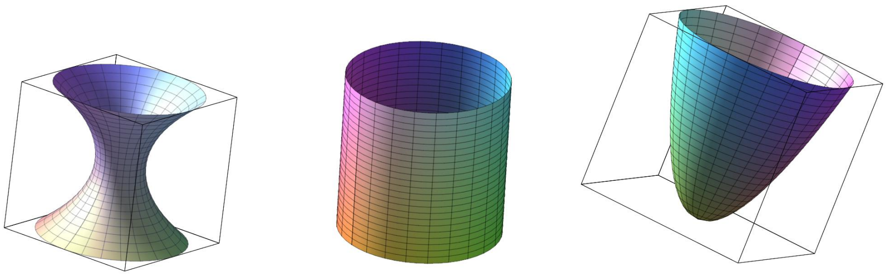
Axial symmetry

We say that (M, g_{ab}) has axial symmetry if there is a KVF with closed orbits.

$$ds^2 = g_{ab}(x) + \rho^2(x) d\phi^2, \quad \phi \sim \phi + 2\pi, \quad \text{length of KVF orbits: } \ell = 2\pi\rho(x_o)$$

There can be no symmetry center. If there is, $\rho(x_o)$ will not be the distance to it.
Left: $ds^2 = (1 + 4a^2z^2)dz^2 + (az^2 + b)^2 d\phi^2$.



Center: $ds^2 = dz^2 + R^2 d\phi^2$.
Right: $ds^2 = (1 + 4a^2\rho^2)d\rho^2 + \rho^2 d\phi^2$

Spherical symmetry

We say that (M, g_{ab}) has spherical symmetry if (note that it has the KVF os the sphere)

$ds^2 = g_{ab}(x) + r^2(x)(d\theta^2 + \sin^2\theta d\phi^2)$, area of $x^a = x^a_o$ symmetry orbits: $A = 4\pi r(x_o)^2$

There can be no symmetry center. If there is, $r(x_o)$ will not be the distance to it.

A region of a spacetime is *stationary* if it has a timelike KVF.

A region of a spacetime is *static* if it has a timelike KVF which is orthogonal to hypersurfaces.

In the static case can put coordinates $\{x^1, x^2, x^3\}$ on an hypersurface Σ_o orthogonal to the timelike KVF:

$$ds^2_{\Sigma_o} = h_{ij}(x)\ dx^i\ dx^j, i,j = 1,2,3 \text{ signature is} +++$$

Use coordinates $\{t, x^1, x^2, x^3\}$ for the point $\Phi_t(x^1, x^2, x^3)$

$$ds^2 = -f(x)dt^2 + h_{ij}(x)\ dx^i\ dx^j, \quad \text{timelike KVF is } \frac{\partial}{\partial t} \tag{1}$$

Static + spherically symmetric spacetime

Static + spherically symmetric gives Schwarzschild ansatz:

$$ds^2 = -f(r)dt^2 + \underbrace{h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)}_{h(x)_{ij}dx^i dx^j}, \quad f(r), h(r) > 0, \tag{2}$$

See the magic of using all symmetries in the ansatz (from $g_{ab}(x^c)$ to $f(r)$ and $g(r)$)

Stationary + spherically symmetric spacetime

Stationary + spherically symmetric gives Weyl-Lewis-Papapetrou ansatz

$$ds^2 = -f(dt + Ad\phi)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad f(\rho, z), \ g(\rho, z), A(\rho, z) \quad (3)$$

Black Hole region

Comment on subtleties in colloquial /non-technical definitions
Comment of problems with too formal definitions (Penrose's null infinity)
Comment on trapped surfaces?

Minkowski in spherical coordinates,

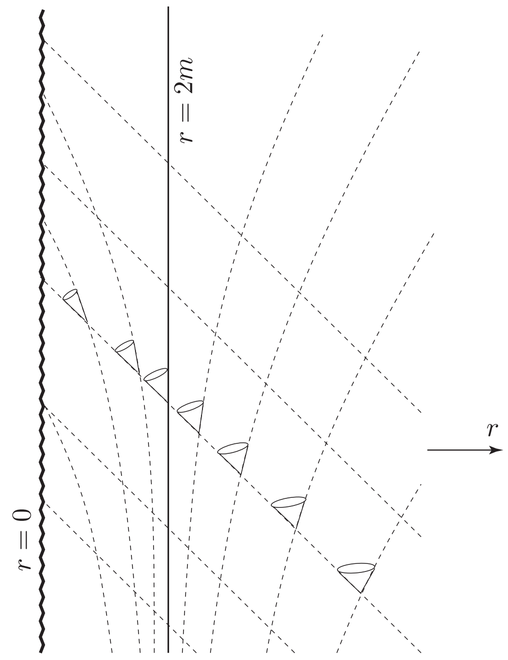
$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) & (4)\\ &= -dv^2 + 2dv\, dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) & (5) \end{aligned}$$

the region where future null geodesics go corresponds to $v \equiv t + r \to \infty$ while $u \equiv t - r$ is kept finite.

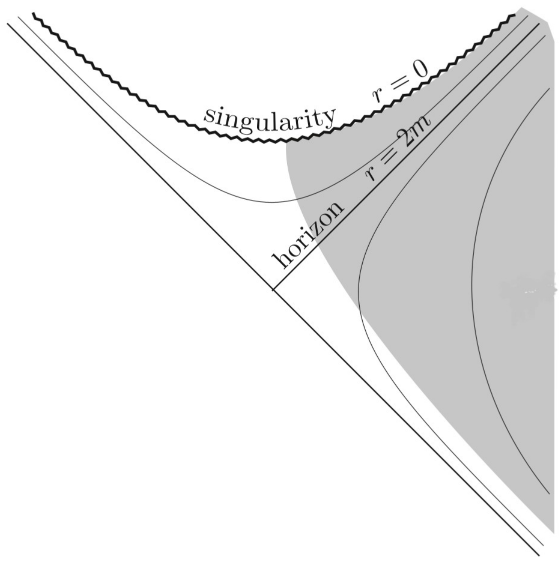
In *Eddingstein* spacetime

$$\begin{aligned} ds^2 = -(1 - 2M/r)\, dv^2 + 2dv\, dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \\ \simeq -dv^2 + 2dv\, dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \text{ for } r >> M \quad (6) \end{aligned}$$

future null infinity corresponds to $(v, r) = (u_o + 2s, s)$, $s \to \infty$



Eddington-Finkelstein solution:
Dotted straight lines are $v =$ constant curves (incoming radial null geodesics).
Dotted curved lines are the "outgoing" null radial geodesics defined by $dv/dr = 2/f(r)$.
Plotted half cones are future pointing.



