PHYSICAL PROPERTIES OF PERTURBED BLACK-HOLE INITIAL DATA

(ON HYPERBOLOIDAL SLICES)

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Class. Quantum Grav. 31 075017 (2014)

Class. Quantum Grav. 31 165001 (2014)

GRK 1523
Quantum and
Gravitational Fields

SFB/Transregio 7

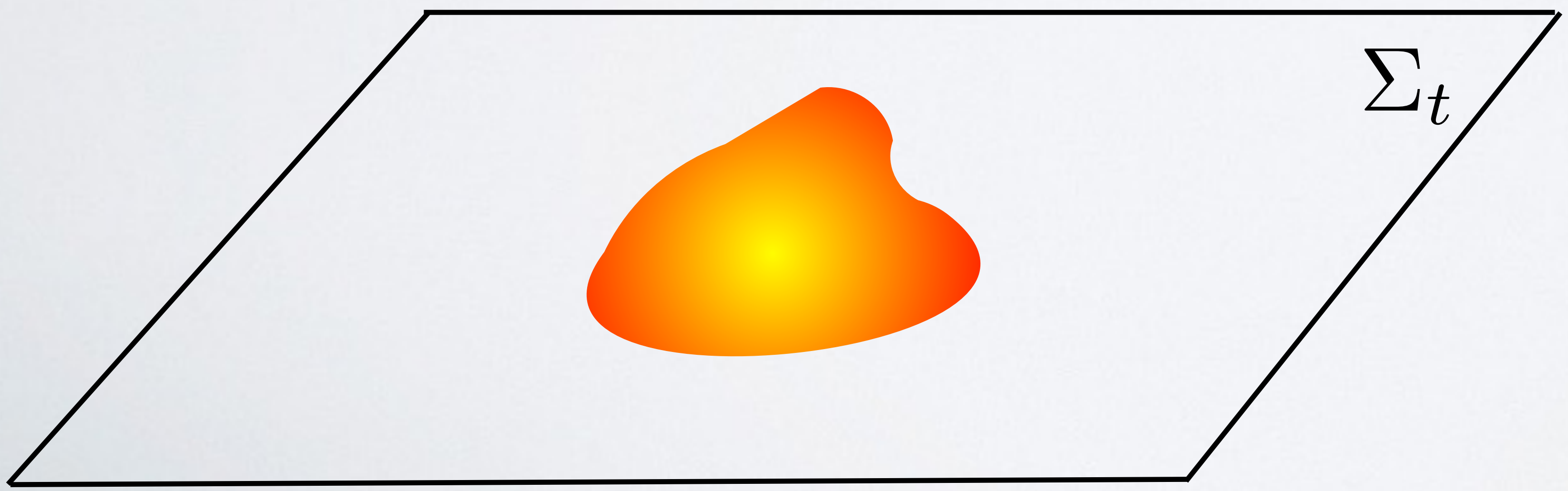
