Cosmological Parameters in Modified Theories of Gravity

S.V. Chervon
Ulyanovsk State Pedagogical University
Bauman Moscow State Technical University
Kazan Federal University

WACG-UFES, 6 March 2020

The action of a General Modified Gravitational Theories (GMGT):

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}F(\phi)R + \omega(\phi)X - V(\phi) - \xi(\phi)\mathcal{G} - G(\phi, X)\Box\phi\right] \tag{1}$$

- g – a determinant of the space-time metric $g_{\mu\nu}(x)$,
- R – a scalar curvature,
- $X = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ – a kinetic term of a scalar field ϕ,
- $F(\phi),\ \omega(\phi),\ \xi(\phi)$ – differentiable functions of ϕ,
- $G(\phi, X)$ depend on both ϕ and X,
- $\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ – the Gauss-Bonnet term,
- $V(\phi)$ – the potential energy (potential) of the field ϕ,
- $\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$.

In the work [A. De Felice et al, ArXiv:1105.4685] one can find

- Background equations for General Modified Gravitational Theories (GMGT)

$$E_1 \equiv 3M_{pl}^2 FH^2 + 3M_{pl}^2 H\dot{F} - \omega X - V - 24H^3\dot{\xi} - 6H\dot{\phi}XG_{,\phi} + 2XG_{,\phi} = 0$$

$$E_2 \equiv M_{pl}^2 \left[3FH^2 + 2H\dot{F} + 2F\dot{H} + \ddot{F}\right] + \omega X - V - 16H^3\dot{\xi} - 16H\dot{H}\dot{\xi} - 8H^2\ddot{\xi} - G_{,X}\dot{\phi}\dot{X} - G_{,\phi}\dot{\phi}^2 = 0$$

- Slow-roll parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{\dot{F}}{2HF} + \frac{\ddot{F}}{2H^2F} + \frac{\omega X}{M_{pl}^2 H^2 F} + \frac{4H\dot{\xi}}{M_{pl}^2 F} - \ldots$$

- New slow-roll parameters connected with $F, X, \xi, \ldots$ functions in the action

$$\delta_F \equiv \frac{\dot{F}}{HF}, \ \delta_X \equiv \frac{\omega X}{M_{pl}^2 H^2 F}, \ \delta_\xi \equiv \frac{H\dot{\xi}}{M_{pl}^2 F}, \ \ldots$$

The action (1) for the ADM perturbed metric leads to the second-order action

$$S_2 = \int dtd^3xa^3Q\left[\dot{\mathcal{R}}^2 - \frac{c_s^2}{a^2}\left(\partial\mathcal{R}\right)^2\right] \tag{2}$$

where $\mathcal{R}$ is the curvature perturbations, $Q \equiv \omega_1(4\omega_1\omega_3 + 9\omega_2^2)/(3\omega_2^2)$, $c_s^2 = f(\omega_i, \dot{\omega_k})$ and

$$\omega_1 \equiv M_{pl}^2 F - 8H\dot{\xi},\ \omega_2 \equiv M_{pl}^2(2HF + \dot{F}) - 2\dot{\phi}XG_{,X} - 24H^2\dot{\xi},\ \omega_4 \equiv M_{pl}^2 F - 8\ddot{\xi}$$

$$\omega_3 \equiv -9M_{pl}^2 FH^2 - 9M_{pl}^2 H\dot{F} + 3\omega X + 144H^3\dot{\xi} + 18H\dot{\phi}(2XG_{,X} + X^2G_{,XX}) - 6(XG_{,\phi} + X^2G_{,\phi X})$$

One can express ω_i $(i = 1, ..., 4)$ in terms of slow-roll parameters.

The power spectrum of curvature perturbation is given by

$$\mathcal{P}_s \equiv \frac{H^2}{8\pi^2 Q c_s^3}$$

which gives the scalar spectral index

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}_s}{d\ln k}\Big|_{c_s k = aH} = -2\epsilon - \delta_Q - 3s, \quad \delta_Q = \frac{\dot{Q}}{HQ}, \ s = \frac{\dot{c}_s}{Hc_s}$$

The tensor power spectrum and the tensor spectral index can be written in the same way.