EF in different perspective: null lines are at 45 degrees, future directed timelike curves don't exceed this angle.
Shaded region models outer region of a spherically collapsing object whose surface is the boundary of this region. Non shaded region has to be replaced by star interior. Note the BH region.

Kerr spacetime

$$ds^2 = -\frac{(\Delta_r - a^2\sin^2\theta)}{\Sigma}\mathrm{d}t^2 + \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \frac{4aMr\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi + \Sigma\mathrm{d}\theta^2 + \frac{\sin^2\theta[(r^2+a^2)^2 - a^2\sin^2\theta\Delta_r]}{\Sigma}\mathrm{d}\phi^2, \quad (7)$$

where

$$\Delta_r = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad M > a > 0$$
$$t \in \mathbb{R}, \quad r > m + \sqrt{M^2 - a^2}, \quad 0 < \theta < \pi, \quad \phi \sim \phi + 2\pi$$

- Metric elements do not depend on (t, ϕ), then $\partial/\partial t$ and $\partial/\partial\phi$ are KVFs
- If $a = 0$ reduces to Schwarzschild (i.e., Kerr's solution generalizes Schwarschid's)
- For large r approaches Minkowski's: $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
- In the region $r > M + \sqrt{M^2 - a^2\cos^2\theta}$, the KVF $\partial/\partial t$ is timelike: thus Kerr's solution is axially symmetric (KVF is $\frac{\partial}{\partial\phi}$) and stationary (KVF is $\partial/\partial t$) –but not static– in this region.
- For large r it models the exterior of a rotating central object.

- It has a true singularity (spacetime boundary points) at $r = 0, \theta = \pi/2$ (that is, where $\Sigma = 0$)

$$R^{ab}{}_{cd}\,\epsilon^{cdef}\,R_{efab} = \frac{M^2 r\, a\cos\theta(3a^2\cos^2\theta - r^2)(a^2\cos^2\theta - 3r^2)}{(r^2 + a^2\cos^2\theta)^6}$$

- The region $r < r_+$ is a BH

The following Theorem follows from results of Carter, Robinson, Hawking and Wald:

Theorem: If (M, g) is an asymptotically-flat stationary vacuum spacetime containing a BH, then it is a member of the two-parameter Kerr family (the parameters are the mass M and angular momentum per unit mass $a = J/M$).

Linear perturbation theory

Comment on:

- Use of perturbation theory to model nearly ideal situations.
- Use of perturbation theory to analyze feasibility of time independent solutions (e.g., Mechanics and stability problems)
- A few monsters in the Kerr family: super extremal solutions ans their naked singularities, negative mass Schwazschild, maximal analytic extensions... beyond Cauchy horizons (predictability?). Causal pathologies beyon inner radius, etc...
- Are all the above pathological solutions unstable?
- If GR models real BHs, then Kerr is the geometry. Are the Kerr and Schwarzschild outside regions stable?

Perturbing warped spacetimes (such as non rotating BHs)

A warped product metric (here $\alpha = (a, A), \beta = (b, B)$, etc) looks like this:

$$\underbrace{g_{\alpha\beta}dz^\alpha dz^\beta}_{M,\nabla_\alpha} = \underbrace{g_{ab}(x)dx^a dx^b}_{O,\,\tilde{D}_a} + r^2(x)\ \underbrace{g_{AB}(y)dy^A dy^B}_{K,\,\hat{D}_A}$$

In our case $K = S^2$ and $g_{AB}(y)dy^A dy^B = d\theta^2 + \sin^2\theta\ d\phi^2$.

We need some facts

1. If K is closed (compact without boundary), any vector field V^A on K decomposes as
$$V^A = U^A + \hat{D}^A S, \quad D_B U^B = 0$$

2. If, furthermore, g_{AB} is Einstein ($R_{AB} \propto g_{AB}$), then $S_{AB} = S_{(AB)}$ decomposes as
$$S_{AB} = \underbrace{Q_{AB}}_{TT} + \underbrace{D_{(A}V_{B)}}_{\hat{D}_C V^C = 0} + \left(D_A D_B - \tfrac{1}{n} g_{AB} \Box\right) S + \tfrac{1}{n} g_{AB} S_C{}^C$$

   TT means transverse traceless: $Q_A{}^A = 0, \hat{D}^B Q_{AB} = 0$ and $n = \dim K$.

