Dark matter particles velocity dispersion

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Outline

Collaborators

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Have a look here:
`http://www.cosmo-ufes.org`

Standard model of cosmology

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \frac{\Omega_k}{a^2} + \frac{\Omega_c}{a^3} + \frac{\Omega_b}{a^3} + \frac{\Omega_r}{a^4}$$

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Comparison of *Planck*-only and *WMAP*-only Six-Parameter ΛCDM Fits[a]

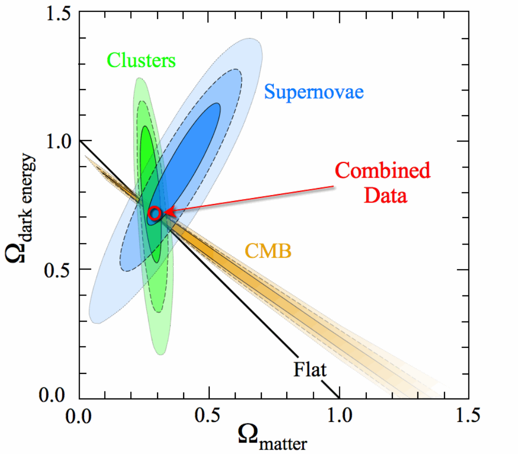
Parameter	*Planck* ("CMB+Lens")	*WMAP* (9-year)	Difference value	Difference *WMAP* σ
$\Omega_b h^2$	0.02217 ± 0.00033	0.02264 ± 0.00050	-0.00047	0.9
$\Omega_c h^2$	0.1186 ± 0.0031	0.1138 ± 0.0045	0.0048	1.1
Ω_Λ	0.693 ± 0.019	0.721 ± 0.025	-0.028	1.1
τ	0.089 ± 0.032	0.089 ± 0.014	0	0
t_0 (Gyr)	13.796 ± 0.058	13.74 ± 0.11	56 Myr	0.5
H_0 (km s^{-1}Mpc^{-1})	67.9 ± 1.5	70.0 ± 2.2	-2.1	1.0
σ_8	0.823 ± 0.018	0.821 ± 0.023	0.002	0.1
Ω_b	0.0481[b]	0.0463 ± 0.0024	0.0018	0.7
Ω_c	0.257[b]	0.233 ± 0.023	0.024	1.0

[a]The new *Planck* results strongly favor the standard six-parameter ΛCDM model with parameter values that are consistent with *WMAP* parameters, as shown in this table which compares results derived entirely from *Planck* data with those derived entirely from *WMAP* data.

[b]Parameters derived from quoted values. No error estimate is given for this data/model combination.

`http://lambda.gsfc.nasa.gov/common/images/planck_wmap_table.svg`

Cosmic concordance



https://www.astro.virginia.edu/class/whittle/astr553/Topic01/t1_cos_combined.gif

Open questions in cosmology

Fundamental ones

- What are dark matter and dark energy?

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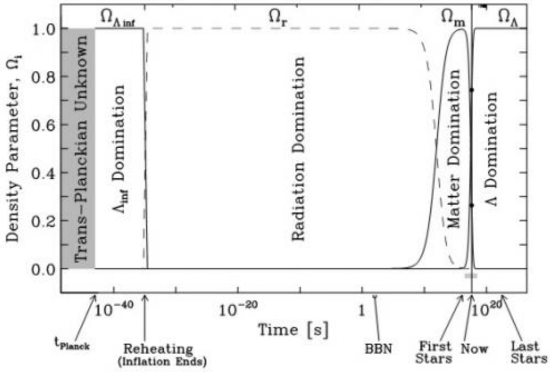

http://cdn.phys.org/newman/gfx/news/2011/1-cosmiccoinci.jpg

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- Too big to fail problem [Boylan-Kolchin et al., 2011]