Friedrich-Schiller-Universität Jena

OUTLINE

1. INITIAL DATA
2. (ON HYPERBOLOIDAL SLICES)
3. PERTURBED BLACK-HOLE
4. PHYSICAL PROPERTIES OF

INITIAL-DATA PROBLEM

3+1 Decomposition : $\{x^0; x^i\} \quad i = 1 \ldots 3$



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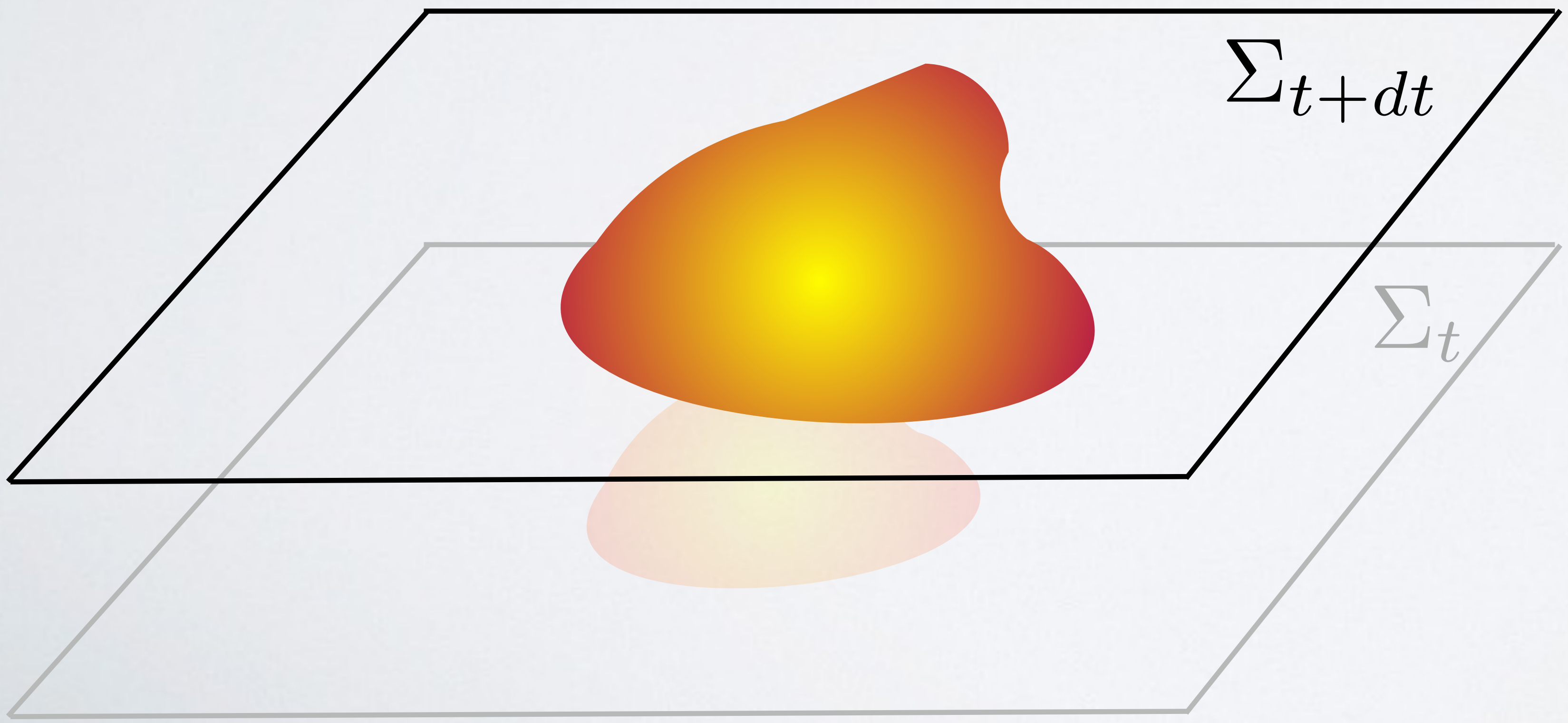