It follows that vector fields on K contain a scalar field and a divergence free vector field:
$$V^A = U^A + \hat{D}^A S, \quad (\hat{D}_C U^C = 0) \tag{8}$$
Similarly, symmetric tensor fields on K contain a scalar field, a divergence free vector field and a TT tensor field:
$$T_{AB} = Q_{AB} + D_{(A}V_{B)} + \left(\hat{D}_A \hat{D}_B - \tfrac{1}{n} g_{AB} \Box\right) S + \tfrac{1}{n} g_{AB} J \tag{9}$$
The terms in the sums (8) and (9) are orthogonal (under $(U, V) = \int_K U^C V_C$, etc)

Any symmetric tensor field on $M = O \times_{r^2} K$, such as the metric perturbation $h_{\alpha\beta}$

$$h_{\alpha\beta} = \begin{pmatrix} h_{ab} & h_{aB} \\ h_{Ab} & h_{AB} \end{pmatrix} \tag{10}$$

can be decomposed from the K viewpoint into a set of $K-$scalar, divergence free vector and TT tensor fields as follows
(Number of each type of field below corresponds to the 4D case)

$$\begin{aligned} h_{ab} &= h_{ab} \quad \text{(3 scalar fields in 4D)} \\ h_{aB} &= h_{aB} + \hat{D}_B h_a, \quad \text{(2 vector fields and 2 scalars)} \\ h_{AB} &= h_{AB} + \hat{D}_{(A} h_{B)} + \left(\hat{D}_A \hat{D}_B - \tfrac{1}{n} g_{AB} \Box \right) S + \tfrac{1}{n} g_{AB} J, \\ &\quad \text{(2 scalars, 1 vector, 1 tensor)} \end{aligned} \tag{11}$$

Maxwell fields $F_{\alpha\beta} = \nabla_{[\alpha} A_{\beta]}$ similarly decomposed using (8) of $A_\alpha = (A_a, A_A) = (A_a, U_A + D_A S)$.

A geometric gravity theory in any dimensions:

$$\mathcal{G}_{\alpha\beta}[g] = \kappa^2 T_{\alpha\beta}[g, \Phi], \quad \text{(e.g. } G_{ab}[g] + \Lambda g_{ab} = 8\pi T_{ab}[g, \Phi] \text{)} \tag{12}$$

$(g_\epsilon, \Phi_\epsilon)$ is a monoparametric family of solutions through $(g = g_{\epsilon=0}, \Phi = \Phi_{\epsilon=0})$, then

$$\mathcal{G}_{\alpha\beta}[g_\epsilon] = \kappa^2 T_{\alpha\beta}[g_\epsilon, \Phi_\epsilon] \quad \text{for all } \epsilon \tag{13}$$

Linearization of the equation (12) around the solution (g, Φ) is obtained by taking the derivative of (13) with respect to ϵ and evaluating at $\epsilon = 0$

We symbolize this operation with an overdot, e.g., $\Phi_\epsilon = \Phi + \epsilon\dot{\Phi} + \mathcal{O}(\epsilon^2)$. Exception: $h_{\alpha\beta} = \dot{g}_{\alpha\beta}$ and we rise the h_{ab} indices with t unperturbed metric inverse $g^{\alpha\beta}$, this implies $\dot{g}^{\alpha\beta} = -h^{\alpha\beta}$.

The linear equations

$$\dot{\mathcal{G}}_{\alpha\beta}[h] = \kappa^2 \dot{T}_{\alpha\beta}[h, \dot{\Phi}]$$

are linear in the perturbation fields $(\dot{g}_{ab} = h_{ab}, \dot{\Phi})$ and depend on how the "unperturbed" or "background" solution (g_{ab}, Φ) is.

Group theory arguments indicate that scalar, vector and tensor modes of h_{ab} enter respectively analogous components of $\dot{\mathcal{G}}_{\alpha\beta}[h]$ and $\dot{T}_{\alpha\beta}[h, \dot{\Phi}]$, leading to 3 separate untangled set of equations: scalar mode equations (historically *even*), vector modes (historically *axial*) and tensor modes (absent in 4D with spherical horizons).

In the case of 4D spherically symmetric spacetimes TT modes are absent and we are left with vector (-) and scalar (+) modes:

A generic smooth metric perturbation admits the following decomposition:

$$h_{\alpha\beta} = h^{(-)}_{\alpha\beta} + h^{(+)}_{\alpha\beta} \tag{14}$$

where the <u>odd</u> piece of $h_{\alpha\beta}$ is

$$h^{(-)}_{\alpha\beta} = \begin{pmatrix} 0 & \widehat{\epsilon}_B{}^C \widehat{D}_C h_a \\ \widehat{\epsilon}_A{}^C \widehat{D}_C h_b & 2r^2 \widehat{\epsilon}_{(A}{}^C \widehat{D}_{B)} \widehat{D}_C k \end{pmatrix}. \tag{15}$$

h_a and k are unique if they are required to belong to $L^2(S^2)_{>0}$ and $L^2(S^2)_{>1}$ respectively (a condition that will be assumed).
The <u>even</u> piece of the metric perturbation is

$$h^{(+)}_{\alpha\beta} = \begin{pmatrix} h_{ab} & \widehat{D}_B q_a \\ \widehat{D}_A q_b & r^2 \left[\frac{J}{2} \widehat{g}_{AB} + (2\widehat{D}_A \widehat{D}_B - \widehat{g}_{AB} \widehat{D}^C \widehat{D}_C) G \right] \end{pmatrix}. \tag{16}$$

The fields (h_{ab}, q_a, J, G) are uniquely defined if we require that $q_a \in L^2(S^2)_{>0}$ and $G \in L^2(S^2)_{>1}$ (a condition that will be assumed).