The action in the Jordan frame

$$S_J = \int d^4x\sqrt{-g_J}\left[f(\phi^C)R_J - \frac{1}{2}h^J_{AB}\phi^A_{,\mu}\phi^B_{,\nu}g^{\mu\nu}_J - V_J(\phi^c)\right] \quad (3)$$

by conformal transformation $g^J_{\mu\nu} = \Omega^2(x)g^E_{\mu\nu}$ with $\Omega^2(x) = \frac{2}{M^2_{pl}}f(\phi^C)$ can be transformed to the action in the Einstein frame

$$S_E = \int d^4x\sqrt{-g_E}\left[\frac{M^2_{pl}}{2}R_E - \frac{1}{2}h^E_{AB}\phi^A_{,\mu}\phi^B_{,\nu}g^{\mu\nu}_J - V_E(\phi^c)\right] \quad (4)$$

where $h^E_{AB} = \frac{M^2_{pl}}{2f}\left[h^J_{AB} + 3f_{,A}f_{,B}/f\right]$ and $V_E = \frac{M^4_{pl}}{4f^2}V_J$.
Thus we need algorithm for calculations of cosmological parameters for the chiral cosmological model (4).

In the work Naruko et al (2015) it was considered the most general form of the gravitational theory with Ricci scalar and its higher derivatives. Such gravitational theory is described by the action

$$S = \int d^4x\sqrt{-g}\, f\left(R, (\nabla R)^2, \Box R\right), \tag{5}$$

where $(\nabla R)^2 = g^{\mu\nu}\nabla_\mu R \nabla_\nu R$ and $\Box R = g^{\mu\nu}\nabla_\mu\nabla_\nu R$. Excluding the f-dependence on $\Box R$ we have

$$S_1 = \int d^4x\sqrt{-g} f\left(R, (\nabla R)^2\right) \tag{6}$$

The situation with dependence $f\left(R, (\nabla R)^2\right) = f_1(R) + X(R) g^{\mu\nu}\nabla_\mu R \nabla_\nu R$ studied in the work Chervon et al, 2018. Let us approach to such model using Naruko et al results.

The action (6), using the set of Lagrange multipliers (and associated auxiliary fields), after special transformations can be reduced to the following form

$$S_2 = \int d^4x \sqrt{-g_j} \left[f\left(\phi, (\nabla\phi)^2\right) - \lambda(\phi - R_J) \right]. \tag{7}$$

The action (7) contains the interaction of the Lagrange multiplier λ to scalar curvature R. Thus the model belongs to the Jordan frame.
Now, by the standard way, we move from one frame (J-frame) to another (E-frame) by means of the conformal transformation $g^E_{\mu\nu} = \Omega^2 g^J_{\mu\nu}$ to E-frame choosing $\Omega^2 = 2\lambda$.

The resulting action is

$$S_8 = \int d^4x\sqrt{-g_E}\left(\frac{R_E}{2} - \frac{3}{4\lambda^2}g_E^{\mu\nu}\lambda_{,\mu}\lambda_{,\nu} + \frac{f_1(\phi)}{4\lambda^2} + \frac{X(\phi)\phi_{,\mu}\phi_{,\nu}g_E^{\mu\nu}}{2\lambda} - \frac{\phi}{4\lambda}\right) \qquad (8)$$

Let us introduce new scalar field $\lambda = \exp\left(\sqrt{2/3}\chi\right)$, with the aim to obtain a canonical form for the scalar field χ in (8).

Finally the action S_8 is transformed to the form

$$S_9 = \int d^4x\sqrt{-g_E}\left(\frac{R_E}{2} - \frac{1}{2}g_E^{\mu\nu}\chi_{,\mu}\chi_{,\nu} + \frac{1}{4}f_1(\phi)e^{-2\sqrt{\frac{2}{3}}\chi} - \frac{1}{4}\phi e^{-\sqrt{\frac{2}{3}}\chi} + \frac{1}{2}Xe^{-\sqrt{\frac{2}{3}}\chi}g_E^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right) \tag{9}$$

To study correspondence to observation data we need the algorithm of cosmological parameters calculation.