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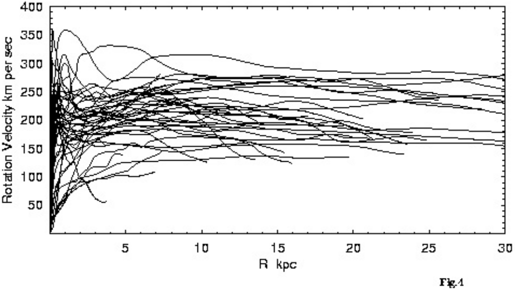

http://ned.ipac.caltech.edu/level5/Sept05/Sofue/Figures/figure4.jpg

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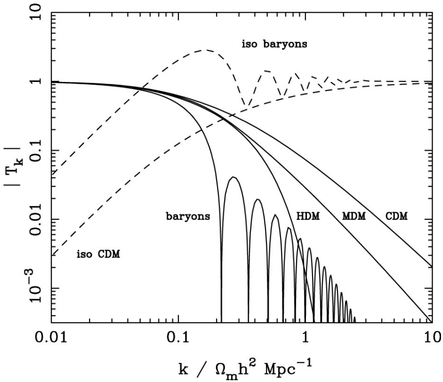

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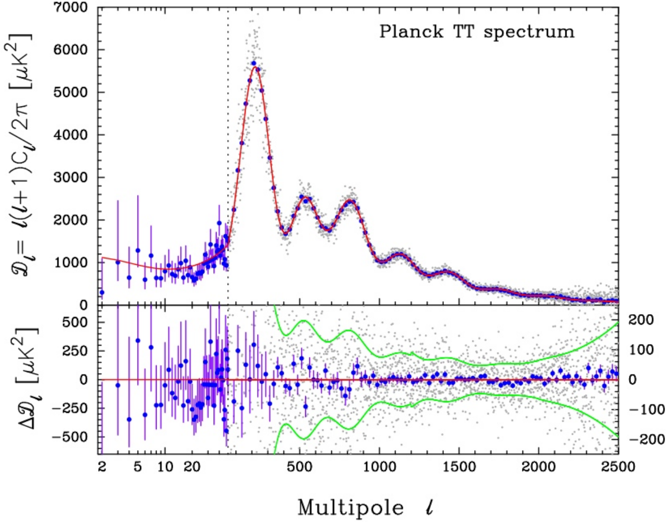

What is dark matter?

- New particles [Bertone et al., 2005]

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Modified gravity

- MOND [Milgrom, 1983]

$$F = m\mu(a/a_0)a$$

- TeVeS [Bekenstein, 2004]
- Conformal gravity [Mannheim, 2002]

$$\mathcal{S} = \int d^4x\sqrt{-g}C_{abcd}C^{abcd}$$

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Non-trivial GR effects:

- Non-Newtonian effects from an axial symmetric geometry [Balasin and Grumiller, 2006]

Dark matter particles

Some famous candidates:

- Sterile Neutrino ($m \gtrsim 10$ keV) [Dodelson and Widrow, 1994]

$$\mathcal{L} = \mu \left(\frac{\phi}{v} \right) \bar{\nu}_L \nu_R + M \nu_R \nu_R + \text{h.c.}$$

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- Axion ($m \lesssim 0.01$ eV, $s = 0$, non-thermally produced) [Peccei and Quinn, 1977]
- Neutralino (MSSM) ($m \gtrsim 100$ GeV) or WIMP in general.

$$\Omega_{\text{WIMP}} \approx 0.2 \frac{(m/T_{\text{cd}})/25}{\langle \sigma_{\text{ann}} v \rangle / 1 \text{ pb}}$$

Stable relics from the electroweak scale

Primordial evolution of DM

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For WIMPS $T_{\rm kd} < T_{\rm cd} < T_{\rm nr}$ and $5 \lesssim T_{\rm kd}/\mathrm{MeV} \lesssim 3000$
[Profumo et al., 2006]
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For the neutralino, $T_{\rm cd} \approx 4$ GeV ($z \approx 10^{13}$) and $T_{\rm kd} \approx 25$ MeV ($z \approx 10^{11}$).