Gauge issue

$$h_{\alpha\beta} \sim h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_b \xi_\alpha, \quad \xi_\alpha = \xi_a^+ + \xi_\alpha^- = (\xi_a, \widehat{D}_A X) + (0, \epsilon_{AB} \widehat{D}^B Y)$$

Can construct gauge invariant fields $H_a = h_a - r^2 \tilde{D}_a k$, H_{ab}, J and S and write perturbations in Regge-Wheeler gauge

$$^{RW}h_{\alpha\beta}^{(-,>1)} = \begin{pmatrix} 0 & \widehat{\epsilon}_B{}^C \widehat{D}_C H_a \\ \widehat{\epsilon}_A{}^C \widehat{D}_C H_b & 0 \end{pmatrix}, \quad H_a \in L^2(S^2)_{>1} \tag{17}$$

$$^{RW}h_{\alpha\beta}^{(+,>1)} = \begin{pmatrix} H_{ab} & 0 \\ 0 & \frac{r^2}{2} \widehat{g}_{AB} \mathcal{J} \end{pmatrix}. \tag{18}$$

Modal decomposition

Introduce the operator J_1, J_2, J_3 canonical basis of KVFs on sphere, e.g. $J_3 = \frac{\partial}{\partial \phi}$

$$\mathbf{J}^2 = \sum_{m=1}^{3} (\pounds_{J_{(m)}})^2. \tag{19}$$

We can further decompose $h_{\alpha\beta}^{\pm}$ into harmonic modes:

$$\mathbf{J}^2 h_{\alpha\beta}^{(\ell,m,\pm)} = -\ell(\ell+1) h_{\alpha\beta}^{(\ell,m,\pm)}, \tag{20}$$

$$P_* h_{\alpha\beta}^{(\ell,m,\pm)} = \pm(-1)^\ell h_{\alpha\beta}^{(\ell,m,\pm)}. \tag{21}$$

Schwarzschild BH, low harmonic modes:

- $\ell = 0$ is a spherical perturbation, solutions of the LEE can only give a change of the mass parameter: $M \to \delta M$.
- $\ell = 1$ modes give axially symmetric perturbations of Schwarzschild: if adjusted such that KVF is Schwarzschild $\partial/\partial\phi$ we get "slowly rotating Kerr":

  $$ds^2 = -f(r)drt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4M\epsilon}{r}\sin^2\theta\ dtd\phi$$

  Let ϵ be $a = J/M$....

- Truly dynamical, interesting modes are those with $\ell > 1$, for which, for the odd/vector modes we find

$$^{(RW)}h_{\alpha\beta}^{(\ell,m,p=\pm)} = \mathcal{D}_{\alpha\beta}^{(\ell,m,p=\pm)}\left[\phi^{\pm}_{(\ell,m)}, S_{(\ell,m)}\right] \tag{22}$$

$$(\partial_t^2 - \partial_x^2 + fU_\ell^{RW})\phi^{-}_{(\ell,m)} = 0, \quad \ell \geq 2, \tag{RWE}$$

where $\partial_x = f\partial_r$, $f = 1 - 2M/r$, and

$$U_\ell^{RW} = \frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}. \tag{23}$$

whereas for the even/scalar modes

$$(\partial_t^2 - \partial_x^2 + fU_{\ell)}^{Z})\phi^{+}_{(\ell,m)} = 0, \quad \ell \geq 2, \tag{ZE}$$

with

$$U_\ell^Z = \frac{[\mu^2\ell(\ell+1) - 24M^2\Lambda]r^3 + 6\mu^2 M r^2 + 36\mu M^2 r + 72M^3}{r^3\,(6M + \mu r)^2}, \quad \mu = (\ell-1)(\ell+2). \tag{24}$$

- Compare: complexity of U_ℓ^Z with that of U_ℓ^{RW}
- Note that U_ℓ^{RW} seem to come from a 4D equation.... but U_ℓ^Z doesn't.
- See that if we apply this to $M < 0$ the U_ℓ^Z gets a pole for $r > 0$, at an ℓ-dependent position!

The fall of a monster: negative mass Schwarzschild spacetime is unstable

- Comment on non global hyperbolicity / ambiguity in dynamics / bc's at $r = 0$
- For vector/odd perturbations there are no ambiguities for boundary conditions at $r = 0$, no instabilities found.
- For scalar/even perturbations, from the viewpoint of the 2D Zerilli wave equation, there is a circle of choices of boundary conditions at $r = 0$ (bc's parametrized by S^1), but only one singled out to avoid inconsistency with linear treatment:

$$C^{abcd}C_{abcd} = \frac{3M^2}{r^6} + \epsilon \frac{C_{(\ell,m)}}{r^6} \ln\left(\frac{r}{|M|}\right) S_{(\ell,m)}(\theta,\phi)\cos(\omega t),$$

- A misleading $w = 0$ mode for $\ell = 2$ scalar/even perturbations was reported (Gibbons, Hartnoll and Ishibashi 2005) and thought to signal the possibility of stability, but there is a flaw in this argument: ϕ: $\partial_x^2\phi$ is discontinuous at the negative mass, Zerilli singularity at $r = r_c$, then the perturbation

$$^{(RW)}h_{\alpha\beta}^{(\ell=2,m,p=+)} = \mathcal{D}_{\alpha\beta}^{(\ell=2,m,p=+)}\left[\phi^{\pm}_{(\ell,m)}, S_{(\ell,m)}\right]$$

  introduces a "thin shell" of matter at $r = r_c$: is not a vacuum solution.

- An unstable scalar mode was found for every harmonic in 2006 (R. Gleiser and G. D, 2006)

- Hard to find observable effects, but, e.g.:

$$R_{abcd;e}R^{abcd;e} = \frac{720(r-2M)M^2}{r^9} - \epsilon\frac{5\ell(\ell-1)(\ell+1)(\ell+2)}{r^8}Y_{\ell m}\theta,\phi)$$
$$\times\,[r\ell(\ell+1)-6M)(r-2M)^{-2Mk}\exp(k(t-r)), \quad k = \frac{(\ell-1)(\ell(\ell+1)(\ell+2)}{6|M|} \tag{25}$$

- The instability also introduces degeneration in double PNDs (B. Araneda and G.D, 2015)
- The same type of instability was found in super-extreme Reissner Nordström solutions and in RN interiors. In all cases the RW-Z formalism is ill defined in the scalar (Zerilli) sector, new variables have to be introducedin the 2D wave formalism (intertwinner operators)

The fall of two monsters: i) super-extreme Kerr, ii) $r < r_-$ region of Kerr BH

Introducing the (separated) Teukolsky fields ($s = 0$ for scalars, $s = \pm 1$ for EM, $s = \pm 2$ for gravity)

$$\Psi_s = R_{\omega,m,s}(r) S^m_{\omega,s}(\theta) \exp(im\phi) \exp(-i\omega t), \tag{26}$$

the test scalar / Maxwell and linearized gravity equations reduce to a coupled system for S and R:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + E - s^2\right) S = 0 \tag{27}$$

$$\Delta \frac{d^2R}{dr^2} + (s+1)\frac{d\Delta}{dr}\frac{dR}{dr} + \left\{\frac{K^2 - 2is(r-M)K}{\Delta} + 4ir\omega s - [E - 2am\omega + a^2\omega^2 - s(s+1)]\right\} R = 0, \tag{28}$$

where $K = (r^2 + a^2)\omega - am$.