The action S_9 (9) can be considered as 2D CCM with the chiral fields $\varphi^1 = \chi$, $\varphi^2 = \phi$.
The target space metric is

$$\left(ds^2 = d\chi^2 - e^{-\sqrt{\frac{2}{3}\chi}} X(\phi) d\phi^2 \right)$$

That is $h_{11} = 1$, $h_{22} = -e^{-\sqrt{\frac{2}{3}}\chi} X(\phi)$; $h_{12} = h_{21} = 0$
The potential is

$$W = \frac{1}{4} e^{-\sqrt{\frac{2}{3}}\chi} \left(\phi - e^{-\sqrt{\frac{2}{3}}\chi} f_1(\phi) \right)$$

The dynamic equations are

$$3H^2 = \frac{1}{2}\dot{\chi}^2 - \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}X(\phi)\dot{\phi}^2 + \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(-\phi + e^{-\sqrt{\frac{2}{3}}\chi}f_1(\phi)\right) \tag{10}$$

$$\dot{H} = -\frac{1}{2}\dot{\chi}^2 + \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}X(\phi)\dot{\phi}^2 \tag{11}$$

$$\ddot{\chi} + 3H\dot{\chi} - \frac{1}{2}\dot{\phi}^2\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}X(\phi) + \frac{1}{4}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi - 2e^{-\sqrt{\frac{2}{3}}\chi}f_1(\phi)\right) = 0 \tag{12}$$

$$X(\phi)\left(-3H\dot{\phi} - \ddot{\phi} + \sqrt{\frac{2}{3}}\dot{\chi}\dot{\phi}\right) - \frac{1}{2}\dot{\phi}^2X'(\phi) - \frac{1}{4} + \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}f_1'(\phi) = 0 \tag{13}$$

The general action

$$S = \int d^4x\sqrt{-g}\, f\left(R, (\nabla R)^2, \Box R\right) \tag{14}$$

can be transformed to the following one

$$S = \int d^4x\sqrt{-g}\,\left[f\left(\phi, (\nabla\phi)^2, \Box\phi\right) - \lambda(\phi - R) - \Lambda(B - \Box\phi)\right], \ \Lambda = f_B = \partial f/\partial B \tag{15}$$

If $f_{BB} \neq 0$, then $B = \Box\phi$

$$S = \int d^4x\sqrt{-g}\,\left[\lambda R - g^{\mu\nu}\partial_\mu f_B \partial_\nu\phi + f - f_B B - \lambda\phi\right] \tag{16}$$

Suggesting $\lambda > 0$ we can transform the action to the E-frame

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}g^{\mu\nu}\left(\chi_{,\mu}\chi_{,\nu} + e^{-\sqrt{\frac{2}{3}}\chi}\varphi_{,\mu}\phi_{,\nu}\right) - \frac{1}{4}\left(e^{-\sqrt{\frac{2}{3}}}(\varphi B - f)\right)\right] \qquad (17)$$

here $B = B(\phi, (\nabla\phi)^2, \varphi)$ and $f_{BB} \neq 0$.

Choosing the linear dependence for kinetic part of the function B as follows

$$B\left(\phi, (\nabla_E \phi)^2, \varphi\right) = 2B_1(\phi,\varphi)\, g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + B_2(\phi,\varphi) \tag{18}$$

we arrive to the Chiral Cosmological Model (CCM) with the metric of the target space

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2B_1(\phi,\varphi) & e^{-\sqrt{2/3}\chi} \\ 0 & e^{-\sqrt{2/3}\chi} & 0 \end{bmatrix}$$

and with the potential

$$W(\chi,\phi,\varphi) = \frac{1}{4}\left[e^{-\sqrt{2/3}\chi}\phi + e^{-2\sqrt{2/3}\chi}\left(B_2(\phi,\varphi)\varphi - f\right)\right] \tag{19}$$

The case $f_{BB}=0$ leads to the action in E-frame

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2}-\frac{1}{2}g^{\mu\nu}\left(\chi_{,\mu}\chi_{,\nu}+\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}G\chi_{,\mu}\phi_{,\nu}\right)+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}K-\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\phi}\right]$$

where $K=K\left(\phi,2e^{\sqrt{\frac{2}{3}}}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)$, $\quad G=G\left(\phi,2e^{\sqrt{\frac{2}{3}}}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)$.

Note, the term $\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}G\Box\phi$ could not alow us to reduce the model above to CCM by standard way. Therefore we may consider special cases.

If we choose

$$K = K_1(\phi) + Y(\phi) e^{\sqrt{\frac{2}{3}\chi}} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}, \quad G = G_1(\phi) = 2e^{\sqrt{\frac{2}{3}}\chi} \tag{20}$$

then the CCM characterised by the target space metric

$$\begin{bmatrix} 1 & 2\sqrt{2/3} \\ 2\sqrt{2/3} & e^{-\sqrt{2/3}\chi} Y(\phi) \end{bmatrix} \tag{21}$$

and the potential

$$V(\chi, \phi) = \frac{1}{2} e^{-2\sqrt{\frac{2}{3}}\chi} \left(\phi/2 - K_1(\phi)\right) \tag{22}$$

To apply standard methods for calculation of cosmological parameters let us suggest the linear dependence between fields

$$\phi(t) = k\chi(t), \quad \dot{\phi}(t) = k\dot{\chi}(t), \quad \ddot{\phi}(t) = k\ddot{\chi}(t), \quad k = const. \tag{23}$$

Derivatives wrt field ϕ are

$$\frac{dX}{d\phi} = k^{-1}\frac{dX}{d\chi}, \quad \frac{df_1}{d\phi} = k^{-1}\frac{df_1}{d\chi} \tag{24}$$

Then the action transforms to the one with single scalar field and can be studied by usual manner.

