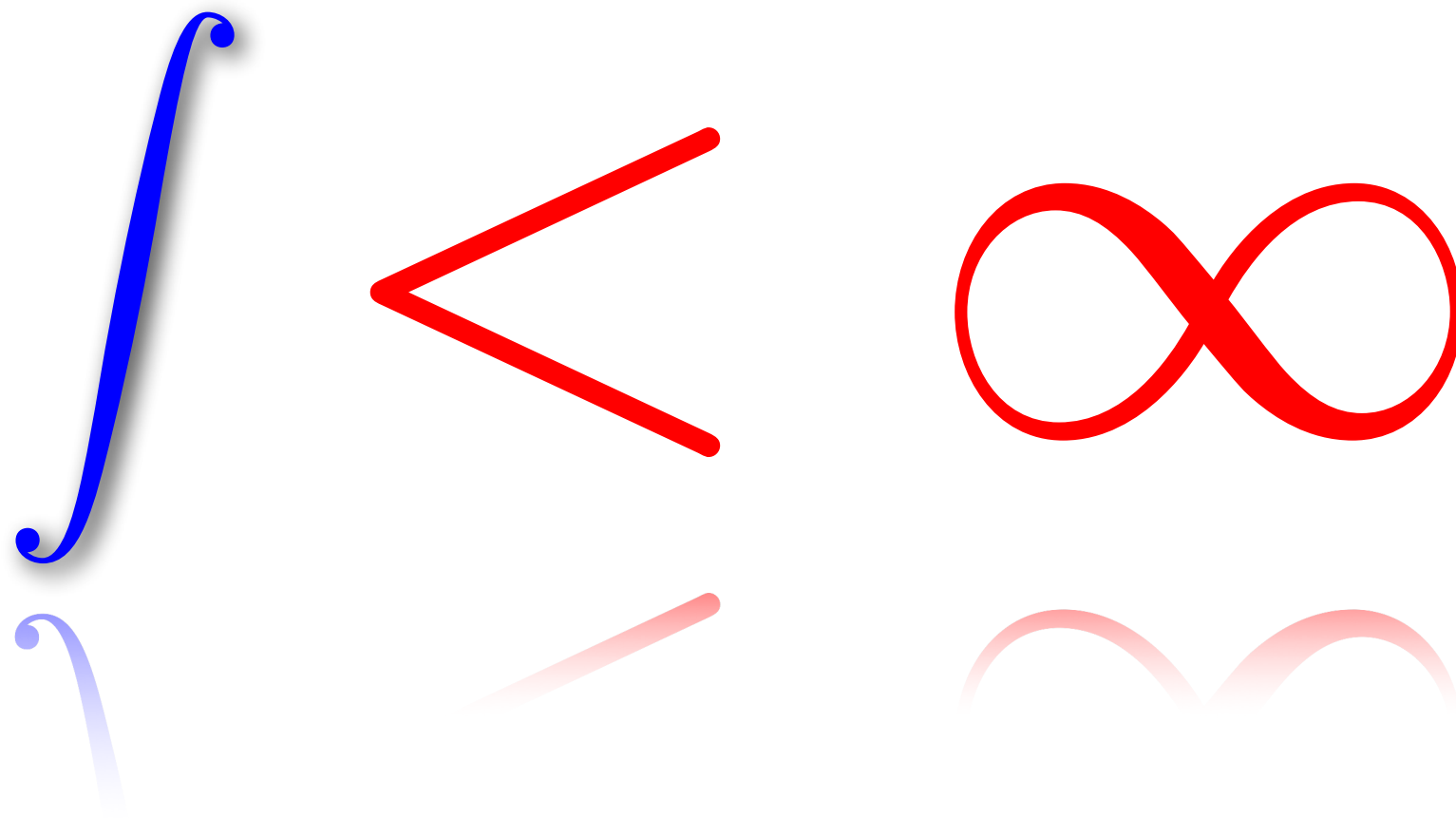
Propagator

$$\mathcal{O}^{-1}(k) = \frac{V(k^2/\Lambda^2)}{k^2} \left(P^{(2)} - \frac{P^{(0)}}{D-2} \right). \quad \mathcal{O}_R^{-1}(k) \sim \frac{V(k^2/\Lambda^2)}{k^2 \left\{ 1 + V(k^2/\Lambda^2) [c_0 k^2 + \dots + c_N k^{2N+2}] \log\left(\frac{k^2}{\mu^2}\right) \right\}}.$$

Power Counting

$$\omega(G) = D - 2\gamma(L - 1).$$

Beyond Renormalizability

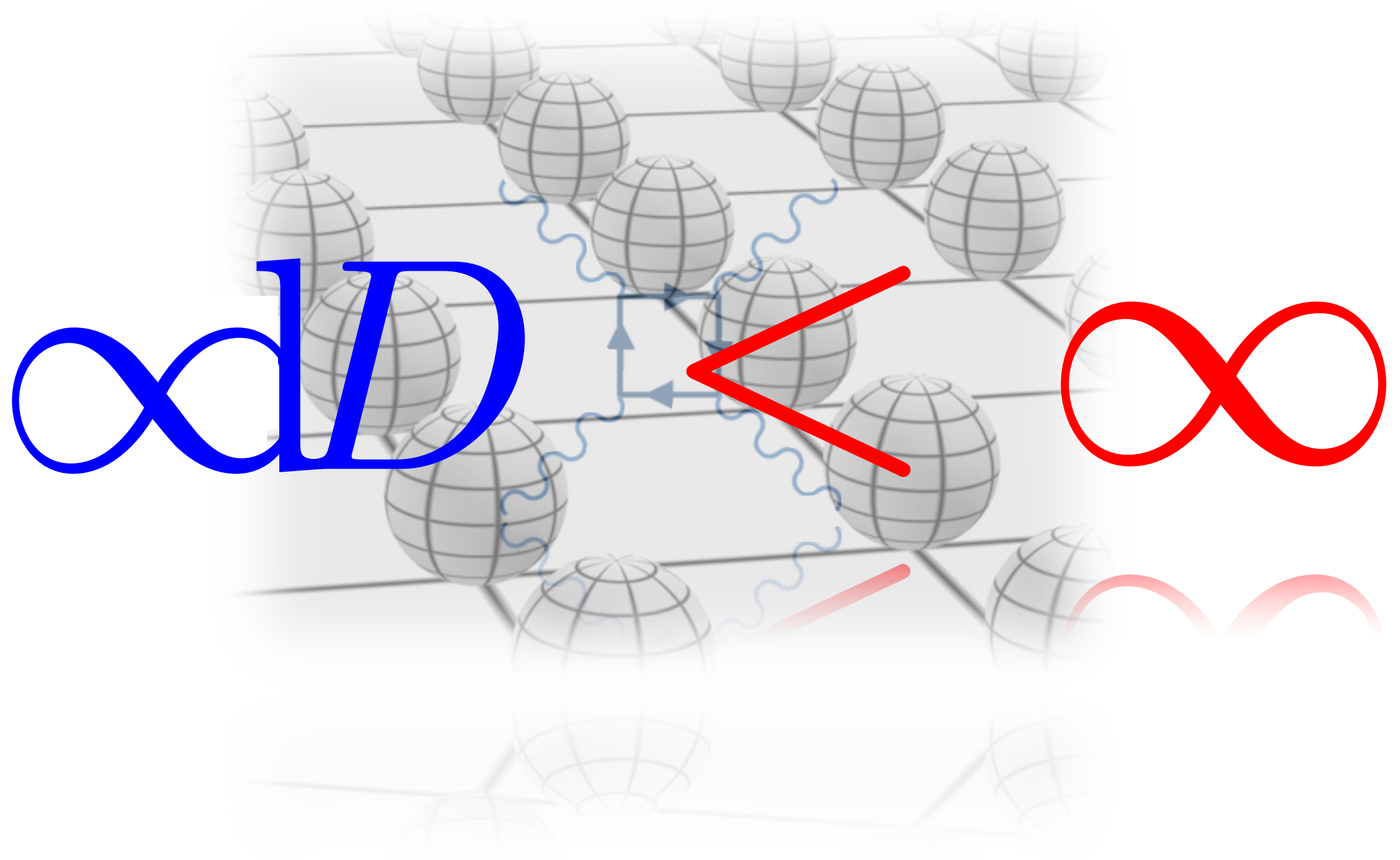


Could we make quantum gravity finite?

$$\beta_{\alpha_i} \equiv 0$$

- Multidimensional Gravity .
- Terminating Curvature Potential $\mathcal{V} \sim O(R^3)$.
- Supergravity .

i. Multidimensional Gravity



1 – Loop

No counter terms in Odd dimension



Finite Quantum Gravity in Odd dimension

$$\beta_i = 0$$

One loop effective action:

$$W_{\text{div}} = -\frac{1}{\epsilon} [c_D A_D(\Delta) + c_{D-2} A_{D-2}(\Delta) + c_{D-4} A_{D-4}(\Delta) + \cdots + c_0 A_0(\Delta)] = 0,$$

$$A_l(\Delta) \propto \int d^D x R_{\mu\nu\rho\sigma}^{l/2}.$$

$$\beta_{a_n} = \beta_{b_n} = \beta_{c_i^{(n)}} = 0, \quad i \in 1, \dots, (\text{number of invariants of order } N), \quad n = 1, \dots, N.$$

$$\mathcal{I}_{k,n} = \int d^D p \frac{(p^2)^k}{(p^2 + C)^n} = i \frac{C^{\frac{D}{2} - (n-k)}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n - k - D/2) \Gamma(k + D/2)}{\Gamma(D/2) \Gamma(n)},$$

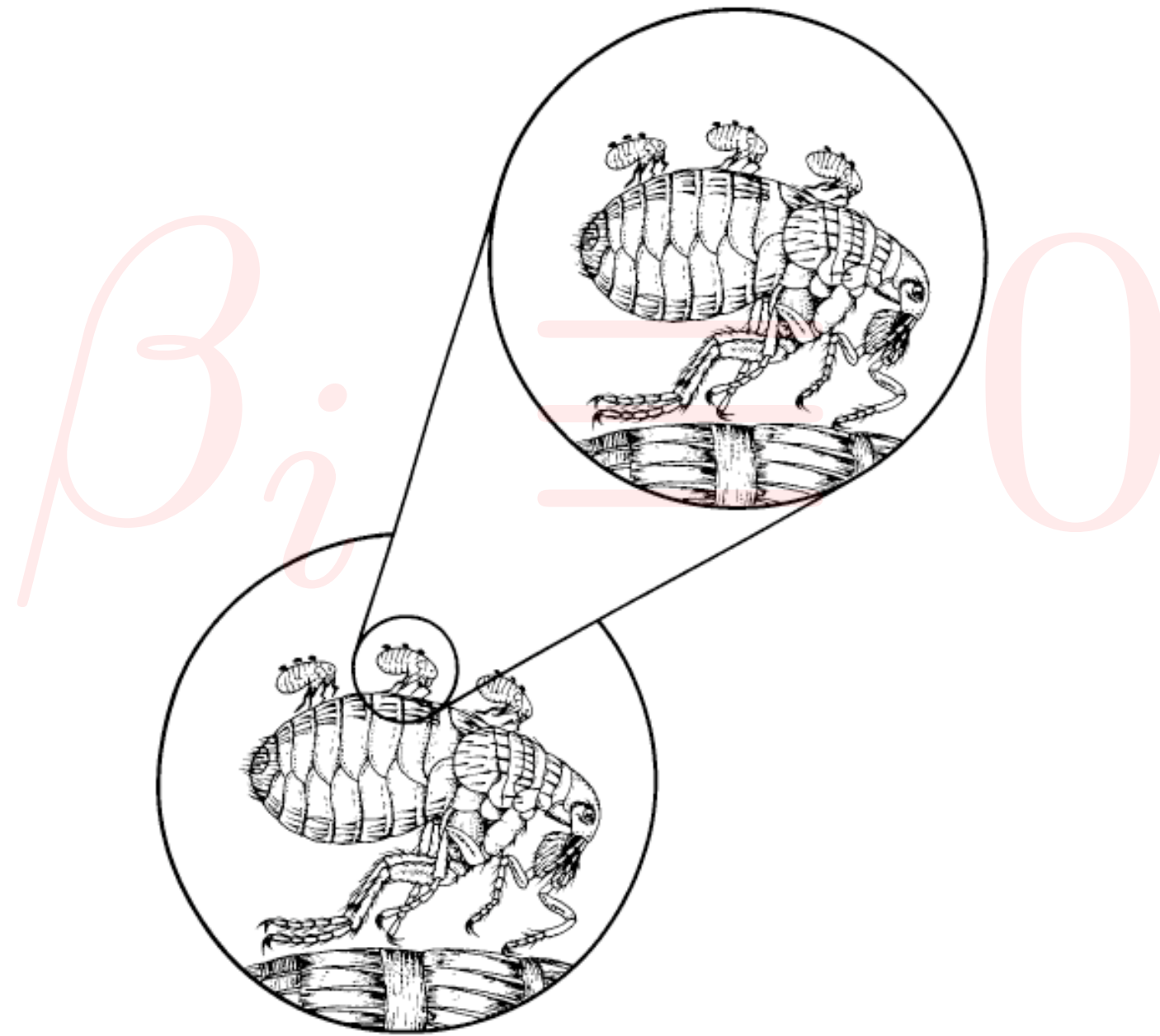
$$\mathcal{I}_{\text{null}} = \int d^D p \frac{1}{p^{2N}} \equiv 0 \quad \text{for } N < D/2.$$

New Gravity

$$\mathcal{L}_{\text{odd}} = \sqrt{|g|} 2\kappa_D^{-2} \left(R + G_{MN} \gamma(\square) R^{MN} \right).$$

$$\gamma(\square) := \frac{\overbrace{V(-\square_\Lambda)^{-1} - 1}}{\square}$$

ii. Scale Invariant Quantum Gravity



Multidimensional Theory

L.M. , Leslaw Rachwal

$$\mathcal{L} = -2 \kappa_D^{-2} \sqrt{|g|} \left(R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} + \underbrace{\sum_{n=3}^{\gamma+N+2} \alpha_{2n} \Lambda^{2-2n} \mathcal{O}_{2n}(\partial_\rho g_{\mu\nu})}_{\mathcal{V}} \right).$$

$$O_6 = \{R^3\},$$

$$O_8 = \{R^4, \nabla^2 R^3\},$$

$$O_{10} = \{R^5, \nabla^2 R^4, \nabla^4 R^3\},$$

...