The system (27)-(28) is coupled by their common eigenvalue E

Strategy: look for intersections of spectra assuming exponentially growing mode

$$\omega = ik, \quad \Psi \sim e^{kt}$$

Can show that this happens for very ℓ (spheroidal) $m = 0$ harmonic

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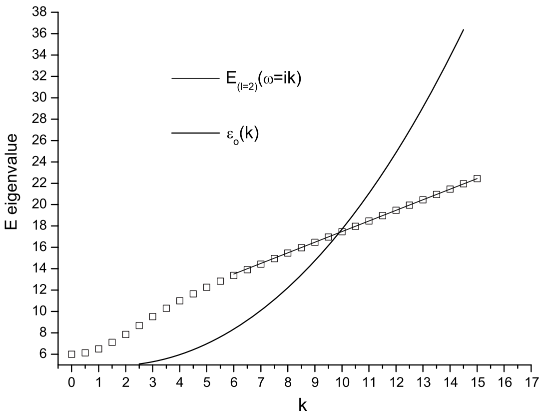
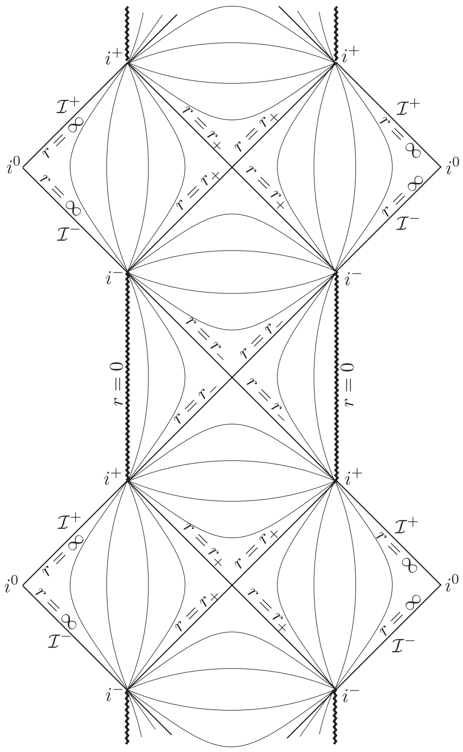


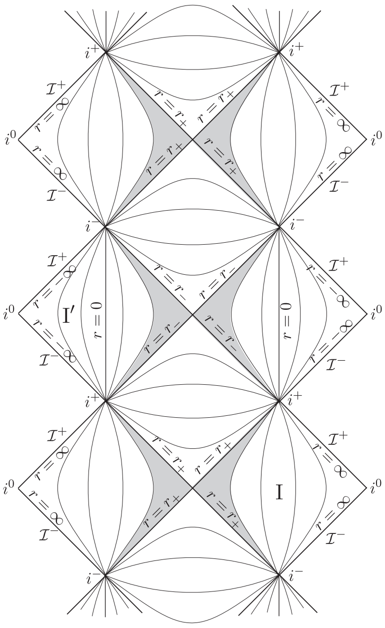
Figure 3. Intersection of the numerically generated curves $E_{(\ell=2)}(a\omega)|_{a\omega=\mathrm{i}k}$ and $\epsilon_o(k)$ for $a/M = 1.4$. Note the agreement with the results in [6], according to which for $a = 1.4$ the intersection should occur at $k \simeq 7.07 \times 1.4 = 9,898$.

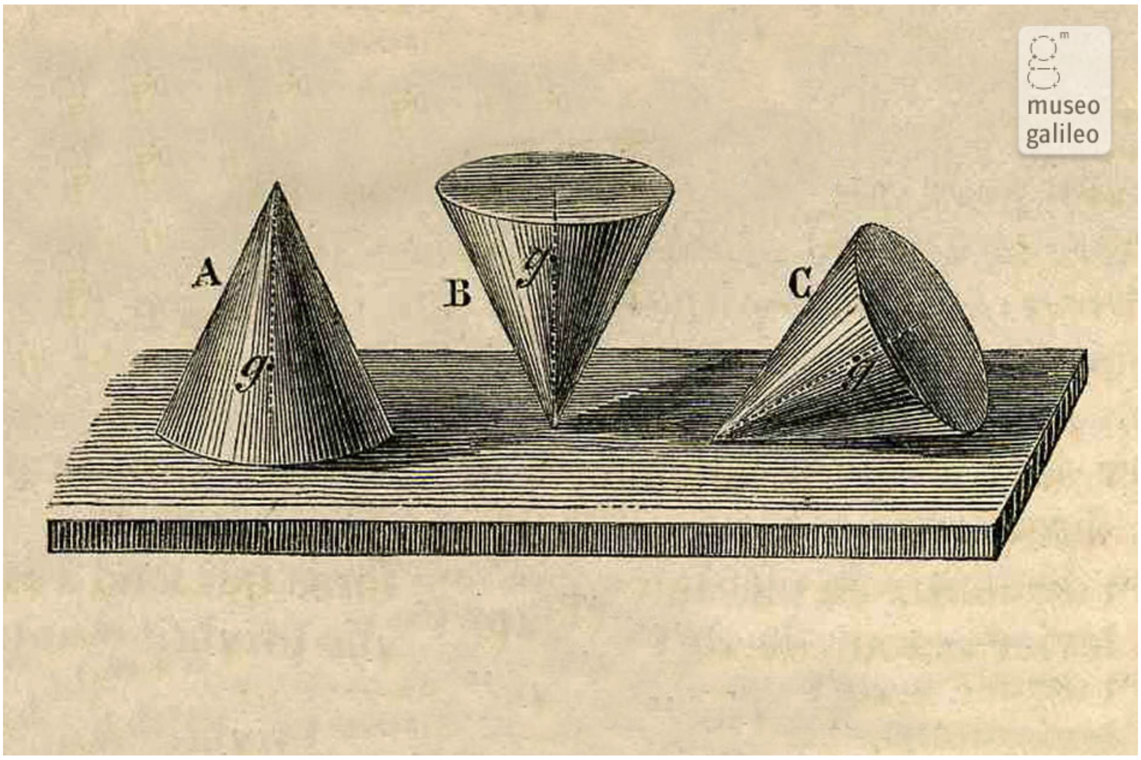
(G.D., R. Gleiser, I.Sandoval and H.Vucetich, 2008, 2012)