Physics of the primordial evolution of DM

When $T_{\mathrm{kd}} < T < T_{\mathrm{cd}}$

- Collisional damping $\rightarrow$ Damping scale [Green et al., 2004]

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Diffusion wavenumber

$$k_{\rm d} \approx \frac{3.8 \times 10^7}{\rm Mpc} \left(\frac{m}{100\ \rm GeV}\right)^{1/2} \left(\frac{T_{\rm kd}}{30\ \rm MeV}\right)^{1/2}$$

Diffusion mass

$$M_{\rm d} \approx 10^{-10} M_\odot$$

Correction to the transfer function

$$D(k) = \exp\left[-\left(\frac{k}{k_{\rm d}}\right)^2\right]$$

Physics of the primordial evolution of DM

When $T_{\mathrm{kd}} < T < T_{\mathrm{cd}}$

- Collisional damping $\rightarrow$ Damping scale [Green et al., 2004]
- Tight coupling $\rightarrow$ Acoustic oscillations [Loeb and Zaldarriaga, 2005]

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$$M_{\mathrm{cut}} \approx 10^{-4} \left(\frac{T_{\mathrm{kd}}}{\mathrm{MeV}} \right)^{-3} M_{\odot}$$

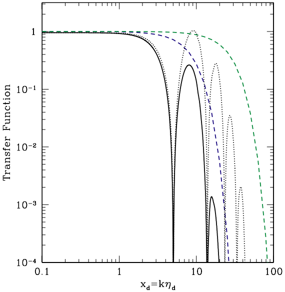

FIG. 2 (color online). Transfer function of the CDM density perturbation amplitude (normalized by the primordial amplitude from inflation). We show two cases: (i) $T_d/M = 10^{-4}$ and $T_d/T_{\mathrm{eq}} = 10^7$; (ii) $T_d/M = 10^{-5}$ and $T_d/T_{\mathrm{eq}} = 10^7$. In each case the oscillatory curve is our result and the other curve is the free-streaming only result that was derived previously in the literature [4,7,8].

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When $T < T_{\rm kd}$

- Free-streaming → Free-streaming length [Green et al., 2004]

Physics of the primordial evolution of DM

When $T < T_{\rm kd}$

- Free-streaming $\rightarrow$ Free-streaming length [Green et al., 2004]

Free-streaming wavenumber

$$k_{\rm fs} \approx \frac{1.7 \times 10^6}{\rm Mpc} \frac{(m/100\ {\rm GeV})^{1/2}\,(T_{\rm kd}/30\ {\rm MeV})^{1/2}}{1 + \ln\left(T_{\rm kd}/30\ {\rm MeV}\right)/19.2}$$

Free-streaming mass

$$M_{\rm fs} \approx 10^{-6} M_\odot$$

Total correction to the transfer function (including collisional damping)

$$D(k) = \left[1 - \frac{2}{3}\left(\frac{k}{k_{\rm fs}}\right)^2\right] \exp\left[-\left(\frac{k}{k_{\rm fs}}\right)^2 - \left(\frac{k}{k_{\rm d}}\right)^2\right]$$

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When $T < T_{\rm kd}$

- Free-streaming $\rightarrow$ Free-streaming length [Green et al., 2004]
- Temperature fluctuations $\rightarrow$ Jeans length [Piattella et al., 2013]

Physics of the primordial evolution of DM

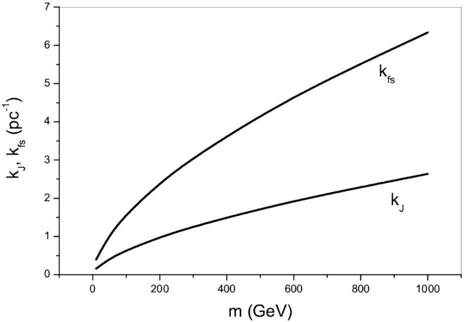
When $T < T_{\mathrm{kd}}$

- Temperature fluctuations → Jeans length [Piattella et al., 2013]

Jeans wavenumber

$$k_{\mathrm{J}} = \frac{12\pi G}{5\sigma} a^2 \rho_{\mathrm{dm}} \text{ with } \sigma = 3T/m$$

Comparison with free-streaming length



Boltzmann equation in GR

Distribution function

$$f\left(t, x^i, P^i = m\frac{dx^i}{d\tau}\right)$$

Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} + \frac{dP^i}{dt}\frac{\partial f}{\partial P^i} = 0$$

GR enters here

$$\frac{dP^i}{d\tau} + \Gamma^i_{\mu\nu}P^\mu P^\nu = 0$$

Boltzmann equation in cosmology

[Bernstein, 1988]

Flat FLRW metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

Proper momentum

$$p^2 = a(t)^2 \delta_{ij} P^i P^j$$

so

$$P^i = \frac{p}{a} \hat{n}^i$$

where $\delta_{ij} \hat{n}^i \hat{n}^j = 1$. VE becomes

$$\frac{\partial f}{\partial t} - H \frac{dp}{dt} \frac{\partial f}{\partial p} = 0$$

Because of isotropy, no dependence on x^i and $\hat{n}^i$.