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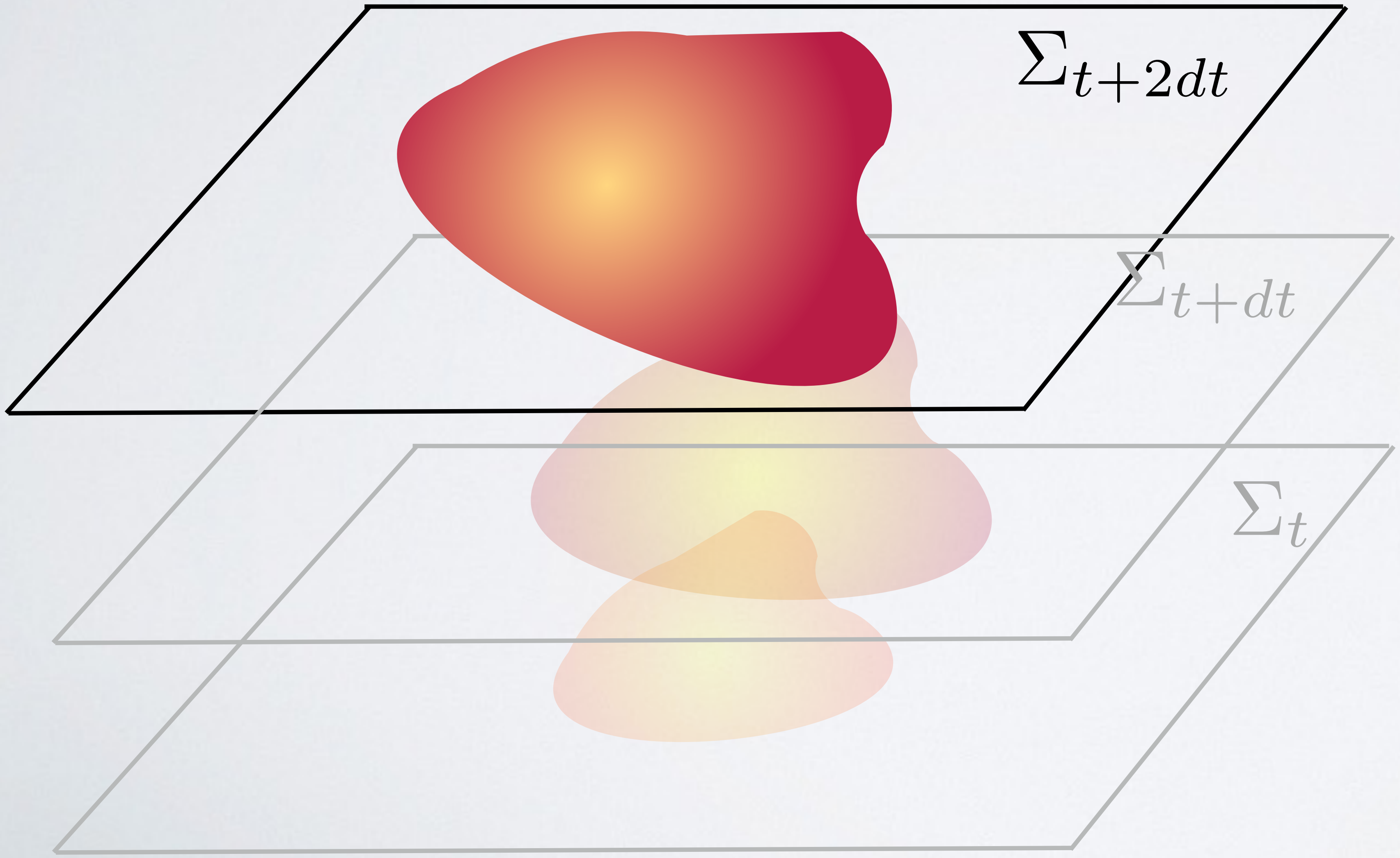