...

$$O_{2\gamma+2N+4} = \{R^{\gamma+N+2}, \nabla^2 R^{\gamma+N+1}, \nabla^4 R^{\gamma+N} \dots, \nabla^{2\gamma+2N-4} R^4, \nabla^{2\gamma+2N-2} R^3\},$$

$$\mathcal{V} = \sum_i s_i R^{\frac{D}{2}}_{\dots} \nabla^{2\gamma-4} R^2_{\dots}.$$

Theory in $D = 4$

L.M. , Leslaw Rachwal

$$\mathcal{L}_g = -2\kappa_D^{-2} \sqrt{-g} \left(R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} + s_1 R^2 \square^{\gamma-2} R^2 + s_2 R_{\mu\nu} R^{\mu\nu} \square^{\gamma-2} R_{\rho\sigma} R^{\rho\sigma} \right).$$

$$\Rightarrow -2\kappa_4^{-2} \sqrt{|g|} \left[\omega_1 R \square^\gamma R + \omega_2 R_{\mu\nu} \square^\gamma R^{\mu\nu} + \textcolor{red}{s_1} R^2 \square^{\gamma-2} R^2 + \textcolor{red}{s_2} R_{\mu\nu} R^{\mu\nu} \square^{\gamma-2} R_{\rho\sigma} R^{\rho\sigma} \right],$$

$$\omega_2 = -2\omega_1 = e^{\gamma E/2} / \Lambda^{2\gamma+2}.$$

$$\Gamma_{\text{div}}^{(1)} = \frac{1}{\epsilon} \left(\beta_{R_{\mu\nu}^2} R_{\mu\nu} R^{\mu\nu} + \beta_{R^2} R^2 \right),$$

$$\begin{aligned} \beta_{R^2} &:= a_1 \textcolor{red}{s_1} + a_2 \textcolor{red}{s_2} + c_1 = 0 \\ \beta_{R_{\mu\nu}^2} &:= b_2 \textcolor{red}{s_2} + c_2 = 0 \end{aligned} \Rightarrow \begin{aligned} s_1 &= \frac{-(3\omega_1 + \omega_2)(40c_1\omega_1 + 10c_2\omega_1 + 14c_1\omega_2 - c_2\omega_2)}{48(20\omega_1 + 7\omega_2)}, \\ s_2 &= \frac{-3c_2\omega_2(3\omega_1 + \omega_2)}{2(20\omega_1 + 7\omega_2)}. \end{aligned}$$

1-loop Effective Action

Asorey, Lopez, Shapiro

Background Field Method : $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$,

$$S_{\text{gf}} = \int d^D x \sqrt{|\bar{g}|} \frac{1}{2} \chi_\mu(g) C^{\mu\nu}(\bar{g}) \chi_\nu(g),$$

$$\chi_\mu(g) = \nabla_\sigma h_\mu^\sigma - \beta_g \nabla_\mu h, \quad C^{\mu\nu}(\bar{g}) = -\frac{1}{\alpha_g} (\bar{g}^{\mu\nu} \square + \gamma_g \nabla^\mu \nabla^\nu - \nabla^\nu \nabla^\mu) \square_\Lambda^{N+\gamma},$$

$$S_{\text{gh}} = \int d^D x \sqrt{|\bar{g}|} \left[\bar{C}_\alpha M^\alpha{}_\beta(\bar{g}) C^\beta + \frac{1}{2} b_\alpha C^{\alpha\beta} b_\beta \right], \quad M^\alpha{}_\beta(\bar{g}) = \square \delta_\beta^\alpha + \nabla_\beta \nabla^\alpha - 2\beta_g \nabla^\alpha \nabla_\beta.$$

$$Z[g] = \int \mu(g, h) \prod_{\mu \leq \nu} \mathcal{D}h_{\mu\nu} \prod_\alpha \mathcal{D}\bar{C}_\alpha \prod_\beta \mathcal{D}C^\beta \prod_\gamma \mathcal{D}b_\gamma e^{i[S_{\text{HD}} + S_{\text{gf}} + S_{\text{gh}}]},$$

$$Z[g] = e^{iS_g[\bar{g}]} \left\{ \text{Det} \left[\frac{\delta^2 (S_{\text{HD}}[\bar{g} + h] + S_{\text{gf}}[\bar{g} + h])}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \right]_{h=0} \right\}^{-\frac{1}{2}} (\text{Det } M^\alpha{}_\beta(\bar{g})) (\text{Det } C^{\mu\nu}(\bar{g}))^{\frac{1}{2}}.$$

$$\hat{H}^{\mu\nu,\rho\sigma} = \frac{\delta^2 S_{\text{HD}}}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \Big|_{h=0} + \frac{\delta \chi_\delta}{\delta h_{\mu\nu}} C^{\delta\tau} \frac{\delta \chi_\tau}{\delta h_{\rho\sigma}} \Big|_{h=0}.$$

Following Shapiro and BOS:

$$\begin{aligned} \hat{H}^{\mu\nu,\alpha\beta} = & \left(\frac{\omega_{\text{Ric}}}{4} g^{\mu(\rho} g^{\nu)\sigma} - \frac{\omega_{\text{Ric}}(\omega_{\text{Ric}} + 4\omega_{\text{R}})}{16\omega_{\text{R}}} g^{\mu\nu} g^{\rho\sigma} \right) \times \left\{ \delta_{\rho\sigma}^{\alpha\beta} \square^{\gamma+2} + V_{\rho\sigma}{}^{\alpha\beta,\lambda_1 \dots \lambda_{2\gamma+2}} \nabla_{\lambda_1} \dots \nabla_{\lambda_{2\gamma+2}} + \right. \\ & \left. + W_{\rho\sigma}{}^{\alpha\beta,\lambda_1 \dots \lambda_{2\gamma+1}} \nabla_{\lambda_1} \dots \nabla_{\lambda_{2\gamma+1}} + U_{\rho\sigma}{}^{\alpha\beta,\lambda_1 \dots \lambda_{2\gamma}} \nabla_{\lambda_1} \dots \nabla_{\lambda_{2\gamma}} + O(\nabla^{2\gamma-1}) \right\}, \end{aligned}$$

The one-loop effective action is :

$$\Gamma^{(1)}[g] = -i \ln Z[g] = S_{\text{HD}}[g] + \frac{i}{2} \ln \text{Det}(\hat{H}) - i \ln \text{Det}(\hat{M}) - \frac{i}{2} \ln \text{Det}(\hat{C}).$$

$$\begin{aligned}
\text{Tr} \ln H_{\mu\nu,\alpha\beta} &= (\gamma + 2) \text{Tr} \ln [\delta_{\mu\nu,\alpha\beta} \square] + \text{Tr} V_{\mu\nu,\alpha\beta}^{\lambda_1 \lambda_2 \dots \lambda_{2\gamma+2}} \nabla_{\lambda_1} \nabla_{\lambda_2} \dots \nabla_{\lambda_{2\gamma+2}} \frac{1}{\square^{\gamma+2}} \\
&+ \text{Tr} W_{\mu\nu,\alpha\beta}^{\tau_1 \tau_2 \dots \tau_{2\gamma+1}} \nabla_{\tau_1} \nabla_{\tau_2} \dots \nabla_{\tau_{2\gamma+1}} \frac{1}{\square^{\gamma+2}} + \text{Tr} U_{\mu\nu,\alpha\beta}^{v_1 v_2 \dots v_{2\gamma}} \nabla_{v_1} \nabla_{v_2} \dots \nabla_{v_{2\gamma}} \frac{1}{\square^{\gamma+2}} \\
&- \frac{1}{2} \delta^{\rho\sigma,\delta\phi} \text{Tr} V_{\mu\nu,\rho\sigma}^{\lambda_1 \dots \lambda_{2\gamma+2}} \nabla_{\lambda_1} \dots \nabla_{\lambda_{2\gamma+2}} \frac{1}{\square^{\gamma+2}} V_{\delta\phi,\alpha\beta}^{\lambda'_1 \dots \lambda'_{2\gamma+2}} \nabla_{\lambda'_1} \dots \nabla_{\lambda'_{2\gamma+2}} \frac{1}{\square^{\gamma+2}} \\
&+ (\text{terms of higher background dimension}).
\end{aligned}$$

$$\begin{aligned}
S_{\text{HD}} &= \int d^4x \sqrt{|g|} \left[\underbrace{\bar{\lambda} - 2\kappa^{-2}R}_{\text{no div.}} + \omega_{2,\gamma} R_{\mu\nu} \square^\gamma R^{\mu\nu} + \omega_{2,\gamma-1} R_{\mu\nu} \square^{\gamma-1} R^{\mu\nu} + \omega_{2,\gamma-2} R_{\mu\nu} \square^{\gamma-2} R^{\mu\nu} + \underbrace{O(\omega_{2,\gamma-3})}_{\text{no div.}} \right. \\
&\quad + \omega_{0,\gamma} R \square^\gamma R + \omega_{0,\gamma-1} R \square^{\gamma-1} R + \omega_{0,\gamma-2} R \square^{\gamma-2} R + \underbrace{O(\omega_{0,\gamma-3})}_{\text{no div.}} \\
&\quad \left. + s_1 R^2 \square^{\gamma-2} R^2 + s_2 R_{\mu\nu} R^{\mu\nu} \square^{\gamma-2} R_{\alpha\beta} R^{\alpha\beta} + s_3 R^2 \square^{\gamma-1} R + s_4 R^2 \square^{\gamma-2} R \right].
\end{aligned}$$

$$\mathbf{V} \subset \left\{ \frac{\omega_\gamma}{\omega_\gamma} \mathcal{R}, \frac{s_3}{\omega_\gamma} \mathcal{R}, \frac{\omega_{\gamma-1}}{\omega_\gamma} \right\},$$

$$\mathbf{W} \subset \left\{ \frac{\omega_\gamma}{\omega_\gamma} \nabla \mathcal{R}, \frac{s_3}{\omega_\gamma} \nabla \mathcal{R} \right\},$$

$$\mathbf{U} \subset \left\{ \frac{\omega_\gamma}{\omega_\gamma} \mathcal{R}^2, \frac{s_{1,2,3}}{\omega_\gamma} \mathcal{R}^2, \frac{s_3}{\omega_\gamma} \nabla^2 \mathcal{R}, \frac{\omega_\gamma}{\omega_\gamma} \nabla^2 \mathcal{R}, \frac{\omega_{\gamma-1}}{\omega_\gamma} \mathcal{R}, \frac{s_4}{\omega_\gamma} \mathcal{R}, \frac{\omega_{\gamma-2}}{\omega_\gamma} \right\}.$$

Scattering Amplitudes

Scattering Amplitudes

P. Donà, S. Giaccary, L.M., L. Rachwal, Yiwei-Zhu.

$$\mathcal{L}_g = -2\kappa_D^{-2} \sqrt{g} [R + R \gamma_0(\square) R + R_{\mu\nu} \gamma_2(\square) R^{\mu\nu} + R_{\mu\nu\rho\sigma} \gamma_4(\square) R^{\mu\nu\rho\sigma}] .$$

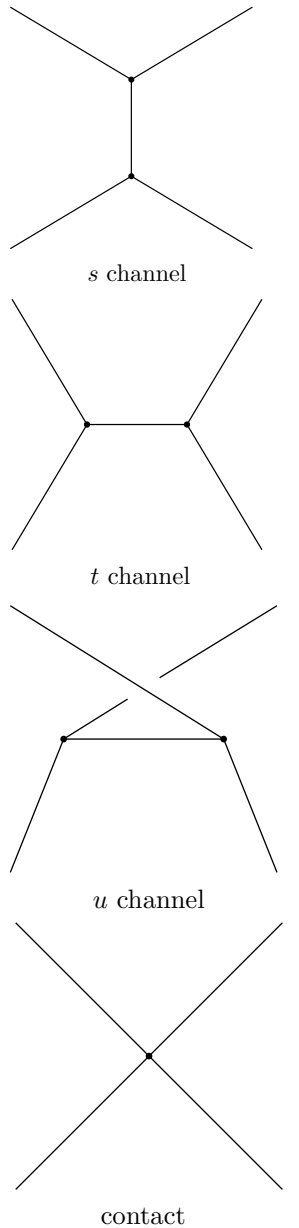
$$\gamma_4(\square) = 0,$$

$$\mathcal{A}_s(++++) = -g \frac{9}{8} \frac{t(s+t)}{s} + \frac{9}{32} \gamma_2(s) (s^2 + (s+2t)^2) + \frac{9}{8} s^2 \gamma_0(s) ,$$

$$\mathcal{A}_t(++++) = -\frac{g}{8} \frac{(s^3 - 5s^2t - st^2 + t^3)(s+t)^2}{s^3t} + \frac{1}{16} \gamma_2(t) \frac{(2s^4 - 10s^3t - s^2t^2 + 4st^3 + t^4)(s+t)^2}{s^4} + \frac{1}{8} \gamma_0(t) \frac{t^2(s+t)^4}{s^4} ,$$

$$\mathcal{A}_u(++++) = -\frac{g}{8} \frac{(s^3 - 5s^2u - su^2 + u^3)(s+u)^2}{s^3u} + \frac{1}{16} \gamma_2(u) \frac{(2s^4 - 10s^3u - s^2u^2 + 4su^3 + u^4)(s+u)^2}{s^4} + \frac{1}{8} \gamma_0(u) \frac{u^2(s+u)^4}{s^4} ,$$

$$\mathcal{A}_{\text{contact}}(++++) = -\frac{g}{4} \frac{s^4 + s^3t - 2st^3 - t^4}{s^3} + -\frac{9}{32} \gamma_2(s) (s^2 + (s+2t)^2) - \frac{9}{8} s^2 \gamma_0(s) - \frac{1}{16} \gamma_2(t) \frac{(2s^4 - 10s^3t - s^2t^2 + 4st^3 + t^4)(s+t)^2}{s^4} - \frac{1}{8} \gamma_0(t) \frac{t^2(s+t)^4}{s^4} - \frac{1}{16} \gamma_2(u) \frac{(2s^4 - 10s^3u - s^2u^2 + 4su^3 + u^4)(s+u)^2}{s^4} - \frac{1}{8} \gamma_0(u) \frac{u^2(s+u)^4}{s^4} .$$



$$\mathcal{A}(++++, +++) = \mathcal{A}_s(++++) + \mathcal{A}_t(++++) + \mathcal{A}_u(++++) + \mathcal{A}_{\text{contact}}(++++) = \mathcal{A}(++++, +++)_{\text{EH}}$$

Stelle, Weyl gravity, etc.

Scattering Amplitudes in Higher-derivative Gravities

Stelle Quadratic Gravity: $\boxed{\kappa_4^{-2}R + \alpha R^2 + \beta R_{\mu\nu}^2}$ $\rightarrow$ same of $\boxed{\kappa_4^{-2}R}$ (ext. gravitons).