Solution and important quantities

Solution

$$f = f(ap)$$

Particles number density

$$n = \int d^3p f(ap) = \frac{4\pi}{a^3} \int_0^\infty dx\, x^2 f(x)$$

Velocity dispersion

$$\sigma^{ij} = \frac{1}{n} \int d^3p f(ap) \frac{p^2}{E^2} \hat{n}^i \hat{n}^j$$

Call

$$\sigma = \delta_{ij} \sigma^{ij}$$

Momenta

Non-relativistic approximation: Neglect $O\left(p^4/m^4\right)$ terms ($p^4/m^4 \sim 10^{-6}$ for the neutralino).
Zero momentum

$$\int d^3p \left(\frac{\partial f}{\partial t} - H\frac{dp}{dt}\frac{\partial f}{\partial p}\right) = 0 \quad \Rightarrow \quad \frac{\partial n}{\partial t} + 3Hn = 0$$

Second momentum

$$\frac{\partial}{\partial t}\int d^3p f \frac{p^2}{m^2}\hat{n}^i\hat{n}^j - H\int d^3p p \frac{\partial f}{\partial p}\frac{p^2}{m^2}\hat{n}^i\hat{n}^j = 0$$

i.e.

$$\frac{\partial \sigma}{\partial t} + 2H\sigma = 0 \quad \Rightarrow \sigma \propto a^{-2}$$

Density evolution

Density

$$\varepsilon \equiv \int d^3pEf$$

multiplying Vlasov equation by d^3pE and integrating over the momenta gives

$$\frac{\partial\varepsilon}{\partial t} + 3H\left(\varepsilon + \frac{1}{3}mn\sigma\right) = 0$$

$$\varepsilon = mn_{\rm kd}\left(\frac{a_{\rm kd}}{a}\right)^3 + \frac{mn_{\rm kd}\sigma_{\rm kd}}{2}\left(\frac{a_{\rm kd}}{a}\right)^5 \ .$$

For the neutralino $\sigma_{\rm kd} \approx 10^{-3}$. Velocity dispersion acts like a pressure. Does its perturbation act as a pressure perturbation?

Perturbations

[Ma and Bertschinger, 1995]

Synchronous gauge

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right]$$

Scalar perturbations only

$$h_{ij}(\mathbf{x},\tau) = \int d^3k \exp(i\mathbf{k}\cdot\mathbf{x}) \left[\hat{k}_i\hat{k}_j h(\mathbf{k},\tau) + \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right) 6\eta(\mathbf{k},\tau)\right]$$

Fluctuation in the distribution function

$$f(\tau, x^i, P^i) = f_0(\tau, p) + f_1(\tau, x^i, P^i)$$

Perturbed Vlasov-Einstein equation:

$$\frac{\partial f_1}{\partial \tau} - \frac{\dot{a}}{a} p \frac{\partial f_1}{\partial p} + \frac{p}{E}\left(i\mathbf{k}\cdot\hat{\mathbf{n}}\right) f_1 + p\frac{\partial f_0}{\partial p}\left[\dot{\eta} - \frac{\dot{h} + 6\dot{\eta}}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})^2\right] = 0$$