The action (9) reads

$$S=\int d^4x\sqrt{-g_E}\left(\frac{R_E}{2}-\frac{1}{2}\left[1-k^2X(\chi)e^{-\sqrt{\frac{2}{3}}\chi}\right]g_E^{\mu\nu}\chi_{,\mu}\chi_{,\nu}+\frac{1}{4}f_1(\chi)e^{-2\sqrt{\frac{2}{3}}\chi}-\frac{1}{4}k\chi e^{-\sqrt{\frac{2}{3}}\chi}\right) \tag{25}$$

Then the equations of cosmological dynamics (10)-(13) take the form:

$$3H^2=\frac{1}{2}\dot{\chi}^2-\frac{1}{2}k^2e^{-\sqrt{\frac{2}{3}}\chi}X(\chi)\dot{\chi}^2+\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(-k\chi+e^{-\sqrt{\frac{2}{3}}\chi}f_1(k\chi)\right), \tag{26}$$

$$\dot{H}=-\frac{1}{2}\dot{\chi}^2\left(1-k^2e^{-\sqrt{\frac{2}{3}}\chi}X(\chi)\right), \tag{27}$$

The χ-field equation is

$$\ddot{\chi}+3H\dot{\chi}-\frac{1}{2}k^2\dot{\chi}^2 e^{-\sqrt{\frac{2}{3}}\chi}\left[-\sqrt{\frac{2}{3}}X(\chi)+\frac{dX}{d\chi}\right]+$$

$$\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(e^{-\sqrt{\frac{2}{3}\chi}}\left(2\sqrt{\frac{2}{3}}f_1(\chi)-\frac{df_1}{d\chi}\right)-\sqrt{\frac{2}{3}}k\chi+k\right)=0 \qquad (28)$$

Evidently, from (25) the potential takes the form

$$V(\chi)=\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(k\chi-e^{-\sqrt{\frac{2}{3}}\chi}f_1(\chi)\right)$$

From (26) and (27) we have the consequense

$$3H^2+\dot{H}=V(\chi,k\chi)=\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(-k\chi+e^{-\sqrt{\frac{2}{3}}\chi}f_1(k\chi)\right) \qquad (29)$$

In order to apply the ISB approach, we use the equation (27) and the freedom to choose the kinetic function $X(\chi)$. Transforming to the dependence of the Hubble function on the field χ from (27), we have:

$$H'=-\frac{1}{2}\dot{\chi}\left(1-k^2e^{-\sqrt{\frac{2}{3}}\chi}X(\chi)\right), \qquad (30)$$

Then, the function selection

$$X(\chi)=e^{\sqrt{\frac{2}{3}}\chi} \qquad (31)$$

allows us to reduce the equation (29) to the following

$$3H^2+\frac{2H'}{1-k^2}=\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(-k\chi+e^{-\sqrt{\frac{2}{3}}\chi}f_1(\chi)\right) \qquad (32)$$

Choosing the function $f_1(\chi)$ as

$$f_1(\chi) = 4\chi e^{\sqrt{\frac{2}{3}}\chi}\left(1+\frac{k}{4}-\frac{k_1}{6}\right) - 4k_1 e^{\sqrt{\frac{2}{3}}\chi}\left(\frac{1}{4\chi}-\frac{1}{\sqrt{6}}\right) \tag{33}$$

where $k \neq 1, \quad k_1 = \frac{2}{3(1-k^2)}$ one can find the Hubble parameter

$$H(\chi) = \sqrt{\frac{\chi}{3}e^{-\frac{\chi}{\sqrt{6}}}} \tag{34}$$

and e-folds number

$$N = (k^2-1)\sqrt{\frac{3}{2}}\left[\chi - \frac{3}{2}\ln|2\chi+\sqrt{6}|\right] \tag{35}$$