Weyl Gravity $\boxed{\alpha C_{\mu\nu\rho\sigma}^2}$ $\rightarrow$ same of $\boxed{0}$ (ext. gravitons).

Nonlocal Gravity: $\boxed{R + R\gamma_0(\square)R + R_{\mu\nu}\gamma_0(\square)R^{\mu\nu}}$ $\rightarrow$ same of $\boxed{\kappa_4^{-2}R}$.

Theorem. All the n-point functions in any gravitational theory
(in particular super-renormalizable or finite) with an action

$$\mathcal{L}_{\text{gr}} = -2\kappa_D^{-2} \sqrt{-g} [\mathbf{R} + \mathbf{R} \gamma_0(\square) \mathbf{R} + \mathbf{Ric} \gamma_2(\square) \mathbf{Ric} + \mathbf{V}(\mathbf{R}, \mathbf{Ric}, \mathbf{Riem}, \nabla)] ,$$

give the same on-shell tree-level amplitudes as the Einstein-Hilbert theory,

$$\mathcal{L}_{\text{EH}} = -2\kappa_D^{-2} \sqrt{-g} \mathbf{R}, \quad \text{provided that the potential } \mathbf{V} \text{ is at least quadratic in } \mathbf{Ric} \text{ and/or } \mathbf{R}.$$

In particular for any theory in which we can recast the potential in the following form

$$\mathbf{V} = \mathbf{Ric} \cdot \tilde{\mathbf{V}} \cdot \mathbf{Ric} \equiv R_{\mu\nu} [\tilde{\mathbf{V}}(\mathbf{R}, \mathbf{Ric}, \mathbf{Riem}, \nabla)]^{\mu\nu\rho\sigma} R_{\rho\sigma} ,$$

the theorem is valid.

Proof. Based on the Anselmi's field redefinition theorem.

$$S(g') \equiv S_{\text{EH}}(g') \equiv S_{\text{EH}}(g) + R_{\mu\nu}(g) F^{\mu\nu,\rho\sigma}(g) R_{\rho\sigma}(g) = S'(g) \equiv S_{\text{gr}}(g).$$

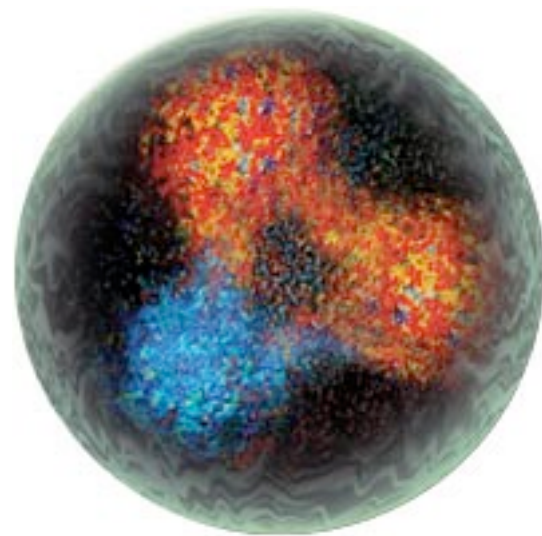
$$g'_i = g_i + \Delta_{ij} E_j = g_i + (F_{ij} + \dots) E_j .$$

Question. Is the theory really nonlocal (perturbatively)?

No!

- * The tree-level n-point scattering amplitudes are the same of the local Einstein theory.
- * At one-loop the non locality makes the difference, but any theory is nonlocal at quantum level.

Gauge Theory



Universal Gauge & Gravitational Theories

$$\boxed{-\frac{1}{2g^2} \int d^4x \operatorname{tr} F^2 \text{ in } D=4} \quad \text{like} \quad \boxed{-\frac{2}{\kappa_2^2} \int d^2x \sqrt{|g|} R \text{ in } D=2}$$

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4g_{\text{YM}}^2} \left[\operatorname{tr} F e^{H(-\mathcal{D}_\Lambda^2)} F + \mathcal{V}_{\text{YM}} \right] \quad (\text{GAUGE} - \text{THEORY}).$$

$$\mathcal{L}_{\text{gr}} = -2\kappa_D^{-2} \sqrt{|g|} \left[R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} + \mathcal{V}_{\text{gr}} \right] \quad (\text{GRAVITY}).$$

Finite Yang-Mills Gauge theory

L.M. , M. Piva, L. Rachwal.

$$\mathcal{L}_{\text{fin, gauge}} = -\frac{\alpha}{4} \text{tr} \left[\mathbf{F} e^{H(\mathcal{D}_\Lambda^2)} \mathbf{F} + \frac{s_g}{\Lambda^4} \mathbf{F}^2 (\mathcal{D}_\Lambda^2)^{\gamma-2} \mathbf{F}^2 \right]$$

Killer Operators

1. $-\frac{s_g}{4g^2 \Lambda^4} F_{\mu\nu}^a F_a^{\mu\nu} \square_\Lambda^{\gamma-2} (F_{\rho\sigma}^b F_b^{\rho\sigma}) ,$
2. $-\frac{s_g}{4g^2 \Lambda^4} F_{\mu\nu}^a F_b^{\mu\nu} (\mathcal{D}_\Lambda^2)^{\gamma-2} (F_{\rho\sigma}^b F_a^{\rho\sigma}) .$

$$\mathcal{L}_{\text{ct}} := -\frac{\alpha}{4}(Z_\alpha - 1) F_a^{\mu\nu} F_{\mu\nu}^a = -\mathcal{L}_{\text{div}} = -\frac{1}{\epsilon} \beta_\alpha F_a^{\mu\nu} F_{\mu\nu}^a.$$

$$\beta_\alpha^{(\gamma)} = -\frac{5 + 3\gamma + 12\gamma^2}{192\pi^2} C_2(G), \quad \gamma \geq 2,$$

$$\textcolor{red}{1.} \quad \beta_\alpha^{(s_g)} = \frac{s_g}{2\pi^2\omega},$$

$$\textcolor{red}{2.} \quad \beta_\alpha^{(s_g)} = \frac{s_g}{4\pi^2\omega} (1 + N_G).$$

$$\beta_\alpha^{(\gamma)} + \beta_\alpha^{(s_g)} = 0 \quad \Longrightarrow \quad s_g^* = -2\pi^2\omega\beta_\alpha^{(\gamma)}.$$

$$\mathcal{L}_{\text{fin, gauge}} = -\frac{\alpha}{4} \left[F_{\mu\nu}^a e^{H(\mathcal{D}_\Lambda^2)} F_a^{\mu\nu} + \omega \frac{(5 + 3\gamma + 12\gamma^2)}{96\Lambda^4} C_2(G) F_{\mu\nu}^a F_a^{\mu\nu} \square_\Lambda^{\gamma-2} (F_{\rho\sigma}^b F_b^{\rho\sigma}) \right].$$

Asymptotic Freedom, Finiteness, and Landau Pole

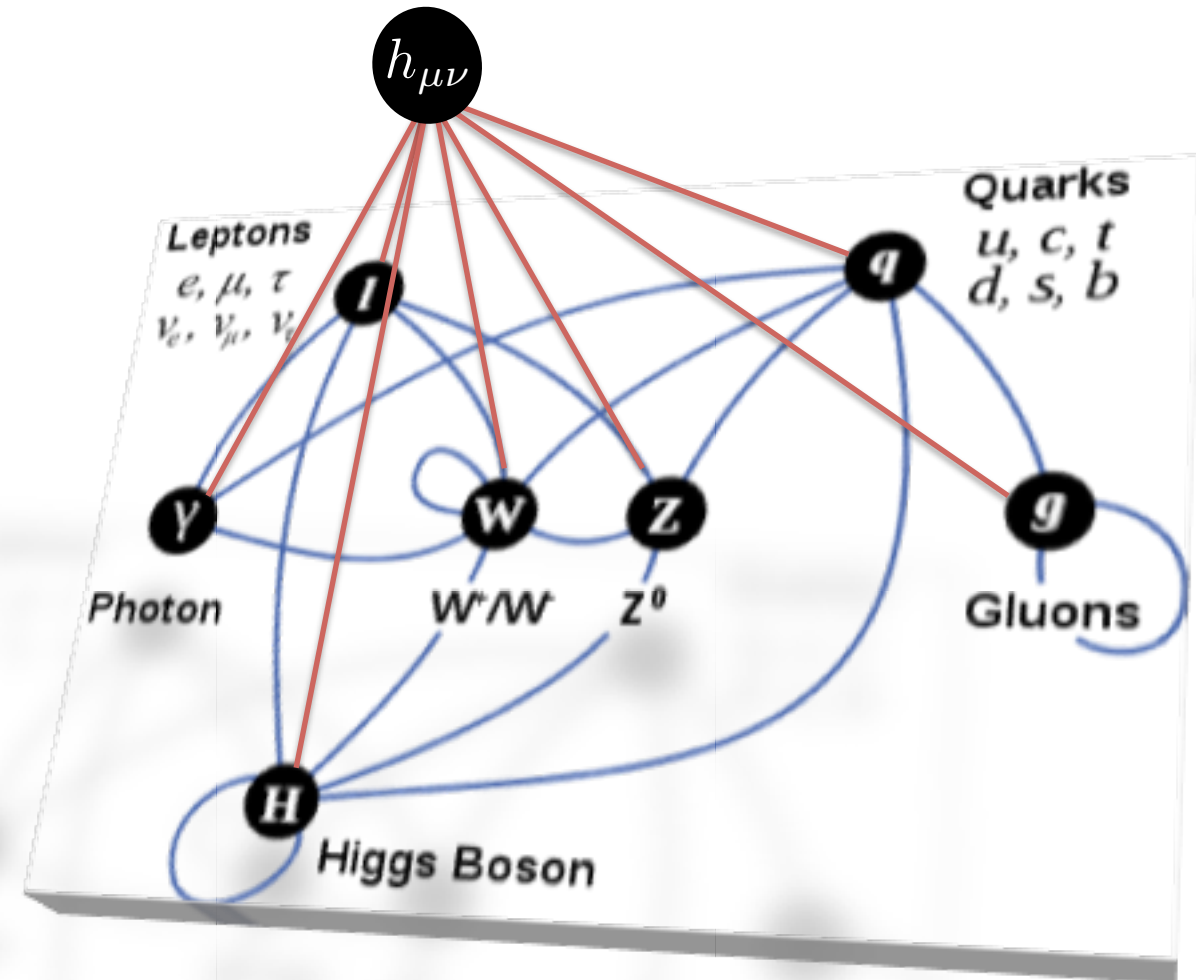
$$s_g = \begin{cases} < \frac{5+3\gamma+12\gamma^2}{96} \omega C_2(G), & \text{Landau pole,} \\ = \frac{5+3\gamma+12\gamma^2}{96} \omega C_2(G) \equiv s_g^*, & \text{finiteness,} \\ > \frac{5+3\gamma+12\gamma^2}{96} \omega C_2(G), & \text{asymptotic freedom.} \end{cases}$$