Reissner-Nordström maximal analytic extension (unstable!)



Kerr maximal analytic extension (unstable!)

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Exterior BH linear stability: limitations of the modal approach

Standard linear stability notions are concerned with the behavior of the 2D potentials $\phi^{\pm}_{(\ell,m)}(t,r)$ of isolated (ℓ,m) modes:

$$\left[\partial_t^2 - \partial_x^2 + V_\ell^{\pm}(r)\right] \phi^{\pm}_{(\ell,m)} = 0,\ \partial_x = f(r)\partial_r$$

- In (Regge, 1957) it was shown that separable solutions $\phi^{-}_{(\ell,m)} = \Re\, e^{i\omega t}\chi^{-}_{(\ell,m)}(r)$ that do not diverge as $r \to \infty$ require $\omega \in \mathbb{R}$, ruling out exponentially growing solutions in the odd sector.

- In (Zerilli, 1970) it was shown that separable solutions $\phi^{+}_{(\ell,m)} = \Re\, e^{i\omega t}\chi^{+}_{(\ell,m)}(r)$ that do not diverge as $r \to \infty$ require $\omega \in \mathbb{R}$, ruling out exponentially growing solutions in the even sector.

- In (Price, 1971) it was shown that, for large t and fixed r, $\phi^{\pm}_{(\ell,m)}(t,r)$ decays as $t^{-(2\ell+2)}$ (an effect known as "Price tails").

- In (Wald, 1979), a point wise bound on the RW and Zerilli potentials was found in the form
$$|\phi^{\pm}_{(\ell,m)}(t,r)| < C^{\pm}_{(\ell,m)},\ \ r > 2M,\ \text{all } t, \tag{29}$$
where the constants $C^{\pm}_{(\ell,m)}$ are given in terms of the initial data $(\phi^{\pm}_{(\ell,m)}(t_o,r), \partial_t\phi^{\pm}_{(\ell,m)}(t_o,r))$

To understand tthe limitations of these results take, e.g., the case of the vector/odd Schwarzschid modes: in the RW gauge the metric perturbation is a series

$$\sum_{(\ell,m)} h_{\alpha\beta}^{(\ell,m,-)}[\phi^-_{(\ell,m)}]$$

$$\left[\begin{array}{cc|cc} 0 & 0 & \frac{\epsilon\,(-r+2\,M)\left(\phi^-_{(\ell,m)}(t,r)+r\,\frac{\partial}{\partial r}\phi^-_{(\ell,m)}(t,r)\right)\frac{\partial}{\partial \phi}Y(\theta,\phi)}{r\sin(\theta)} & -\frac{\epsilon\,(-r+2\,M)\left(\phi^-_{(\ell,m)}(t,r)+r\,\frac{\partial}{\partial r}\phi^-_{(\ell,m)}(t,r)\right)\sin(\theta)\,\frac{\partial}{\partial \theta}Y(\theta,\phi)}{r} \\ 0 & 0 & \frac{\epsilon\,r^2\left(\frac{\partial}{\partial t}\phi^-_{(\ell,m)}(t,r)\right)\frac{\partial}{\partial \phi}Y(\theta,\phi)}{(-r+2\,M)\sin(\theta)} & -\frac{\epsilon\,r^2\left(\frac{\partial}{\partial t}\phi^-_{(\ell,m)}(t,r)\right)\sin(\theta)\,\frac{\partial}{\partial \theta}Y(\theta,\phi)}{-r+2\,M} \\ \hline sym & sym & 0 & 0 \\ sym & sym & 0 & 0 \end{array}\right]$$

Limitations of the modal approach:
Two derivatives of the metric perturbation $\sum_{(\ell,m)} h_{\alpha\beta}^{(\ell,m,-)}$ are required to calculate the perturbed Riemann tensor. The relation of the individual potentials $\phi^{\pm}_{(\ell,m)}$ to geometrically meaningful quantities is remote, the usefulness of bounds on each $\phi^{\pm}_{(\ell,m)}(t,r)$ to measure the strength of the perturbation is far from obvious.