With our choice: $X = e^{\sqrt{\frac{2}{3}}\chi}$ the potential takes the form:

$W = \frac{1}{4} e^{-\sqrt{\frac{2}{3}}\chi} \left(kx - f_1(\phi) e^{-\sqrt{\frac{2}{3}}\chi} \right)$. Let us introduce new variable : $\chi = \frac{\varphi}{\sqrt{1-k^2}}$

For the case of massive inflation:

$$V(\varphi) = \frac{m^2 \varphi^2}{2} - \frac{m^2}{3\kappa} \tag{36}$$

$$H(\varphi) = \sqrt{\frac{\kappa}{6}} m\varphi \tag{37}$$

$$\varphi = -m \sqrt{\frac{2}{3\kappa}} t \tag{38}$$

we have

$$\chi = -\frac{m}{\sqrt{1-k^2}} \sqrt{\frac{2}{3\kappa}} t = -m_k \sqrt{\frac{2}{3\kappa}} t \tag{39}$$

Now we are searching for the *exact inflationary* parameters, defined in [Chervon et al, 2019]. First parameter ($\kappa = 1$) we can find from the relation

$$\epsilon = 2\left(\frac{H'_\varphi}{H}\right)^2 = -\frac{\dot{H}}{H^2}. \tag{40}$$

The result for the massive inflation is

$$\epsilon = \frac{3}{m^2 t^2}. \tag{41}$$

Next, we turn to Hubble parameter dependence on the number of e-foldings $N(t) = -\int H(t)dt = \frac{m^2 t^2}{6}$. Using (40) in $H = H(N)$ representation

$$\epsilon = -\frac{dH}{dN}\frac{\dot{N}}{H^2} \tag{42}$$

and using dependencies on time we obtain $\frac{dH}{dN}$ in the form:

$$\frac{dH}{dN} = \frac{1}{t} \tag{43}$$

Taking into account the Hubble parameter dependence on time $H(t) = \frac{m^2}{2}t$ we find the dependence $t(N) = \frac{\sqrt{6N}}{m}$. Thus we found the dependence $\epsilon = \epsilon(N)$ in the form $\epsilon = \frac{1}{2N}$. If we take $N = 60$ then $\epsilon = 0.0083$. Calculation of the second exact inflationary parameter $\delta = 2\frac{H''}{H} = -\frac{\ddot{H}}{2H\dot{H}}$ gives us $\delta = 0$.

To define the scalar spectral parameter n_S we will use the refined formula

$$n_S - 1 = 2\left(\frac{\delta - 2\epsilon}{1-\epsilon}\right). \tag{44}$$

As the result we obtain $n_S \simeq 0.9664$. Such a way we see that the result corresponds with good accuracy to the observational data of the Planck space mission.

Tensor spectral parameter for this model is defined from the relation

$$n_T = -\frac{2\epsilon}{1-\epsilon} \tag{45}$$

and equal to $n_T = -0.0167$. Tensor-to-scalar ratio we define by formula $r = 4s\epsilon$, where $s = 1$ or $s = 4$ is the value of normalization of the tensor perturbations. If $s = 1$ we obtain $r = 0.0332 < 0.1$ and $r = 0.0332 < 0.065$ in the correspondence with observational data. If $s = 4$, $r = 0.1328$, this result does not correspond to observations.
To coordinate the model according to the power spectrum of scalar perturbations, we use the refined formula

$$\mathcal{P}_S(k) = \frac{1}{2\epsilon}\left(\frac{H}{2\pi}\right)^2 \tag{46}$$

and the experimental value $\mathcal{P}_S(k) = 2.14 \times 10^{-9}$. As a result, we obtain that the mass of the scalar field φ is determined from the relation $m^2 = 6.25 \times 10^{-11}$. From the comparison of the action (25), taking into account (31), we determine the relationship between the masses of the fields φ and χ : $m_\varphi = m_\chi/\sqrt{1 \mp k^2}$. That is, if it is necessary to coordinate with a specific particle mass, the mass m_φ can be decreased or increased.

Power spectrum is

$$P_s(k) = 3600\frac{3600m_k^2}{24\pi^2} \approx 15,2136m_k^2 \tag{47}$$

Confronting with observational data:

$$15,2136m_k^2 = 2,1 \cdot 10^{-9} \tag{48}$$

we get:

$$\frac{m}{\sqrt{1 \mp k^2}} \simeq 1,38 \cdot 10^{-10} \tag{49}$$

Power law and intermediate inflation was considered as well with described algorithm.