Dressed Propagator

$$-i \frac{e^{-H(k^2)}}{k^2 (1 + \beta_\alpha e^{-H(k^2)} \log(k^2/\mu_0^2))}.$$

Matter

$$\begin{aligned} \mathcal{L}_{\text{StandardModel}} = & -\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4}g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \\ & \frac{1}{2}ig_s^2 (\bar{q}_i^\alpha \gamma^\mu q_j^\alpha) g_\mu^a + \bar{G}^a \partial^2 G^a + g_s f^{abc} \partial_\mu \bar{G}^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- - \\ & M^2 W_\mu^+ W_\mu^- - \frac{1}{2}\partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2c_w^2} M^2 Z_\mu^0 Z_\mu^0 - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \\ & \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h \left[\frac{2M^2}{g^2} + \right. \\ & \left. \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-) \right] + \frac{2M^4}{g^2} \alpha_h - igc_w [\partial_\nu Z_\mu^0 (W_\mu^+ W_\nu^- - \\ & W_\mu^- W_\nu^+) - Z_\mu^0 (W_\mu^+ \partial_\nu W_\nu^- - W_\mu^- \partial_\nu W_\nu^+) + Z_\mu^0 (W_\nu^+ \partial_\mu W_\mu^- - \\ & W_\nu^- \partial_\mu W_\mu^+)] - ig s_w [\partial_\nu A_\mu (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+) - A_\nu (W_\mu^+ \partial_\mu W_\mu^- - \\ & W_\mu^- \partial_\mu W_\mu^+) + A_\mu (W_\nu^+ \partial_\nu W_\nu^- - W_\nu^- \partial_\nu W_\nu^+)] - \frac{1}{2}g^2 W_\mu^+ W_\mu^- W_\nu^+ W_\nu^- + \\ & \frac{1}{2}g^2 W_\mu^+ W_\mu^- W_\nu^+ W_\nu^- + g^2 c_w^2 (Z_\mu^0 W_\mu^+ Z_\nu^0 W_\nu^- - Z_\mu^0 Z_\nu^0 W_\mu^+ W_\nu^-) + \\ & g^2 s_w^2 (A_\mu W_\mu^+ A_\nu W_\nu^- - A_\mu A_\nu W_\mu^+ W_\nu^-) + g^2 s_w c_w [A_\mu Z_\nu^0 (W_\mu^+ W_\nu^- - \\ & W_\mu^- W_\nu^+) - 2A_\mu Z_\mu^0 W_\nu^+ W_\nu^-] - g\alpha [H^3 + H\phi^0 \phi^0 + 2H\phi^+ \phi^-] - \\ & \frac{1}{8}g^2 \alpha_h [H^4 + (\phi^0)^4 + 4(\phi^+ \phi^-)^2 + 4(\phi^0)^2 \phi^+ \phi^- + 4H^2 \phi^+ \phi^- + 2(\phi^0)^2 H^2] - \\ & gMW_\mu^+ W_\mu^- H - \frac{1}{2}g \frac{M}{c_w^2} Z_\mu^0 Z_\mu^0 H - \frac{1}{2}ig[W_\mu^+ (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\ & W_\mu^- (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \frac{1}{2}g[W_\mu^+ (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - W_\mu^- (H \partial_\mu \phi^+ - \\ & \phi^+ \partial_\mu H)] + \frac{1}{2}g \frac{1}{c_w} (Z_\mu^0 (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - ig \frac{M}{c_w} Z_\mu^0 (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \\ & ig s_w M A_\mu (W_\mu^+ \phi^- - W_\mu^- \phi^+) - ig \frac{1-2c_w^2}{2c_w} Z_\mu^0 (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + \\ & ig s_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \frac{1}{4}g^2 W_\mu^+ W_\mu^- [H^2 + (\phi^0)^2 + 2\phi^+ \phi^-] - \\ & \frac{1}{4}g^2 \frac{1}{c_w^2} Z_\mu^0 Z_\mu^0 [H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2 \phi^+ \phi^-] - \frac{1}{2}g^2 \frac{s_w^2}{c_w} Z_\mu^0 \phi^0 (W_\mu^+ \phi^- + \\ & W_\mu^- \phi^+) - \frac{1}{2}ig^2 \frac{s_w^2}{c_w} Z_\mu^0 H (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \frac{1}{2}g^2 s_w A_\mu \phi^0 (W_\mu^+ \phi^- + \\ & W_\mu^- \phi^+) + \frac{1}{2}ig^2 s_w A_\mu H (W_\mu^+ \phi^- - W_\mu^- \phi^+) - g^2 \frac{s_w}{c_w} (2c_w^2 - 1) Z_\mu^0 A_\mu \phi^+ \phi^- - \\ & g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma^\partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma^\partial \nu^\lambda - \bar{u}_j^\lambda (\gamma^\partial + m_u^\lambda) u_j^\lambda - \\ & \bar{d}_j^\lambda (\gamma^\partial + m_d^\lambda) d_j^\lambda + ig s_w A_\mu [-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) - \frac{1}{3}(\bar{d}_j^\lambda \gamma^\mu d_j^\lambda)] + \\ & \frac{ig}{2c_w} Z_\mu^0 [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (\frac{4}{3}s_w^2 - \\ & 1 - \gamma^5) u_j^\lambda) + (\bar{d}_j^\lambda \gamma^\mu (1 - \frac{8}{3}s_w^2 - \gamma^5) d_j^\lambda)] + \frac{ig}{2\sqrt{2}} W_\mu^+ [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + \\ & (\bar{u}_j^\lambda \gamma^\mu (1 + \gamma^5) C_{\lambda\kappa} d_j^\kappa)] + \frac{ig}{2\sqrt{2}} W_\mu^- [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}_j^\kappa C_{\lambda\kappa}^\dagger \gamma^\mu (1 + \\ & \gamma^5) u_j^\lambda)] + \frac{ig}{2\sqrt{2}} \frac{m_h^2}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] - \\ & \frac{g}{2} \frac{m_h^2}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 - \gamma^5) d_j^\kappa) + \\ & m_u^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 + \gamma^5) d_j^\kappa)] + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^\kappa (\bar{d}_j^\kappa C_{\lambda\kappa}^\dagger (1 + \gamma^5) u_j^\kappa) - m_u^\kappa (\bar{d}_j^\kappa C_{\lambda\kappa}^\dagger (1 - \\ & \gamma^5) u_j^\kappa)] - \frac{g}{2} \frac{m_h^2}{M} H (\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2} \frac{m_h^2}{M} H (\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_h^2}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \\ & \frac{ig}{2} \frac{m_h^2}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda) + \bar{X}^+ (\partial^2 - M^2) X^+ + \bar{X}^- (\partial^2 - M^2) X^- + \bar{X}^0 (\partial^2 - \\ & \frac{M^2}{c_w^2}) X^0 + \bar{Y} \partial^2 Y + igc_w W_\mu^+ (\partial_\mu \bar{X}^0 X^- - \partial_\mu \bar{X}^+ X^0) + ig s_w W_\mu^+ (\partial_\mu \bar{Y} X^- - \\ & \partial_\mu \bar{X}^+ Y) + igc_w W_\mu^- (\partial_\mu \bar{X}^- X^0 - \partial_\mu \bar{X}^0 X^+) + ig s_w W_\mu^- (\partial_\mu \bar{X}^- Y - \\ & \partial_\mu \bar{Y} X^+) + igc_w Z_\mu^0 (\partial_\mu \bar{X}^+ X^+ - \partial_\mu \bar{X}^- X^-) + ig s_w A_\mu (\partial_\mu \bar{X}^+ X^+ - \\ & \partial_\mu \bar{X}^- X^-) - \frac{1}{2}gM[\bar{X}^+ X^+ H + \bar{X}^- X^- H + \frac{1}{c_w} \bar{X}^0 X^0 H] + \\ & \frac{1-2c_w^2}{2c_w} igM[\bar{X}^+ X^0 \phi^+ - \bar{X}^- X^0 \phi^-] + \frac{1}{2c_w} igM[\bar{X}^0 X^- \phi^+ - \bar{X}^0 X^+ \phi^-] + \\ & igMs_w[\bar{X}^0 X^- \phi^+ - \bar{X}^0 X^+ \phi^-] + \frac{1}{2}igM[\bar{X}^+ X^+ \phi^0 - \bar{X}^- X^- \phi^0]. \end{aligned}$$



$$\begin{aligned} & i\partial\mathcal{V}^m[\chi_0 X - \phi_+ - \chi_0 X + \phi_-] + \frac{5}{1}\partial\mathcal{V}[\chi_+ X + \phi_0 - \chi_- X - \phi_0] \cdot \\ & \frac{5c_m}{1-5c_m} i\partial\mathcal{V}[\chi_+ X_0 \phi_+ - \chi_- X_0 \phi_-] + \frac{5c_m}{1}\partial\mathcal{V}[\chi_0 X - \phi_+ - \chi_0 X + \phi_-] + \\ & 9^h \chi_- X_- - \frac{5}{1}\partial\mathcal{V}[\chi_+ X + H + \chi_- X - H + \frac{c_m}{1}\chi_0 X_0 H] + \\ & 9^h \chi_+ X_+ + i\partial c_m \Sigma_0^h (9^h \chi_+ X_+ - 9^h \chi_- X_-) + i\partial z^m \nabla^h (9^h \chi_+ X_+ - \\ & 9^h \chi_- X_-) + i\partial c_m \Lambda_-^h (9^h \chi_- X_0 - 9^h \chi_0 X_+) + i\partial z^m \Lambda_-^h (9^h \chi_- X_- - \\ & \frac{c_m}{1}\chi_0) \chi_0 + \lambda_0 \partial_5 \lambda_+ + \Lambda_+^h (9^h \chi_0 X_- - 9^h \chi_+ X_0) + i\partial z^m \Lambda_+^h (9^h \chi_+ X_+ - \\ & \frac{3}{1}\chi_0) \chi_0 + (\partial^2 + \lambda_+^2) \chi_- Y + (\partial_- - \lambda_-) \chi_+ Y + \chi_- (\partial_- - \lambda_-) \chi_+ Y + \chi_+ (\partial_+ - \lambda_+) \chi_- Y \end{aligned}$$

Coupling to Matter

Di Xu, L. Rachwal, L.M..

- Gravity does not affect the matter beta functions for $\gamma > 1$.

Standard model beta function remain untouched.

- Conversely, matter fields change the gravity beta functions $\rightarrow$ no more super-renormalizability.

The result is exactly the one evaluated for semiclassical gravity (gravitons can only be external lines)
(N_s scalars, N_f spinors, N_v vectors),

$$\Gamma_{\text{div}}^{(1)} = \frac{1}{\epsilon} \frac{1}{16\pi^2} \int d^4x \sqrt{|g|} [\beta_{\bar{\lambda}} + \beta_R R + \beta_{C^2} C_{\mu\nu\rho\sigma}^2 + \beta_{R^2} R^2],$$

$$\beta_{\bar{\lambda}} = \frac{1}{2} m_s^4 - 4 m_f^4,$$

$$\beta_R = N_s m_s^2 \left(\xi - \frac{1}{6} \right) + \frac{2}{3} N_f^2 m_f^2,$$

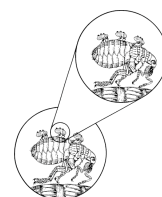
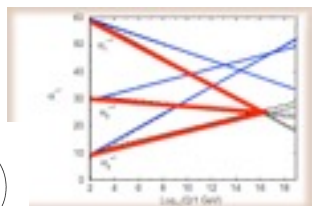
$$\beta_{C^2} = \frac{N_s}{120} + \frac{N_f}{20} + \frac{N_v}{10},$$

$$\beta_{R^2} = \frac{1}{2} N_s \left(\xi - \frac{1}{6} \right)^2.$$

Shapiro (Lectures in Shanghai)

A nonlocal extension of the standard model of particle physics restores super-renormalizability,
(all the leading kinetic and interaction operators for matter and gravity have the same scaling.)

- * In the theory we have now more room to accommodate asymptotic freedom and/or unification for all the four fundamental interactions.
- * Moreover, if other local operators are added to the non-local action we can construct a completely scale invariant standard model of particle physics.



Super-renormalizable or Finite Standard Model

$$\sum_a^{N_f} \bar{\psi}_a i\gamma^\mu \mathcal{D}_{\mu,a} e^{H(\gamma^\mu \mathcal{D}_{\mu,a}^2)} \psi_a + (\mathcal{D}_\mu \Phi)^\dagger e^{H(\mathcal{D})} (\mathcal{D}^\mu \Phi) - \mu^2 \Phi^\dagger e^{H(\mathcal{D})} \Phi - \lambda (\Phi^\dagger \Phi)^2.$$

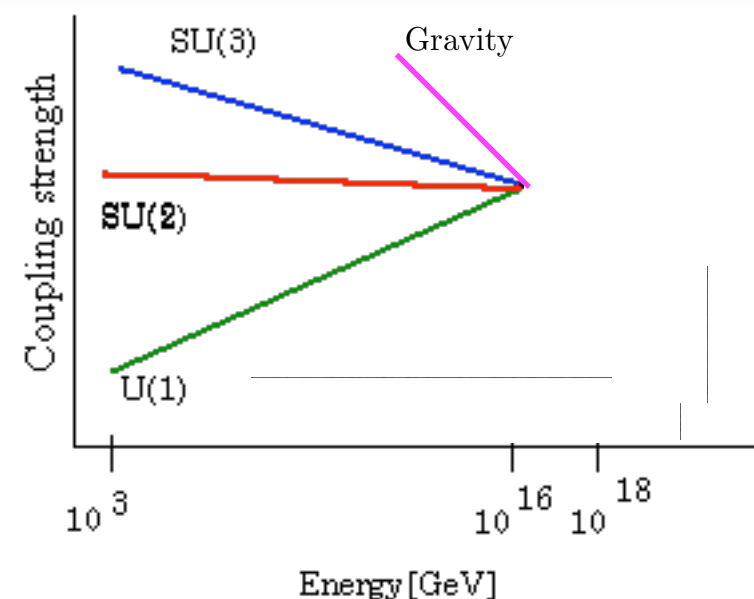
Killers for β_{R^2} and $\beta_{R_{\mu\nu}^2}$: $R^2 \square^{\gamma-2} R^2$ and $R_{\mu\nu}^2 \square^{\gamma-2} R_{\mu\nu}^2$,

Three killers for $\beta_{F^2}^{\text{SU}(3) \times \text{SU}(2) \times U(1)}$:

$$s_A F_A^2 \square^{\gamma_A-2} F_A^2, \quad s_W F_W^2 \square^{\gamma_W-2} F_W^2, \quad s_B F_B^2 \square^{\gamma_B-2} F_B^2.$$

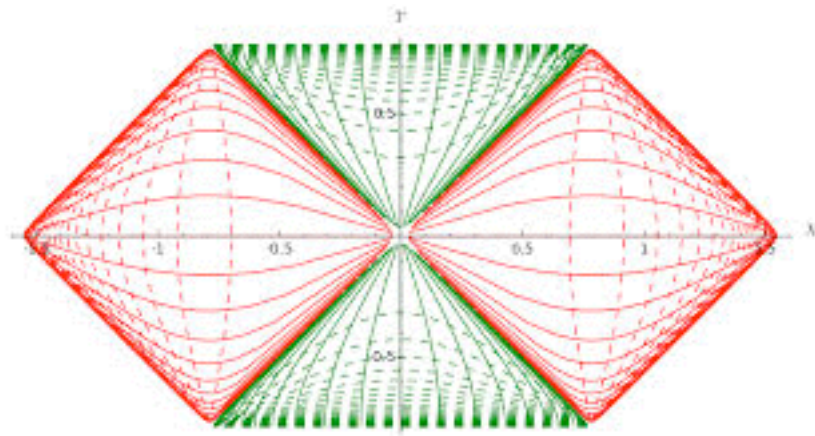
Asorey, Lopez, Shapiro

$s_A, s_B, s_W \implies$ Unification with $G_N/G_{N,0}$.



Exact Solutions, Spacetime Singularities, Conformal Invariance

L.M., L. Rachwal (2014)



Singularity Theorem in non-local gravity

L.M. , Leslaw Rachwal

Given the EOM : $\mathbf{E} = 8\pi G_N \mathbf{T}$,

the following implication turns out to be true,

- $R_{\mu\nu} = 0 \quad \& \quad T_{\mu\nu} = 0 \quad \implies \quad E_{\mu\nu} = 0;$
- $R_{\mu\nu} = 8\pi G_N T_{\mu\nu} \quad \& \quad T^\mu_\mu = 0 \quad \implies \quad E_{\mu\nu} = 8\pi G_N T_{\mu\nu}.$

Therefore,

- all **Ricci flat** spacetimes are exact solutions;
- all the **FRW** spacetime sourced by conformally coupled matter are exact solutions.

The theorem works for any potential at least quadratic in the Bach tensor.

EOM

$$E_{\mu\nu} = \frac{\delta \left[\sqrt{|g|} \left(R + R\gamma_S(\square)R + B_{\alpha\beta}\gamma_B(\square)B^{\alpha\beta} + V(B) \right) \right]}{\sqrt{|g|}\delta g^{\mu\nu}} = 8\pi G_N T_{\mu\nu} ,$$

$$\begin{aligned} & G_{\mu\nu} - \frac{1}{2}g_{\mu\nu} (R\gamma_S(\square)R) - \frac{1}{2}g_{\mu\nu} (B_{\alpha\beta}\gamma_B(\square)B^{\alpha\beta}) \\ & + 2\frac{\delta R}{\delta g^{\mu\nu}} (\gamma_S(\square)R) + \frac{\delta B_{\alpha\beta}}{\delta g^{\mu\nu}} (\gamma_B(\square)B^{\alpha\beta}) + \frac{\delta B^{\alpha\beta}}{\delta g^{\mu\nu}} (\gamma_B(\square)B_{\alpha\beta}) \\ & + \frac{\delta \square^r}{\delta g^{\mu\nu}} \left(\frac{\gamma_S(\square^l) - \gamma_S(\square^r)}{\square^l - \square^r} R R \right) + \frac{\delta \square^r}{\delta g^{\mu\nu}} \left(\frac{\gamma_B(\square^l) - \gamma_B(\square^r)}{\square^l - \square^r} B_{\alpha\beta} B^{\alpha\beta} \right) + \frac{1}{\sqrt{|g|}} \frac{\delta V(B)}{\delta g^{\mu\nu}} = 8\pi G_N T_{\mu\nu}. \end{aligned}$$

Proof.