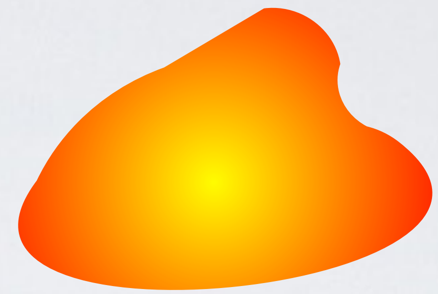
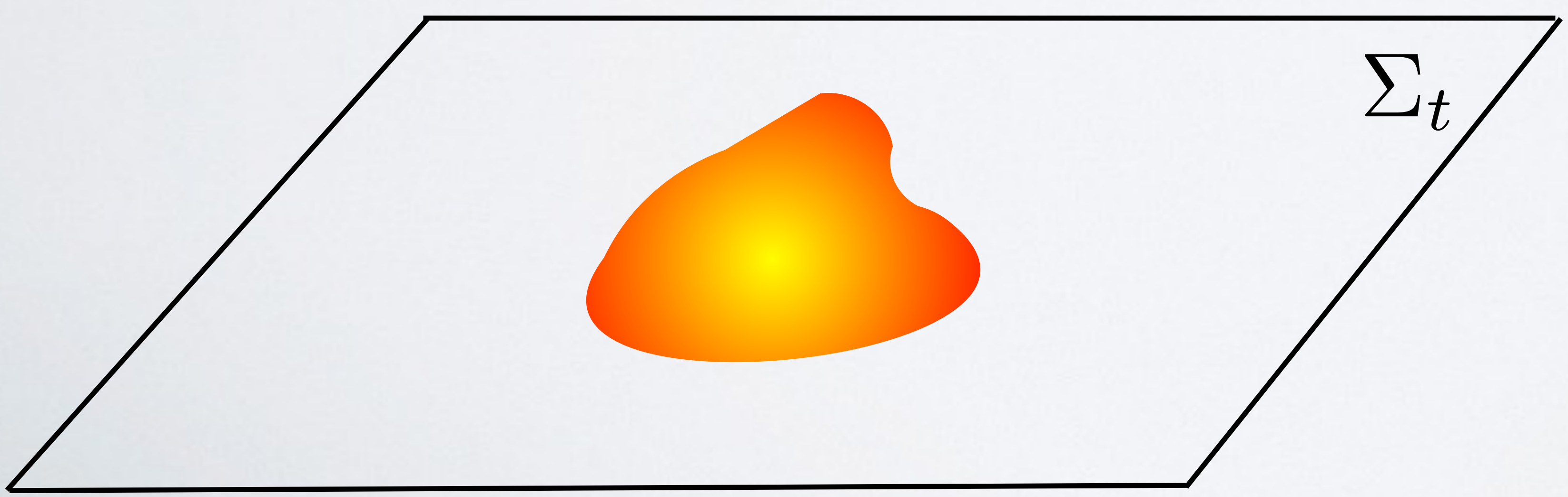