Perturbed quantities

Particle number density

$$n = n_0 + n_1 = \int d^3p f_0 + \int d^3p f_1$$

Peculiar velocity

$$n_0 v^i = \int d^3p \frac{p\hat{n}^i}{m} f_1$$

Perturbed velocity dispersion

$$n\sigma^{ij} = \int d^3p \frac{p^2}{m^2}(f_0 + f_1)\hat{n}^i\hat{n}^j$$

Define v_1 as

$$n_0\sigma_1^{ij} + n_1\sigma_0^{ij} = \int d^3p \frac{p^2}{m^2} f_1 \hat{n}^i\hat{n}^j \equiv n_0 v_1^{ij}$$

v_1 acts as a pressure perturbation. Energy density

$$\rho = n_0 m + n_1 m + \frac{1}{2} n_0 m \sigma_0 + \frac{1}{2} n_0 m v_1$$

Zero momentum

Define

$$\delta_n \equiv \frac{n_1}{n_0} \qquad kv \equiv ik_i v^i$$

Zero momentum

$$\dot{\delta}_n + kv + \frac{\dot{h}}{2} = 0$$

But

$$\delta \equiv \frac{\rho_1}{\rho_0} = \frac{n_1 m + m n_0 v_1/2}{n_0 m + m n_0 \sigma_0/2} = \frac{\delta_n + v_1/2}{1 + \sigma_0/2}$$

So

$$\left(1 + \frac{\sigma_0}{2}\right)\dot{\delta} + \frac{\dot{\sigma}_0}{2}\delta - \frac{\dot{v}_1}{2} + kv + \frac{\dot{h}}{2} = 0$$

First momentum

First momentum

$$\dot{v} + \frac{\dot{a}}{a} v - k \hat{k}_i \hat{k}_j v_1^{ij} = 0$$

Expand v_1^{ij} in its trace and traceless part

$$v_1^{ij} = \frac{v_1}{3} \delta^{ij} + \Sigma^{ij}$$

Then

$$\dot{v} + \frac{\dot{a}}{a} v - \frac{k}{3} v_1 - k\Pi = 0$$

where we have defined the anisotropic shear as

$$\Pi \equiv \hat{k}_i \hat{k}_j \Sigma^{ij}$$

Perturbed Einstein equations

$$k^2\eta - \frac{1}{2}\frac{\dot{a}}{a}\dot{h} = 4\pi G a^2 \delta T^0{}_0$$

$$k^2\dot{\eta} = 4\pi G a^2 i k^i \delta T^0{}_i$$

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} - 2k^2\eta = -8\pi G a^2 \delta T^i{}_i$$

$$\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a}(\dot{h} + 6\dot{\eta}) - 2k^2\eta = 24\pi G a^2 \hat{k}_i \hat{k}^j (\delta T^i{}_j - \delta T^l{}_l \delta^i{}_j/3)$$

Modified contribution for DM

$$\delta T^0{}_0(\mathrm{DM}) = -\rho_0\delta$$

$$i k^i \delta T^0{}_i(\mathrm{DM}) = \rho_0 \frac{kv}{1+\sigma_0/2}$$

$$\delta T^i{}_i(\mathrm{DM}) = \rho_0 \frac{v_1}{1+\sigma_0/2}$$

$$\hat{k}_i \hat{k}^j (\delta T^i{}_j - \delta T^l{}_l \delta^i{}_j/3)(\mathrm{DM}) = \rho_0 \frac{\Pi}{1+\sigma_0/2}$$

Adiabaticity

$$Tds = \frac{d\rho}{n} - \frac{\rho + p}{n^2} dn$$

$Tds = 0$ for DM

$$\frac{\delta}{1+w} = \delta_n \qquad w \equiv \frac{\sigma_0/3}{1+\sigma_0/2}$$

Therefore

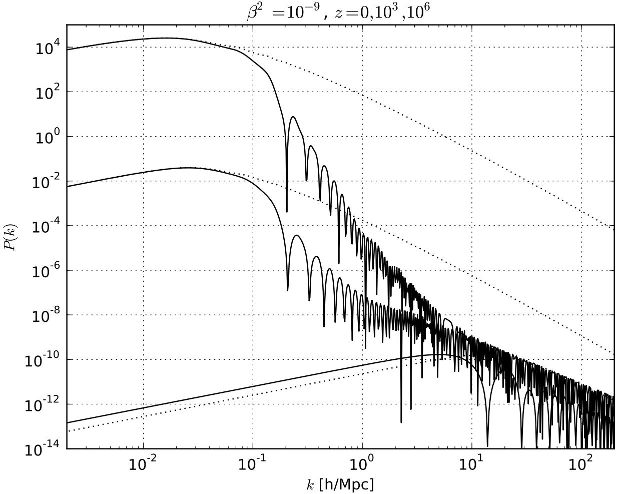
$$v_1 = \frac{5}{3(1+w)}\sigma_0\delta = \frac{1+\sigma_0/2}{1+5\sigma_0/6}\frac{5}{3}\sigma_0\delta$$