- Ricci flat spacetimes :

$$R_{\mu\nu} = 0 \quad \& \quad T_{\mu\nu} = 0 \quad \implies B_{\mu\nu} = 0 \quad \implies E_{\mu\nu} = 0 .$$

Schwarzschild, Kerr, Kasner, etc. are **EXACT SOLUTIONS**.

EOM

$$\begin{aligned}
 & G_{\mu\nu} - \frac{1}{2}g_{\mu\nu} (R\gamma_S(\square)R) - \frac{1}{2}g_{\mu\nu} (B_{\alpha\beta}\gamma_B(\square)B^{\alpha\beta}) \\
 & + 2\frac{\delta R}{\delta g^{\mu\nu}} (\gamma_S(\square)R) + \frac{\delta B_{\alpha\beta}}{\delta g^{\mu\nu}} (\gamma_B(\square)B^{\alpha\beta}) + \frac{\delta B^{\alpha\beta}}{\delta g^{\mu\nu}} (\gamma_B(\square)B_{\alpha\beta}) \\
 & + \frac{\delta \square^r}{\delta g^{\mu\nu}} \left(\frac{\gamma_S(\square^l) - \gamma_S(\square^r)}{\square^l - \square^r} RR \right) + \frac{\delta \square^r}{\delta g^{\mu\nu}} \left(\frac{\gamma_B(\square^l) - \gamma_B(\square^r)}{\square^l - \square^r} B_{\alpha\beta} B^{\alpha\beta} \right) + \frac{1}{\sqrt{|g|}} \frac{\delta V(B)}{\delta g^{\mu\nu}} = 8\pi G_N T_{\mu\nu}.
 \end{aligned}$$

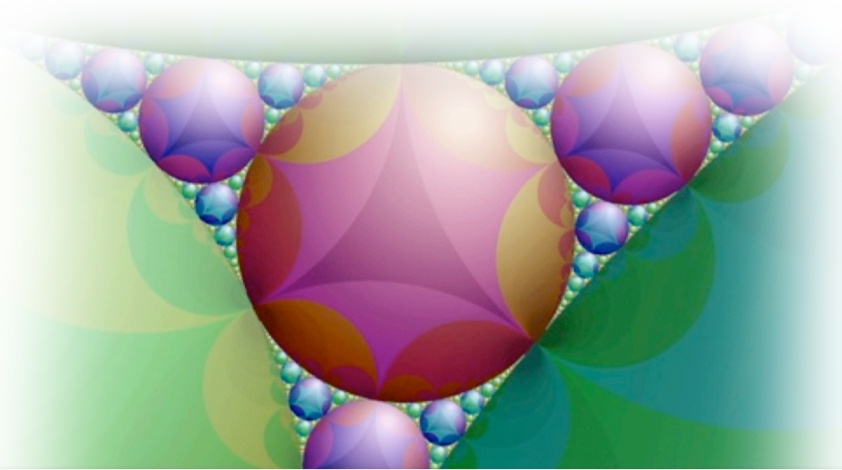
Proof.

- FRW spacetimes :

$$\text{FRW} \implies G_{\mu\nu} - \frac{1}{2}g_{\mu\nu} (R\gamma_S(\square)R) + 2\frac{\delta R}{\delta g^{\mu\nu}} (\gamma_S(\square)R) + \frac{\delta \square^r}{\delta g^{\mu\nu}} \left(\frac{\gamma_S(\square^l) - \gamma_S(\square^r)}{\square^l - \square^r} RR \right) = T_{\mu\nu} .$$

$$\text{tr } \mathbf{T} = 0 \implies R = 0 \implies \mathbf{G} = 8\pi G_N \mathbf{T} \implies \boxed{\text{Big Bang Singularity}} .$$

FRW for Radiation is an **EXACT SOLUTION** .



Conformal Invariant Gravity

$$g_{\mu\nu} = (\phi^2 \kappa_D^2)^{\frac{2}{D-2}} \hat{g}_{\mu\nu}, \quad \hat{g}_{\mu\nu} \rightarrow \Omega^2(x) \hat{g}_{\mu\nu}, \quad \phi \rightarrow \Omega^{\frac{2-D}{2}}(x) \phi.$$

Einstein conformal gravity

$$\mathcal{L}_4 = -2\kappa_4^{-2} \sqrt{|g|} R(g) = -12\phi \left(-\square + \frac{1}{6}R \right) \phi.$$

Nonlocal conformal gravity

$$\mathcal{L}_g = -2 \sqrt{\hat{g}} \left[\phi^2 R(\hat{g}) + \frac{4(D-1)}{D-2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] - \frac{2}{\kappa_D^2} \sqrt{g} [\mathbf{R}(g) \gamma_0(\square) \mathbf{R}(g) + \mathbf{Ric}(g) \gamma_2(\square) \mathbf{Ric}(g) + \mathbf{V}(g)] \Big|_{\phi \hat{g}}.$$

Quantum nonlocal conformal gravity

(Fradkin, Tsytlín, Percacci.)

$$g_{\mu\nu} = \overbrace{(\bar{\phi} + \varphi)^2}^{\phi^2} \left(\overbrace{\bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}}^{\hat{g}_{\mu\nu}} \right),$$

$$\begin{aligned} Z(\bar{g}_{\mu\nu} = \bar{\phi}^2 \bar{g}_{\mu\nu}) &= \int \mathcal{D} \left[\phi(-\hat{g})^{\frac{1}{8}} \right] \mathcal{D} \left[\hat{g}_{\mu\nu}(-\hat{g})^{-\frac{1}{4}} \right] \Delta_{\text{FP}}^{\text{diff}}(g) \delta(\chi(g)^\alpha - \ell^\alpha) \Delta_{\text{FP}}^{\text{conf}}(\phi) \delta(\chi(\phi) - \ell) e^{iS(g)} \Big|_{g_{\mu\nu} = \phi^2 \hat{g}_{\mu\nu}} \\ &= \int e^{\frac{1}{4} \delta^4(0) \int d^4x \log \sqrt{-\hat{g}}} \mathcal{D}\varphi \mathcal{D} \left[\hat{g}_{\mu\nu}(-\hat{g})^{-\frac{1}{4}} \right] \det(C_{\alpha\beta}(\bar{g}))^{\frac{1}{2}} e^{\frac{1}{2} \chi^\alpha(g) C_{\alpha\beta}(\bar{g}) \chi^\beta(g)} \det(M_{\alpha\beta}^{\text{diff}}(\bar{g})) \det(\bar{\phi} + \varphi) \delta(\varphi - \ell) e^{iS(g)} \Big|_{\phi \hat{g}}, \end{aligned}$$

$$\begin{aligned} Z(\bar{g}_{\mu\nu} = \bar{\phi}^2 \bar{g}_{\mu\nu}) &= \det(C_{\alpha\beta}(\bar{g}))^{\frac{1}{2}} \det(M_{\alpha\beta}^{\text{diff}}(\bar{g})) \int \mathcal{D} \left[\hat{g}_{\mu\nu}(-\hat{g})^{-\frac{1}{4}} \right] e^{i[S(g) + S_{\text{gf}}(g)]} \Big|_{g_{\mu\nu} = \bar{\phi}^2 \hat{g}_{\mu\nu}} \\ &= \det(C_{\alpha\beta}(\bar{g}))^{\frac{1}{2}} \det(M_{\alpha\beta}^{\text{diff}}(\bar{g})) \left\{ \int \mathcal{D} \left[g_{\mu\nu}(-g)^{-\frac{1}{4}} \right] e^{i[S(g) + S_{\text{gf}}(g)]} \Big|_{g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}} \right\} \Big|_{\bar{g}_{\mu\nu} = \bar{\phi}^2 \bar{g}_{\mu\nu}} \\ &= \det(C_{\alpha\beta}(\bar{g}))^{\frac{1}{2}} \det(M_{\alpha\beta}^{\text{diff}}(\bar{g})) \left\{ \int e^{-\frac{1}{2} \delta^4(0) \int d^4x \log \sqrt{-g}} \mathcal{D}h_{\mu\nu} e^{i[S(g) + S_{\text{gf}}(g)]} \Big|_{g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}} \right\} \Big|_{\bar{g}_{\mu\nu} = \bar{\phi}^2 \bar{g}_{\mu\nu}} \\ &= \det(C_{\alpha\beta}(\bar{g}))^{\frac{1}{2}} \det(M_{\alpha\beta}^{\text{diff}}(\bar{g})) \left\{ \int \mathcal{D}h_{\mu\nu} e^{i[S(g) + S_{\text{gf}}(g)]} \Big|_{g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}} \right\} \Big|_{\bar{g}_{\mu\nu} = \bar{\phi}^2 \bar{g}_{\mu\nu}}. \end{aligned}$$

Counterterms

$$\sqrt{|g|}R^2, \quad \sqrt{|g|}R_{\mu\nu}^2, \quad \sqrt{|g|}R, \quad \sqrt{|g|} = \sqrt{|\hat{g}|}\phi^4, \quad g_{\mu\nu} = \phi^2 \kappa_4^2 \hat{g}_{\mu\nu}.$$

... all conformal invariant.

In DIMREG :

$$\frac{1}{\epsilon} \phi^{-\epsilon} \mathcal{O}_i(\phi^2 \hat{g}; \epsilon) \approx \frac{1}{\epsilon} (1 - \epsilon \log \phi) \mathcal{O}_i(\phi^2 \hat{g}; \epsilon) = \frac{1}{\epsilon} \mathcal{O}_i(\phi^2 \hat{g}) + \tilde{\mathcal{O}}_i(\phi^2 \hat{g}) - \log(\phi) \mathcal{O}_i(\phi^2 \hat{g}),$$

$$\sqrt{|g|} = \phi^{D-\epsilon} \sqrt{|\hat{g}|} = \overbrace{\phi^D \sqrt{|\hat{g}|}}^{\text{conf. inv.}} \phi^{-\epsilon} = \sqrt{|g|} \phi^{-\epsilon}.$$

Gauge Fixing or Spontaneous Symmetry Breaking

$$\hat{g}_{\mu\nu} \rightarrow \Omega^2(x) \hat{g}_{\mu\nu} , \quad \phi \rightarrow \Omega^{\frac{2-D}{2}}(x) \phi .$$

$$\underbrace{g_{\mu\nu}}_{10} = \underbrace{\phi^2}_{1 \text{ comp.}} \times \underbrace{\hat{g}_{\mu\nu}}_{10 \text{ comp.}} \quad \text{or} \quad \underbrace{g_{\mu\nu}}_{10} = \underbrace{\phi^2}_{1 \text{ comp.}} \times \underbrace{\hat{g}_{\mu\nu}}_{9 \text{ comp.}} .$$

Splitting : $g_{\mu\nu} := e^{2\omega(x)} \hat{g}_{\mu\nu} .$

U(1) Toy Analog Model

$$\mathcal{L} = \frac{1}{4} \mathbf{F}(\mathbf{A})^2 - |D_\mu \phi|^2, \quad 2 (A) + 1 (\phi)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu = \partial_\mu - ig A_\mu, \quad \phi(x) = \frac{1}{\sqrt{2}} v e^{i \frac{\theta(x)}{v}} \quad (v = \text{const. and } [v] = 1),$$

$$\phi'(x) = e^{ig\alpha(x)} \phi(x) \quad \text{or} \quad \theta'(x) = \theta(x) + b v \alpha(x) \quad \text{and,} \quad A'_\mu = A_\mu + \partial_\mu \alpha(x),$$

$$D_\mu \phi = -\frac{1}{\sqrt{2}} ig v e^{i \frac{\theta(x)}{v}} \left(A_\mu - \frac{1}{vg} \partial_\mu \theta(x) \right).$$

$$B_\mu = A_\mu - \partial_\mu \theta(x)/vg \quad \implies \quad \mathcal{L} = \frac{1}{4} \mathbf{F}(\mathbf{B})^2 - \frac{1}{2} g^2 v^2 \mathbf{B}^2. \quad 3 (A) + 0 (\phi)$$

Or imposing $\theta(x) = \text{const.}$,

$$|D_\mu \phi|^2 = \frac{1}{2} g^2 v^2 A_\mu A^\mu \quad \implies \quad \mathcal{L} = \frac{1}{4} \mathbf{F}(\mathbf{A})^2 - \frac{1}{2} g^2 v^2 \mathbf{A}^2.$$

Before spontaneous symmetry breaking:

gauge theory (3 d.o.f.)	gravity (10 d.o.f.)
$A_\mu(x)$ (2 d.o.f.)	$\hat{g}_{\mu\nu}(x)$ (9 d.o.f.)
$\theta(x)$ (1 d.o.f.)	$\phi(x)$ (1 d.o.f.)

After spontaneous symmetry breaking:

gauge theory (3 d.o.f.)	gravity (10 d.o.f.)
$B_\mu(x)$ (3 d.o.f.) gauge invariant	$g_{\mu\nu}(x) = \phi^2 \hat{g}_{\mu\nu}$ (10 d.o.f.) conformal invariant

Equivalently

gauge theory (3 d.o.f.)	gravity (10 d.o.f.)
$\theta(x) = \text{const.}$	$\phi(x) = \text{const.} = 1$
$A_\mu(x) \equiv B_\mu(x)$ (3 d.o.f.)	$\hat{g}_{\mu\nu}(x) = g_{\mu\nu}$ (10 d.o.f.)

Spacetime Singularities and Conformal Invariance

Narlikar, Kembhavi (1977).

If $(\hat{g}_{\mu\nu}, \phi)$ is a solution $\implies (\hat{g}_{\mu\nu}^*, \phi^*)$ is a solution, where $\hat{g}_{\mu\nu}^* = \Omega^2 \hat{g}_{\mu\nu}$, $\phi^* = \Omega^{\frac{2-D}{2}} \phi$.