INITIAL-DATA PROBLEM

How do we obtain  ?

What are the geometrical quantities describing it?

How do we give a physical meaning to it?



INITIAL-DATA PROBLEM

geometrical quantities

- Spatial metric at Σ_t : γ_{ij}

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0i}dx^0dx^i + \gamma_{ij}dx^idx^j$$

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- Spatial metric at Σ_t : γ_{ij}

$$ds^2{}_{|\Sigma_t} = \underbrace{g_{00}(dx^0)^2 + 2g_{0i}dx^0 dx^i}_{\text{Gauge Freedom}} + \gamma_{ij}dx^i dx^j$$

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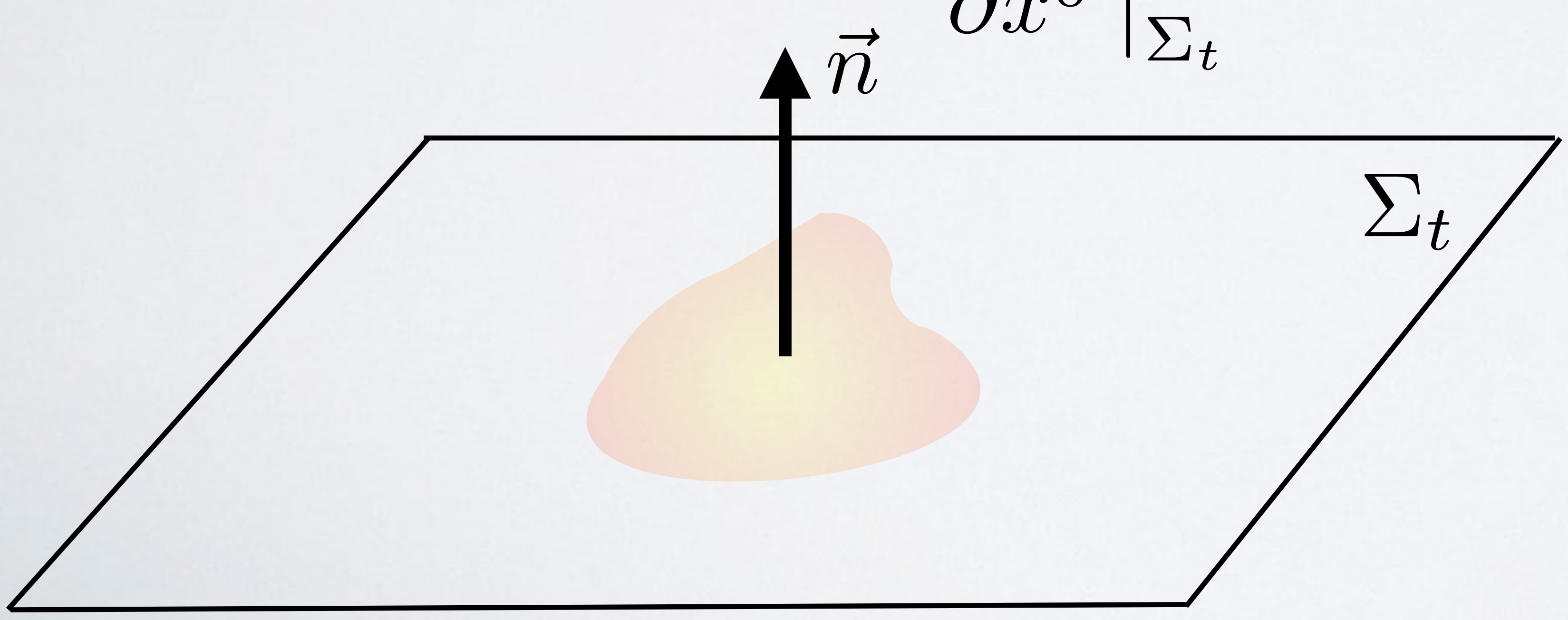
$\Downarrow$

- Extrinsic Curvature : $K_{ij} = \mathcal{L}_{\vec{n}} \gamma_{ij}$

$$K_{ij} \propto \left.\frac{\partial \gamma_{ij}}{\partial x^0}\right|_{\Sigma_t} + (\text{gauge freedom})_{ij}$$