Thus, assuming $\Pi = 0$

$$\ddot{\delta}_n + \frac{\dot{a}}{a}\delta_n + \frac{\ddot{h}}{2} + \frac{\dot{a}}{a}\frac{\dot{h}}{2} + \frac{5}{9}k^2\sigma_0\delta_n = 0$$

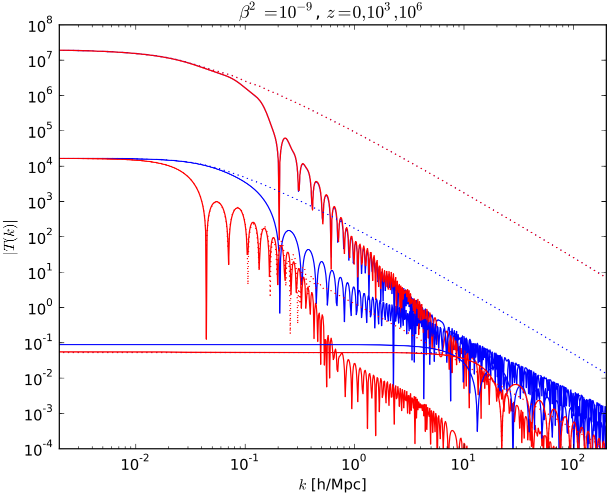
DM power spectrum

$\beta^2 = 10^{-9}$



DM and baryon transfer function

$\beta^2 = 10^{-9}$



DM power spectrum

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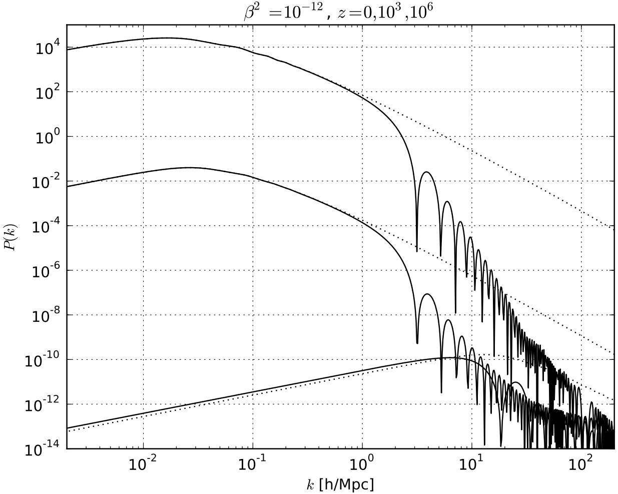

DM and baryon transfer function

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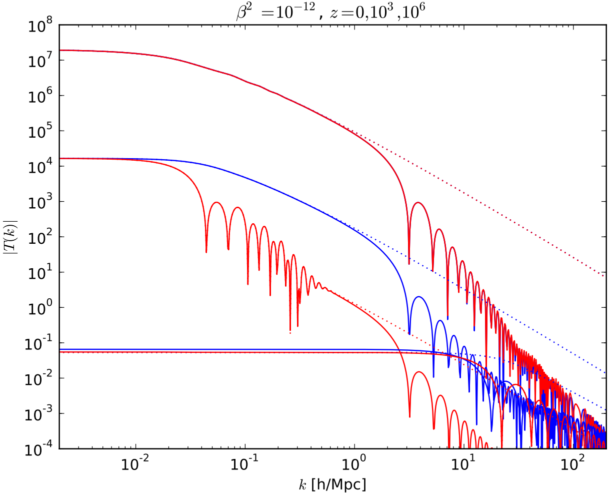