FRW :
$$ds^{*2} = S(t)ds^2, \quad ds^2 = a^2(t)(-dt^2 + d\vec{x}^2);$$
$$S(t)^{-1} = a^2(t).$$

Belinski, Khalatnikov, Lifshitz

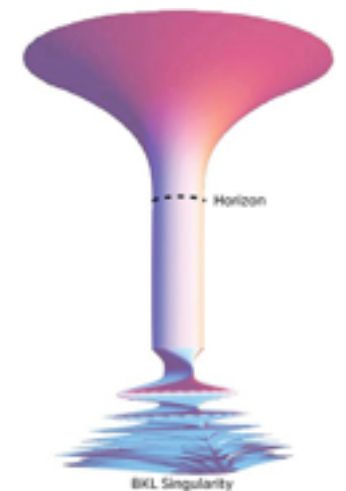
BKL :
$$ds^2 = -dt^2 + a_1(t)dx_1^2 + a_2(t)dx_2^2 + a_3(t)dx_3^2,$$

$$a_i(t) = t^{2p_i} \quad (i = 1, 2, 3),$$

$$\sum_i^3 p_i = \sum_i^3 p_i^2 = 1;$$

$$\hat{\text{Riem}}^2(\hat{\mathbf{g}}_{\mu\nu}^*) = \frac{4 \left(\frac{4(L^2+t^2)^4}{u^2+u+1} - \frac{8(L^2+t^2)^4}{(u^2+u+1)^2} + \frac{4(L^2+t^2)^4}{(u^2+u+1)^3} + 3L^4 (L^4 + 4L^2t^2 + 8t^4) \right)}{(L^2 + t^2)^6}.$$

$$ds^{*2} = S(t) ds^2 = \frac{t^2 + L^2}{t^2} [-dt^2 + a_1(t)dx_1^2 + a_2(t)dx_2^2 + a_3(t)dx_3^2].$$



Schwarzschild Singularity

$$g_{\mu\nu}^{\text{Sch}} = (\phi \kappa_D)^{\frac{4}{D-2}} \hat{g}_{\mu\nu} = (\phi^* \kappa_D)^{\frac{4}{D-2}} \hat{g}_{\mu\nu}^* \iff \hat{g}_{\mu\nu}^* = \Omega^2 \hat{g}_{\mu\nu}, \quad \phi^* = \Omega^{\frac{2-D}{2}} \phi.$$

$$ds^{*2} \equiv \hat{g}_{\mu\nu}^* dx^\mu dx^\nu = S(r) \hat{g}_{\mu\nu} dx^\mu dx^\nu = S(r) \left[\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^{(2)} \right],$$
$$\phi^* = S(r)^{-1/2} \kappa_4^{-1}.$$

Schwarzschild Singularity = one element of the gauge orbit

$$g_{\mu\nu}^{\text{Sch}} = (\phi \kappa_D)^{\frac{4}{D-2}} \hat{g}_{\mu\nu} = (\phi^* \kappa_D)^{\frac{4}{D-2}} \hat{g}_{\mu\nu}^* \iff \hat{g}_{\mu\nu}^* = \Omega^2 \hat{g}_{\mu\nu}, \quad \phi^* = \Omega^{\frac{2-D}{2}} \phi.$$

$$S(r) = \frac{1}{r^2} \left(\frac{L^4}{r^2} + r^2 \right),$$

$$ds^{*2} = -\frac{1}{r^2} \left(\frac{L^4}{r^2} + r^2 \right) \left(1 - \frac{2m}{r} \right) dt^2 + \frac{1}{r^2} \left(\frac{L^4}{r^2} + r^2 \right) \frac{dr^2}{1 - \frac{2m}{r}} + \left(\frac{L^4}{r^2} + r^2 \right) d\Omega^{(2)}.$$

$$\hat{\mathbf{K}} = \frac{1}{(L^4 + r^4)^6} [16r^2(L^{16}(39m^2 - 20mr + 3r^2) + 2L^{12}r^4(66m^2 - 32mr + 3r^2) + L^8r^8(342m^2 - 284mr + 63r^2) + 12L^4m^2r^{12} + 3m^2r^{16})].$$

$$\text{Hawking Temperature : } T_H = \frac{1}{8\pi m} \quad \forall S(r).$$

$$\boxed{r = 0 \text{ Singularity in Conformal Gravity}} \equiv \boxed{r = 2m \text{ Singularity in GR}}.$$

Geodesic Completion

Massive particle probe:

$$S_m = -m \int \sqrt{-\hat{g}_{\mu\nu} dx^\mu dx^\nu} = -m \int \sqrt{-\hat{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda,$$

Proper time gauge : $\frac{d\hat{s}^2}{d\tau^2} = -1.$

Conformally coupled probe:

$$S_m = - \int \sqrt{-f^2 \phi^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu} = - \int \sqrt{-f^2 \phi^2 \hat{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda,$$

Proper time gauge : $\frac{d\hat{s}^2}{d\tau^2} = -1,$

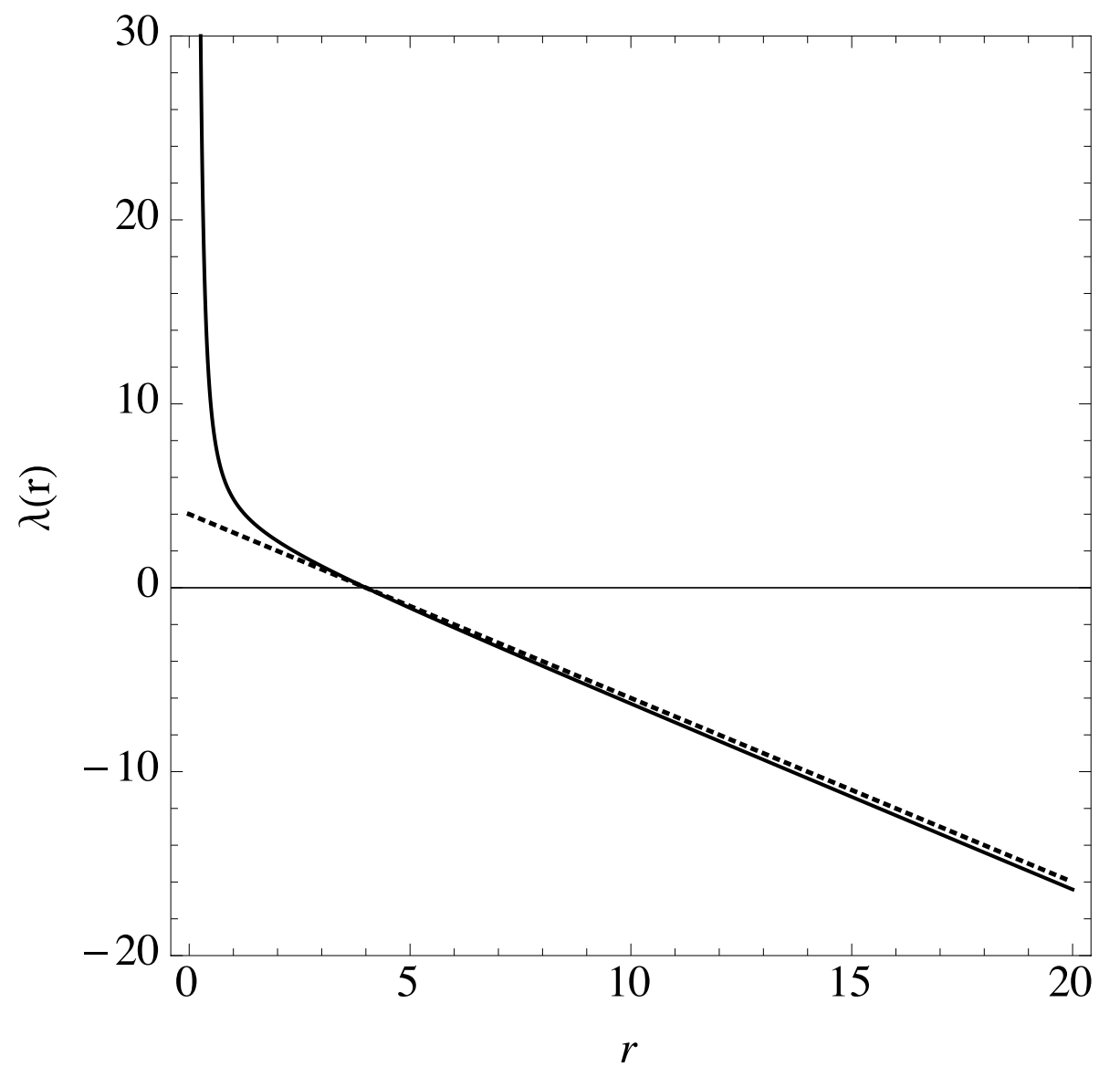
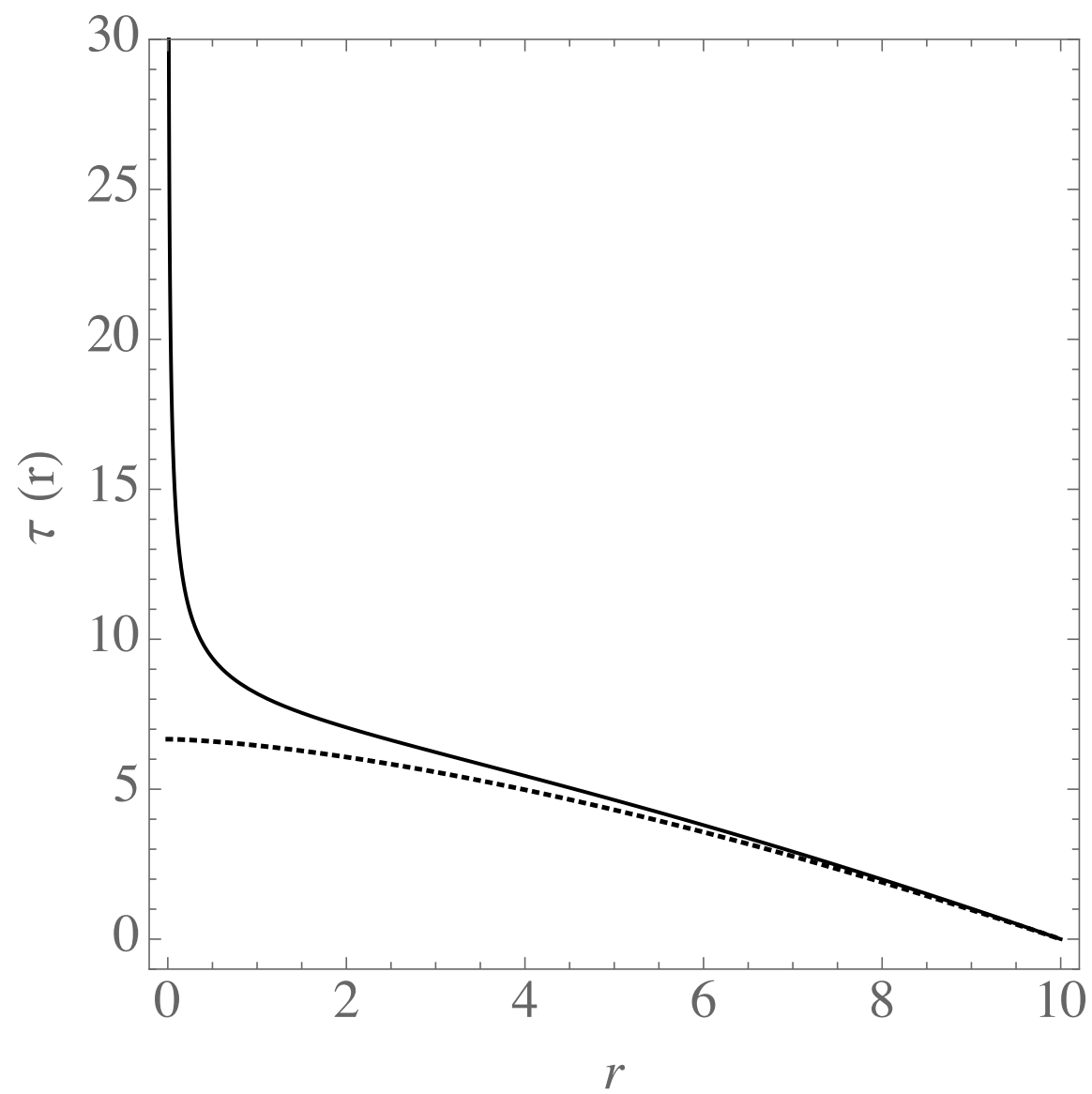
solution : $\tau = \frac{4M^2 - 3L^2}{3M} - \frac{(r^2 - 3L^2) \sqrt{\frac{2r}{M}}}{3r}.$

Light: $d\hat{s}^2 = 0,$

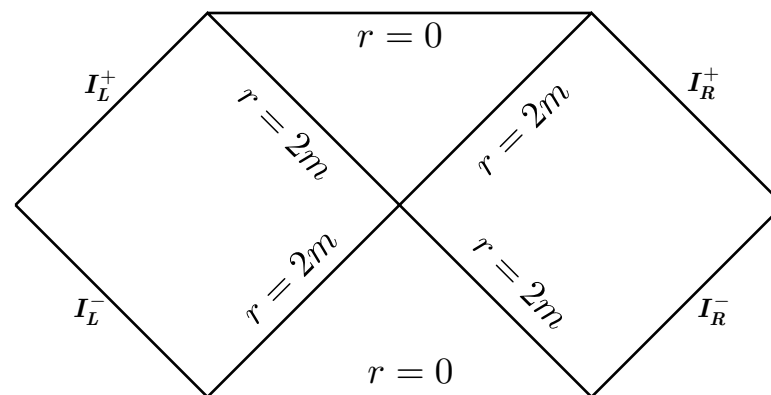
Killing vectors: $\xi^\alpha = (1, 0, 0, 0), \quad \eta^\alpha = (0, 0, 0, 1),$

$$e = -\xi \cdot u = -\xi^\alpha u^\beta \hat{g}_{\alpha\beta} = -\hat{g}_{tt} u^t = S(r) \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} = S(r) \left(1 - \frac{2M}{r}\right) \dot{t},$$

solution : $\lambda(r) = \frac{1}{e} \left[\frac{L^4}{3r^3} - \frac{L^4}{3r_0^3} + \frac{2L^2}{r} - \frac{2L^2}{r_0} - r + r_0 \right].$