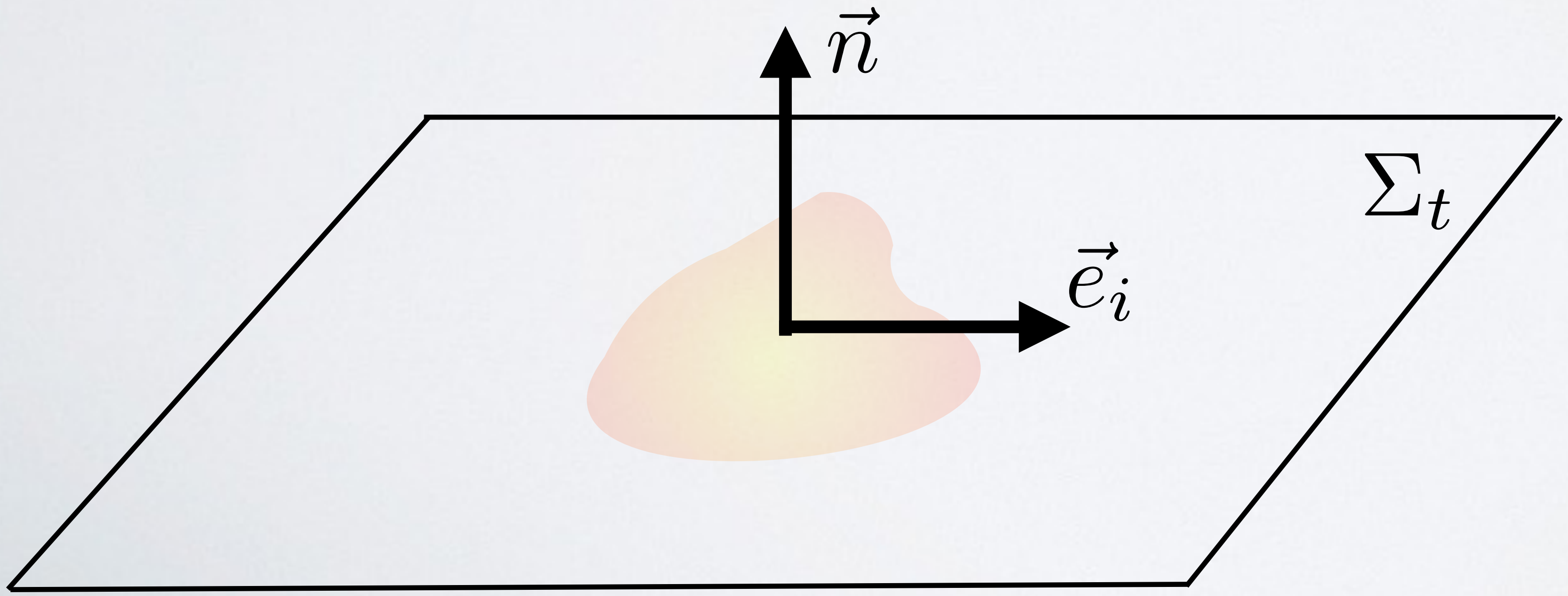
INITIAL-DATA PROBLEM

Physical meaning

- Variables: $\gamma_{ij},\ K_{ij}$
- (Vacuum) Einstein's equations : Constraint Equations

$$\mathbb{H} := n^a n^b G_{ab} = {}^{(3)}R + K^2 - K_{ij}K^{ij} = 0$$
(Hamiltonian Constraint)

$$\mathbb{P}_i := n^a e^b_i G_{ab} = {}^{(3)}\nabla^j \left(K_{ij} - \gamma_{ij} K\right) = 0$$
(Momentum Constraint)



INITIAL-DATA PROBLEM

(Vacuum) Maxwell Equations:

Dynamical Equations:

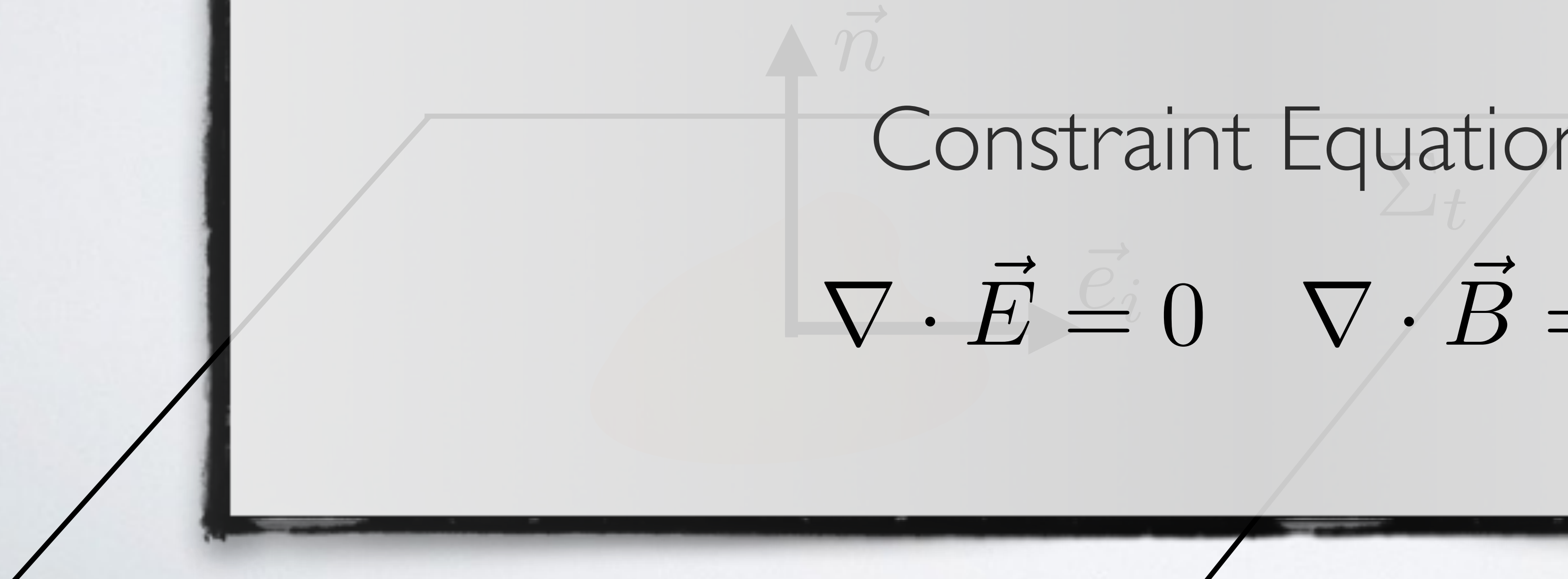
$$\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \quad \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{B} = 0$$