- We need to *undo* angular variable separation, the same way (t,r) separation in $\phi^-_{(\ell,m)} = e^{i\omega t}\chi^-_{(\ell,m)}(r)$ was undone to obtain information on generic (non-separable) solutions of the 1+1 wave equations solutions $\phi^-_{(\ell,m)}$

- There are two difficulties:

  - the complicated way that the $\phi^\pm_{(\ell,m)}$ enter the metric perturbation.
  - There is no natural norm for the metric perturbation $h_{\alpha\beta}$!
    On a Riemannian background we could use, as a pointwise bound, the norm

$$|h| = \sqrt{h_{\alpha\beta}h_{\mu\nu}g^{\alpha\mu}g^{\beta\nu}}$$

    This satisfies $||h|| = 0 \Rightarrow h_{\alpha\beta} = 0$, $||\lambda h|| = |\lambda|\ ||h||$, etc. As an L^2 norm we could use

$$||h|| = \sqrt{\int_M h_{\alpha\beta}h_{\mu\nu}g^{\alpha\mu}g^{\beta\nu}}$$

    Problem is tat in a Lorentzian background $h_{\alpha\beta}h_{\mu\nu}g^{\alpha\mu}g^{\beta\nu}$ is not even positive definite: Consider, e.f., a symmetric tensor $h_{\alpha\beta} = V_\alpha V_b$ where V^α is a null vector, then

$$h_{\alpha\beta}h_{\mu\nu}g^{\alpha\mu}g^{\beta\nu} = 0 = (\lambda h_{\alpha\beta})\ (\lambda h_{\mu\nu})g^{\alpha\mu}g^{\beta\nu}$$

    How "big" is this??

A warning on modal stability

Trivial example of the limitation of non-modal linear stability analysis: (This is a toy model for a problem that arises in the linearized Navier Stokes eqns, see Schmid, Annu. Rev. Fluid Mech. 2007)

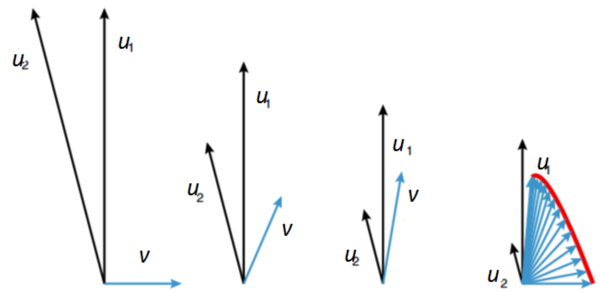
2D state space: $\vec{v} \in \mathbb{R}^2$ satisfying in

$$\frac{d}{dt}\ \vec{v} = A\vec{v}$$

$$A = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}, \quad \text{autoval/vects:} -1, -2 \ \ \vec{\Phi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \ \vec{\Phi}_2 = \begin{pmatrix} 1 \\ -\frac{1}{10} \end{pmatrix},$$

$$\vec{v}(0) = \vec{\Phi}_1 - \vec{\Phi}_2 \Rightarrow \vec{v}(t) = \vec{\Phi}_1 e^{-t} - \vec{\Phi}_2 e^{-2t},$$

$\lim_{t\to\infty} |\vec{v}(t)| = 0$ but there is a transient growth of $|\vec{v}(t)|$.



A second warning on modal stability

- Harmonic modes coefficients are *integrals* on the sphere against a basis of eigenfunctions
- Functions can get narrower and steeper with time. As a consequence, harmonic coefficients will *decrease* with time while the maximum value of the function actually *increases* in time.

Example:

$$f(t,\theta) = g(t)\sin(\theta)^t, \;\; t > 1, \text{ and } 0 < \theta < \pi$$

Its Fourier series is

$$f(t,\theta) = \tfrac{1}{2}a_o(t) + \sum_{n=1}^{\infty} \left[a_n(t)\cos(2n\theta) + b_n(t)\sin(2n\theta)\right]$$

where

$$a_n(t) = \frac{2g(t)}{\pi}\int_0^{\pi} \sin(\theta)^t \cos(2n\theta)\; d\theta, \;\; b_n(t) = \frac{2g(t)}{\pi}\int_0^{\pi} \sin(\theta)^t \sin(2n\theta)\; d\theta$$

It follows that

$$|a_n(t)|, |b_n(t)| < g(t)k(t), \quad \text{where } k(t) \equiv \frac{2}{\pi}\int_0^{\pi} \sin(\theta)^t \; d\theta$$

Note that $k(t)$ positive, decreasing, and $\lim_{t\to\infty} k(t) = 0$, then we can choose a positive, increasing $g(t)$ such that

$$g(t) \to \infty \text{ as } t \to \infty, \quad \text{while} \quad g(t)k(t) \to 0 \;\text{ as } t \to \infty$$

In this case we will have, for all n

$$a_n(t), b_n(t) \to 0 \;\text{ as } t \to \infty$$

in spite of

$$\max_{\theta} f(t,\theta) = g(t) \to \infty \text{ as } t \to \infty$$

That is, every harmonic coefficient tends to zero while the max value of the 2-variable function goes to infinity with time.

A way out of the limitations of the modal decomposition

We consider the effect of a perturbation on the curvature scalars [G.D, 2014, 2016]:

$$Q_{-} = \tfrac{1}{48}{}^{*}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}, \qquad Q_{+} = \tfrac{1}{48}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}, \qquad X = \tfrac{1}{720}\left(\nabla_{\epsilon}C_{\alpha\beta\gamma\delta}\right)\left(\nabla^{\epsilon}C^{\alpha\beta\gamma\delta}\right).$$

The background values are (including cosmological constant)

$$Q_{-S(A)dS} = 0, \quad Q_{+S(A)dS} = \frac{M^2}{r^6}, \quad X_{S(A)dS} = \frac{M^2}{r^9}(r-2M) - \frac{\Lambda M^2}{3r^6}$$

This implies that the following fields are gauge invariant:

$$G_{-} = \delta Q_{-} \quad \text{and} \quad G_{+} = (9M - 4r + \Lambda r^3)\delta Q_{+} + 3r^3\delta X$$

$G_{-} = G_{-}[\delta g^{(-)}]$, this functional depends on up to four derivatives of the $\phi^{-}_{(\ell,m)}$