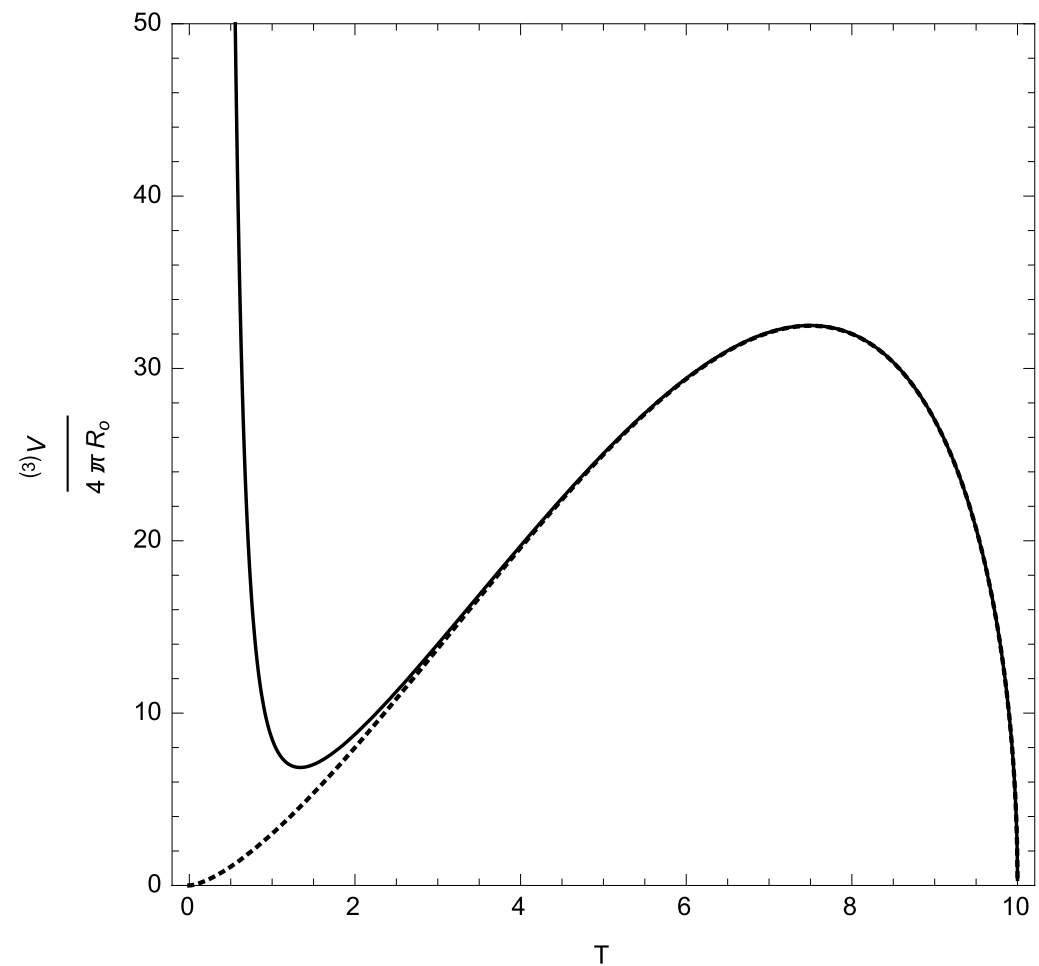
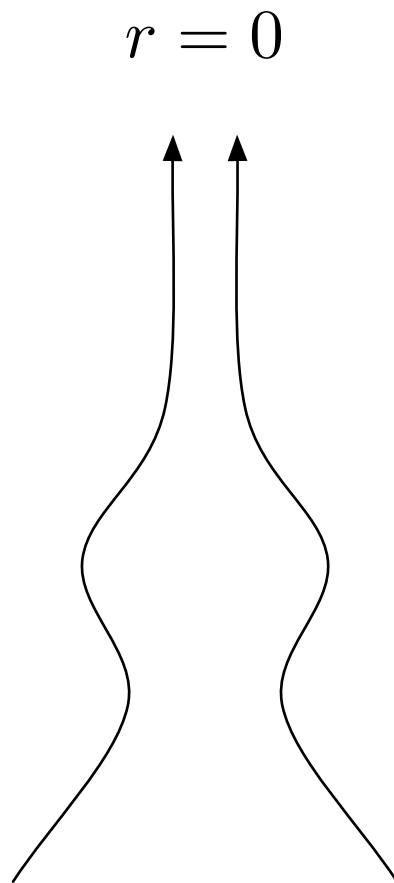


Spacetime Structure



Raychaudhuri Expansion Parameter

Bambi, M., Rachwal, Wang.



$$\lim_{r \rightarrow 0} \theta_{\text{Ray}} = 0.$$

Kerr Singularity

Bambi, M., Rachwal, Wang.

$$ds^{*2} \equiv \hat{g}_{\mu\nu}^* dx^\mu dx^\nu = S(r, \theta) \hat{g}_{\mu\nu}^{\text{Kerr}} dx^\mu dx^\nu ,$$

$$\phi^* = S(r, \theta)^{-1/2} \kappa_4^{-1} ,$$

$$S(r) = \left(1 + \frac{L^2}{r^2 + a^2 \cos^2 \theta} \right)^4 .$$

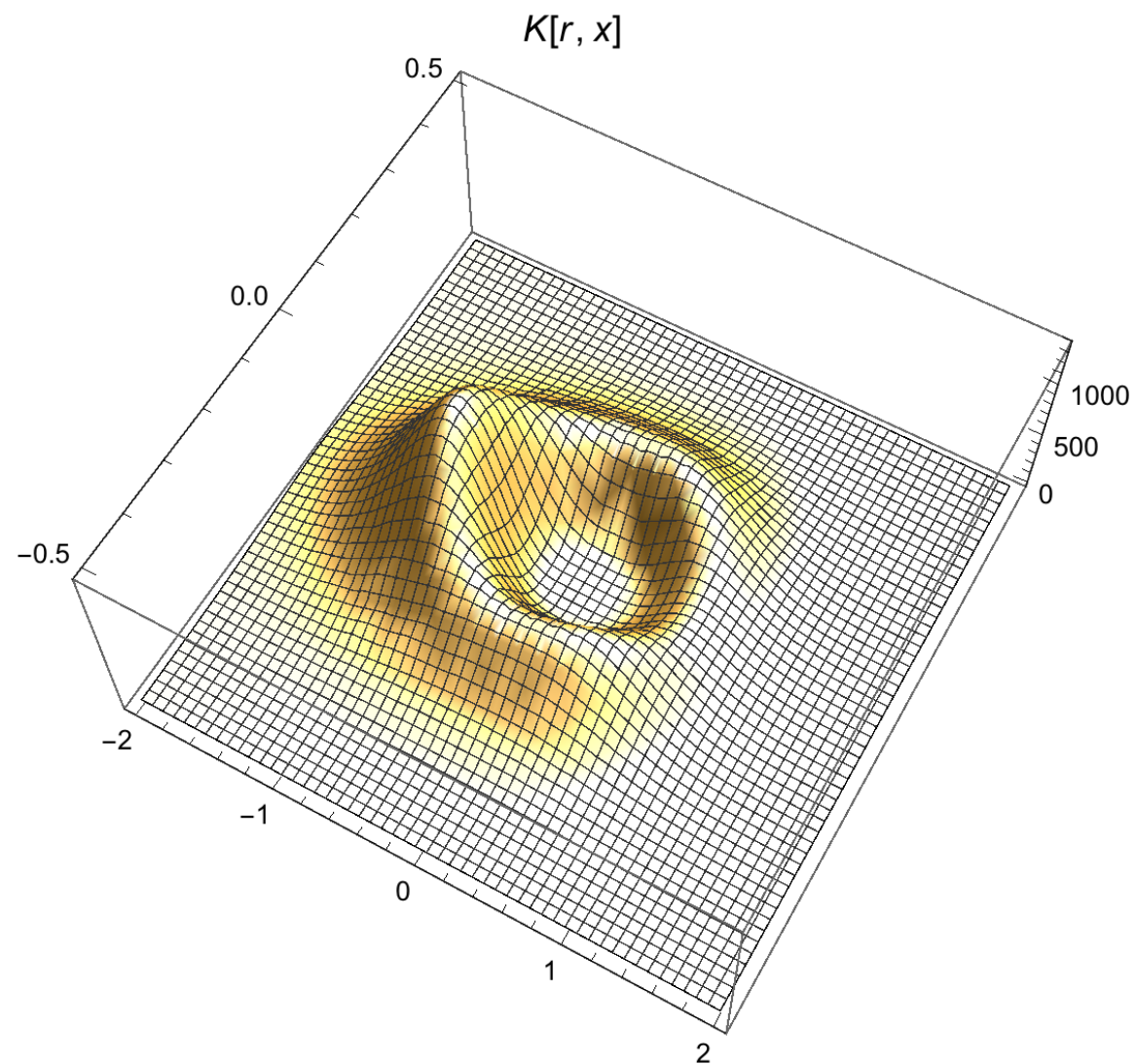
$$\mathbf{K} = \frac{1}{(L^2 + r^2 + a^2 \cos^2 \theta)^{12}} \times \text{polynomial}(r, \theta, m, a, L) .$$

$$L \propto m \quad (\text{black hole mass}).$$

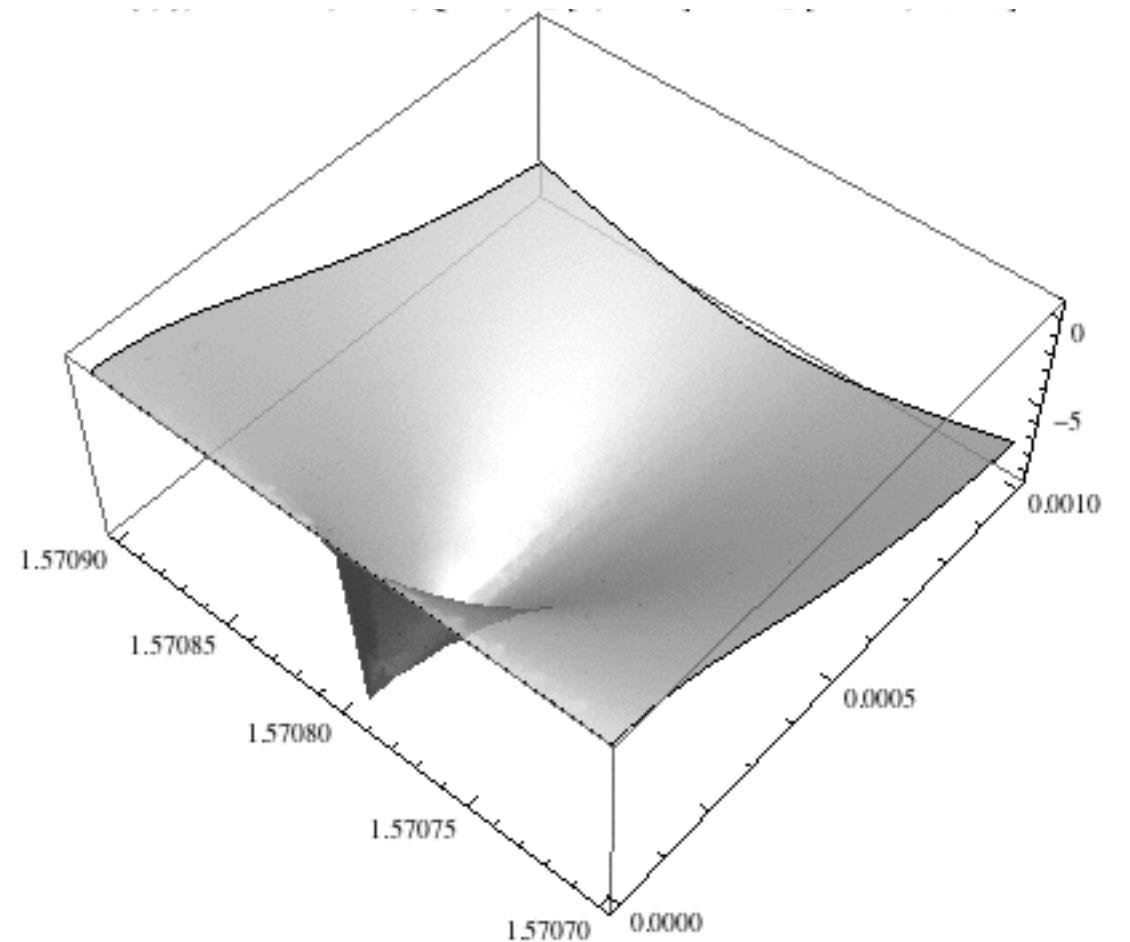
C. Bambi and collaborators: Upper bound on L .

Kretschmann Scalar

Conformal Gravity



Other ...



$$\lim_{r \rightarrow 0} \left(\lim_{\vartheta \rightarrow \pi/2} K(r, \vartheta) \right) = \frac{8M^2}{3\pi\theta^3},$$

$$\lim_{\vartheta \rightarrow \pi/2} \left(\lim_{r \rightarrow 0} K(r, \vartheta) \right) = 0.$$

Analogy ..

Weyl Rescaling

$$\Omega^2 = S(r) = 1 + \frac{L}{r^4},$$

$$ds^2 = S(r) ds_{\text{Sch}}^2.$$

Gullstrand–Painlevé coordinates

$$t = t_r + f(r),$$

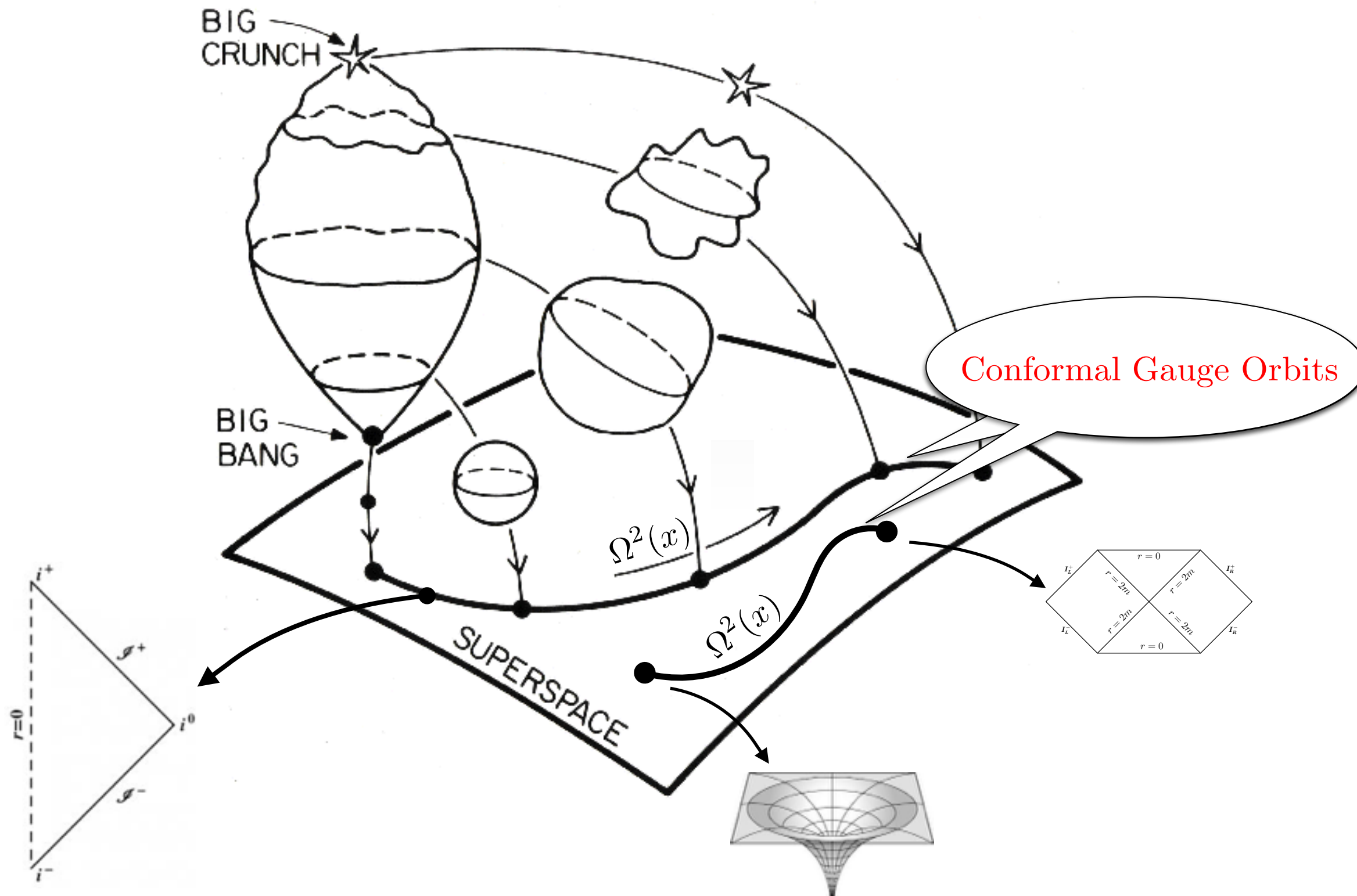
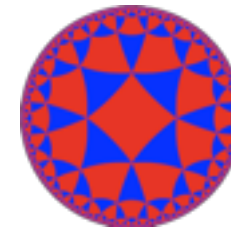
$$f(r) = - \int \frac{\sqrt{2M/r}}{1 - 2M/r} dr = 2M \left(\ln \left(\frac{y+1}{y-1} \right) - 2y \right),$$