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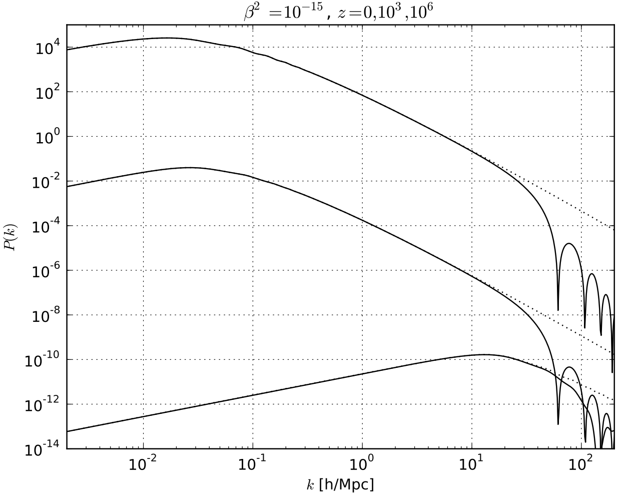

DM and baryon transfer function

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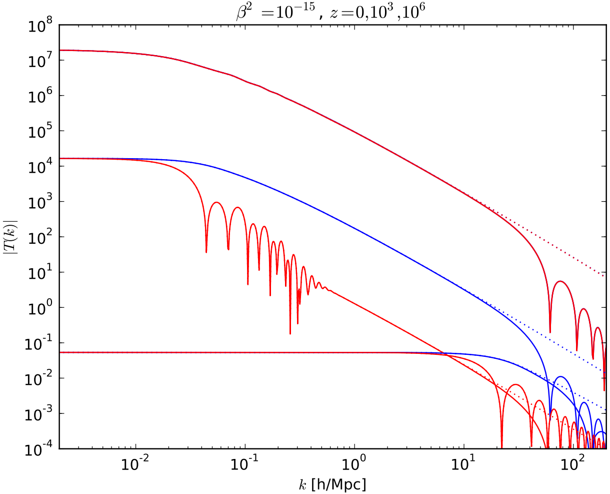

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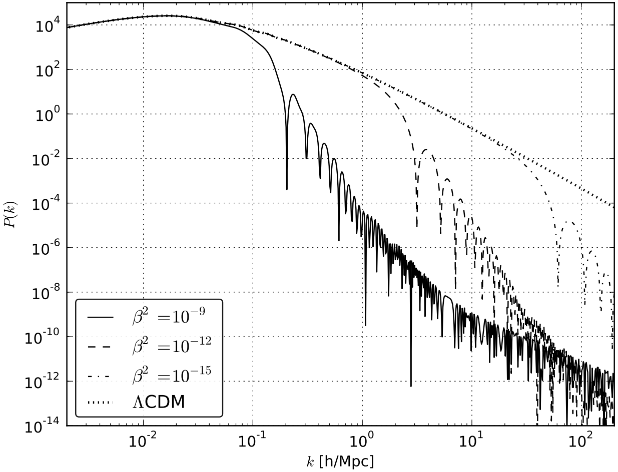
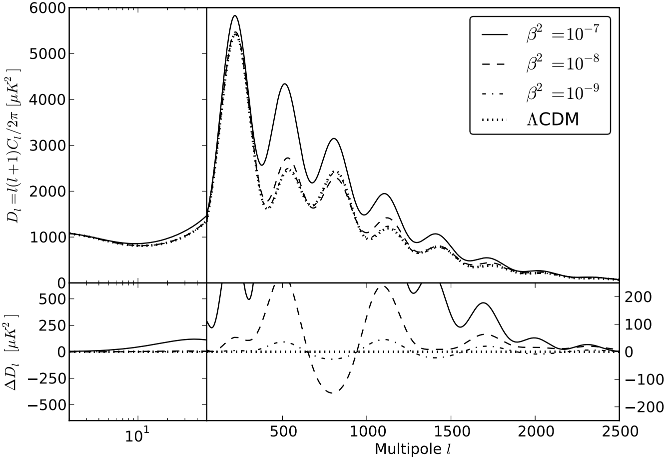


CMB TT spectrum



Summary and Conclusions

1. Velocity dispersion of DM acts as a pressure
2. Fluctuations in the velocity dispersion of DM act as a pressure perturbation
3. A Jeans scale is set
4. This scale is very small, because DM velocity dispersion is relevant at very early times
5. Qualitatively, today β^2 has to be smaller than 10^{-9}.

Future work

1. Spherical collapse
2. Mass function
3. Seeds for cosmological simulations
4. Extension for WDM (it decouples relativistically, so our treatment is no more valid)