Using repeatedly the LEE we arrive at: (recall that $\mathcal{L} = \{\overbrace{\delta j^{(k)}, \phi^{-}_{(\ell,m)}}^{odd}, \overbrace{\delta M, \phi^{+}_{(\ell,m)}}^{even}\}$)

$$G_{-} = -\frac{6M}{r^7}\sqrt{\tfrac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)}S_{(1,m)} - \frac{3M}{r^5}\underbrace{\sum_{(\ell\geq 2,m)}\tfrac{(\ell+2)!}{(\ell-2)!}\tfrac{\phi^{-}_{(\ell,m)}}{r}S_{(\ell,m)}}_{\Phi}$$

Using repeatedly the LEE we arrive at: (recall that $d.o.f. = \{\overbrace{\delta j^{(k)}, \phi^-_{(\ell,m)}}^{odd}, \overbrace{\delta M, \phi^+_{(\ell,m)}}^{even}\}$)

$$G_- = -\frac{6M}{r^7}\sqrt{\tfrac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)} S_{(1,m)} - \frac{3M}{r^5}\underbrace{\sum_{(\ell\geq 2,m)} \tfrac{(\ell+2)!}{(\ell-2)!}\tfrac{\phi^-_{(\ell,m)}}{r} S_{(\ell,m)}}_{\Phi}$$

In view of (RWE) and (23), any 4D field of the form

$$\Phi = \sum_{(\ell\geq 2,m)} c_\ell \tfrac{\phi^-_{(\ell,m)}}{r} S_{(\ell,m)}$$

satisfies a 4DRWE:

$$\nabla^\alpha\nabla_\alpha\Phi + \left(\frac{8M}{r^3} - \frac{2\Lambda}{3}\right)\Phi = 0. \tag{4DRWE}$$

Curiously enough, the $\ell = 1$ piece of G_-, that is, $-\frac{6M}{r^7}\sqrt{\frac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)} S_{(1,m)}$ also satisfies the 4DRWE, and so does G_-

We conclude that

$$\delta Q_- \equiv G_- = -\frac{6M}{r^7}\sqrt{\tfrac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)} S_{(1,m)} - \frac{3M}{r^5}\underbrace{\sum_{(\ell\geq 2,m)} \tfrac{(\ell+2)!}{(\ell-2)!}\tfrac{\phi^-_{(\ell,m)}}{r} S_{(\ell,m)}}_{\Phi}$$

- Contains all the information to allow reconstruct the metric perturbation
- Is a geometrically meaningful scalar field (perturbed scalar of curvature)
- Satisfies a 4D wave equation, then can be pointwise bounded following (Kay, Wald, 1987).
  This was done in (G.D., 2014, 2016), where it was proved that

$$|G_-| < \frac{constant}{r^6}, \quad r > 2M \quad (r_h < r < r_c \text{ for positive } \lambda$$

- For scalar/even modes can use Chandrasekhar notable observation that there is a duality such that, if $\phi^-_{(\ell,m)}$ is a solution of the nice 2D vector (RW) equation, then

$$\phi^+_{(\ell,m)} = [f\partial_r + Z_\ell]\,\phi^-_{(\ell,m)}$$

  solves the scalar (Zerilli) ugly equation!

- Chandrasekhar allows us to write the even curvature scalar perturbation field G_+ as

$$G_+ = -\frac{2M\dot{M}}{r^5} + \frac{M}{2r^3}\left[\partial_t^2\Phi_5 + \sum_{j=0}^{4} r^{-j}\Phi_j + \frac{f(r-3M)}{r^3}\Phi_5 + \frac{f}{r}\Phi_6\right] + \frac{M}{2r^3}\partial_x\Phi_6 + \frac{M(r-3M)}{2r^5}\partial_x\Phi_5, \quad (30)$$

where

$$\Phi_j = \sum_{(\ell\geq 2,m)} P_j(\ell)\frac{\phi^-_{(\ell,m)}}{r}S_{(\ell,m)}, \quad j = 0,1,2,3,4 \text{ ALL solutions of the 4DRWE}$$

- Chandrasekhar duality allows to compress all degrees of freedom of Schwarzschild linear perturbation theory on **two 4DRWE**.
- This was used to prove the *decay* of metric perturbations (Rodnianski, Dafermos and Holzegel, Acta Mathematica 2019), then extended to *slow rotating Kerr.*
- Above technique extrapolates without problem to Reissner-Nordström (J.Tio and G.D, 2016 and 2019)

Current status on full (non-linear) BH stability results

- Full non linear stability of Schwarzschild de-Sitter BH proved and published: *The global non-linear stability of the Kerr–de Sitter family of black holes* P Hintz, A Vasy Acta mathematica 220 (1), 1-206 (2018)

- *The non-linear stability of the Schwarzschild family of black holes*, preprint by M Dafermos, G. Holzegel, I. Rodnianski, M Taylor Apr 16, 2021 **513 pages** e-Print: 2104.08222 [gr-qc]

Conclusions:

- Symmetries allow to find exact solutions.

- Symmetries leave an imprint in the operators involved in the linearized.

- This allows to proceed to "dissect" the problem (for example, for non rotating BHs in any theory and dimension we will have scalar/vector/tensor modes) (Kodama & Ishibashi approach)

- Still, the above separations leaves the very difficult problem of finding "master variables" in the 1+1 orbit space: this is a theory-dependent, difficult task

- For refined results, **we need to undo variable separation**. This is non-modal linear perturbation, which leads to a number of geometric problems, such as identifying proper variables to measure the distortion caused by the perturbation.

- For definite stability studies we have to abandon the linear regime: go to full non linear theory, where the symmetry of the background is lost. Problems gets far away from geometry/physical considerations and falls definitively in the field of functional analysis.

- I have left notes for first half of the course (symmetries and singularities). Expect to review/complete them and add references soon (I will send the updated notes to organizers).
- For those of you interested in discuss the problems I left in these notes, you can contact me at **gustavo.dotti@unc.edu.ar**

Thank you for your attention!