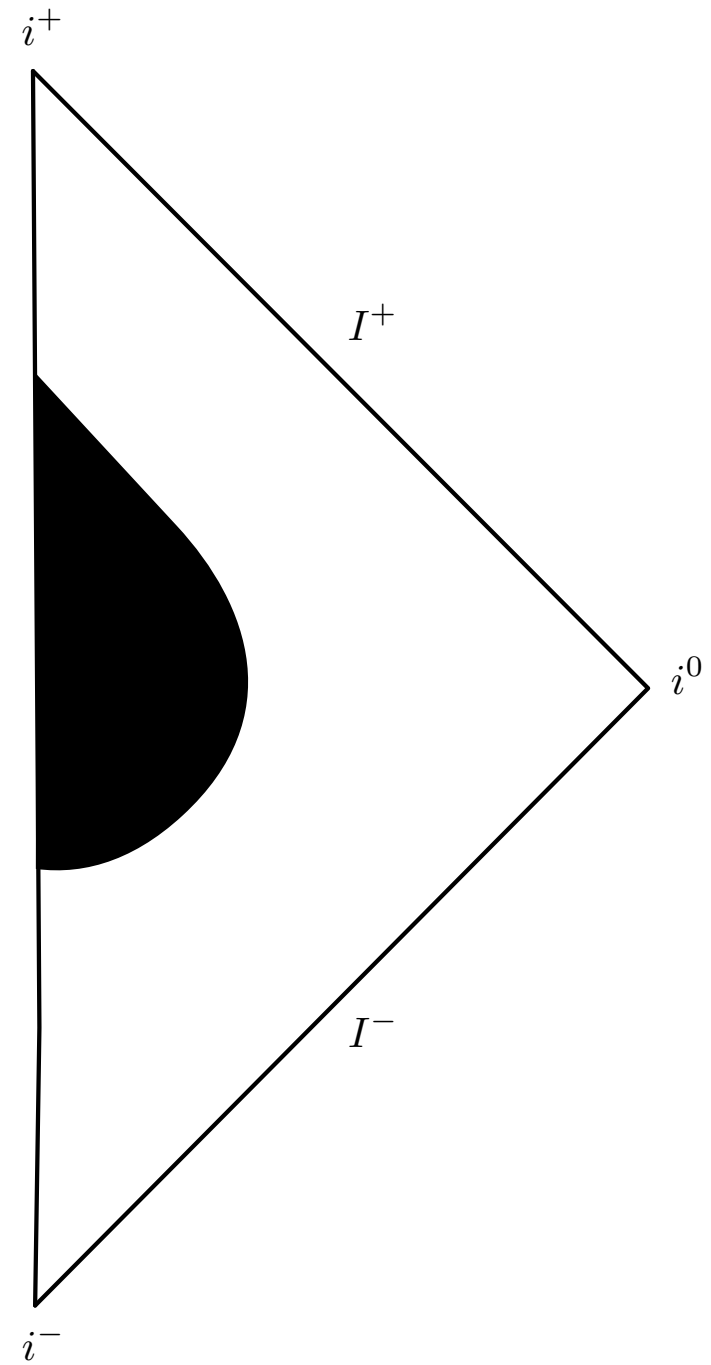
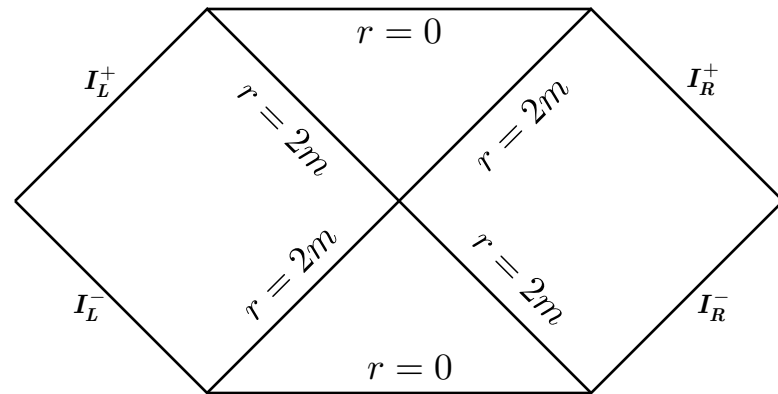
$$y = \sqrt{r/2M},$$

SINGULAR in $r = 2M$,

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt_r^2 + 2\sqrt{\frac{2M}{r}} dt_r dr + dr^2 + r^2 d\Omega^{(2)}.$$

Metric Superspace





V. Frolov, C. Bambi, D. Malafarina, LM
Bambi, Porey, Rachwal.

Quantum Gravity in Cut-Off Regularization Scheme

Stefano Giaccari, LM, Lesław Rachwał, Yiwei Zhu.

$$\mathcal{L}_g = -\frac{2}{\kappa_4^2} \left(R - 2\Lambda_{cc} + G_{\mu\nu} \frac{e^{H(\Box)} - 1}{\Box} R^{\mu\nu} + s_1 R^2 \Box^{\gamma-2} R^2 + s_2 \mathbf{Ric}^2 \Box^{\gamma-2} \mathbf{Ric}^2 + s_\kappa R^2 \Box^{\gamma-1} R \right).$$

$$\beta_\kappa = 0, \quad \beta_{R^2} = 0, \quad \beta_{\mathbf{Ric}^2} = 0,$$

$$\beta_{\Lambda_{cc}} \neq 0.$$

Finite Entanglement Entropy (replica trick)

$$S_{\text{ent}} = (\alpha \partial_\alpha - 1)W(\alpha)|_{\alpha=1}$$

$$E_\alpha = \Sigma \times \mathcal{C}_\alpha,$$

Σ : two dimensional surface ,

$\mathcal{C}_\alpha$: cone with deficit angle $\delta = 2\pi(1 - \alpha)$.

$$W(\alpha) = \frac{1}{2} \log \det \left(\mathcal{H}_{\text{gr}}(g_{\mu\nu}, A_\mu^c, \Phi, \Psi) \right) \propto \underbrace{B_0}_{\Lambda_{\text{cc}}} k^4 + \underbrace{B_2}_R k^2 + \underbrace{B_4}_{R^2, \mathbf{Ric}^2} \log \left(\frac{k^2}{\mu^2} \right) \Big|_{E_\alpha}.$$

$$S_{\text{ent}} = \frac{A}{4G_N} \quad (\text{only from the classical EH operator}).$$

Full agreement between Classical Wald Entropy and Quantum Field Entropy.

Starobinsky Inflation

$$\mathcal{L}_C = \frac{1}{2} \kappa_D^{-2} \sqrt{|g|} \left[\mathbf{R} + \mathbf{C} \gamma_C(\square) \mathbf{C} + \mathbf{R} \gamma_S(\square) \mathbf{R} + V_g(\mathbf{C}) \right],$$

$$\gamma_C = \frac{D-2}{4(D-3)} \frac{e^{H_2(\square)} - 1}{\square}, \quad \gamma_S = -\frac{D-2}{4(D-1)} \frac{e^{H_0(\square)} - 1}{\square},$$

$$D=4: \quad \gamma_S = -\frac{1}{6\square} \left[e^{H_0(\square)} \left(1 - \frac{\square}{M^2} \right) - 1 \right],$$

$$H_0(\square) = \frac{a}{2} \left[\Gamma(0, p_\gamma(\square)^2) + \gamma_E + \log(p_\gamma(\square)^2) \right],$$

$$p_\gamma(\square) = \frac{M^2}{\mathcal{M}^2} \frac{1}{\mathcal{M}^{2\gamma}} \square^{\gamma-5} (\square^2 + M^2 \square + M^4)^2 (\square - M^2), \quad \gamma > 5.$$

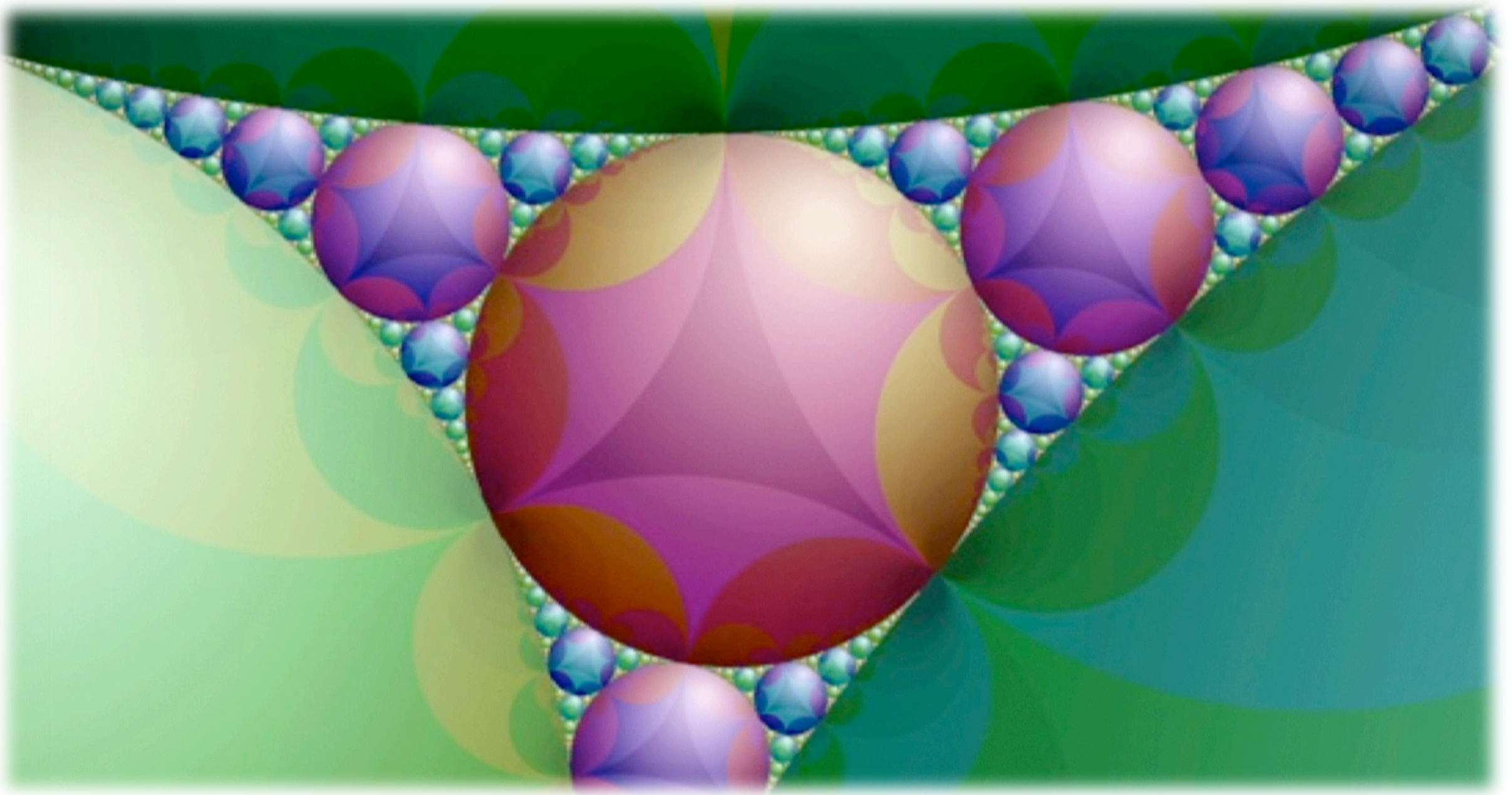
$$a(t) = a_0 \sqrt{t_s - t} e^{\sigma(t_s - t)^2}.$$

Achievements

1. Perturbative Unitarity
2. Super-renormalizability,
3. Finiteness,
4. Non-perturbative Unitarity (work in progress),
5. n-point scattering amplitudes,
6. The beta functions depend at most on 3 parameters,
7. Universal regular Newtonian potential,
8. Quantum Conformal Invariance,
9. Singularity free spacetimes (black holes, FLRW, etc.),
10. Finite Entanglement(-conical) Entropy,
11. Well define theory in (A)dS,
12. Starobinsky Inflation,
13. Finite nonlocal gauge theory,
14. Towards a finite Standard Model.



We have Finite Quantum Gravity !!!



$$\mathcal{L}_g = -2\kappa_D^{-2} \sqrt{-g} \left(R + G_{\mu\nu} \frac{e^{H(-\square_\Lambda)} - 1}{\square} R^{\mu\nu} + s_1 R^2 \square^{\gamma-2} R^2 + s_2 R_{\mu\nu} R^{\mu\nu} \square^{\gamma-2} R_{\rho\sigma} R^{\rho\sigma} \right).$$