Initial Data:

$$\vec{E}(0,\vec{x}), \quad \vec{B}(0,\vec{x})$$

Constraint Equations:

$$\nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0$$



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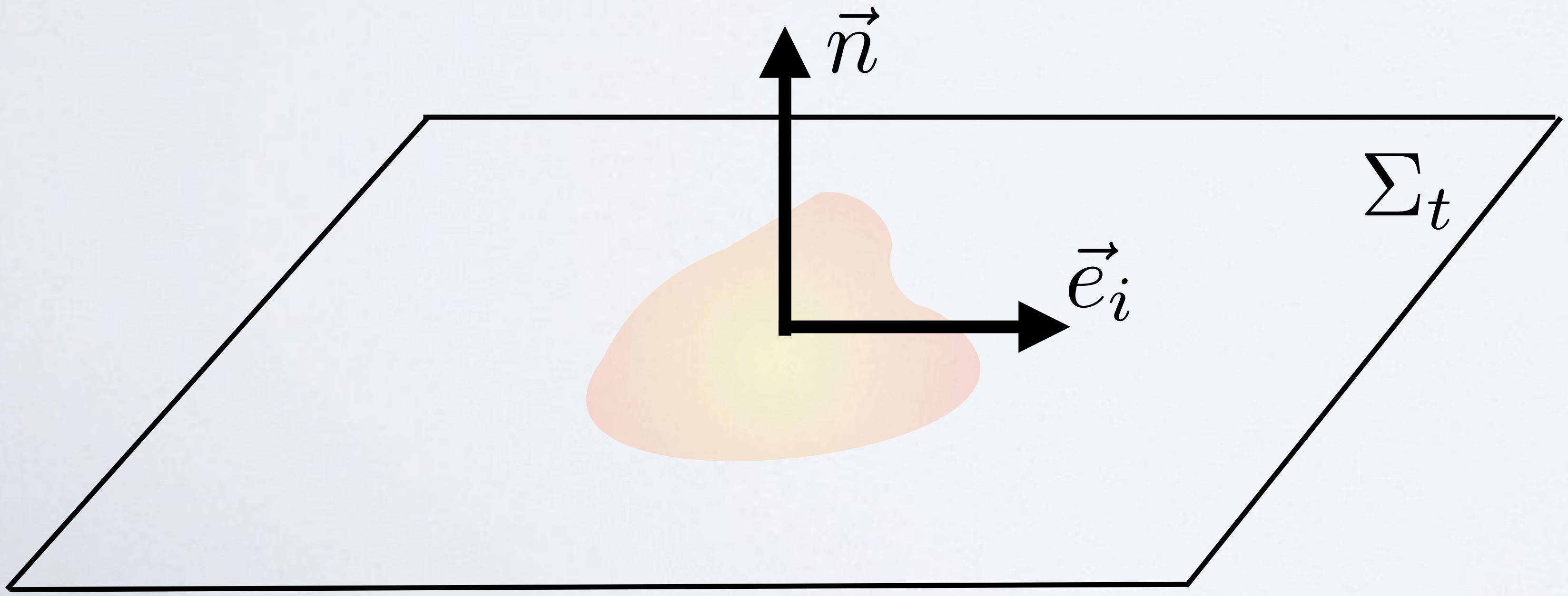

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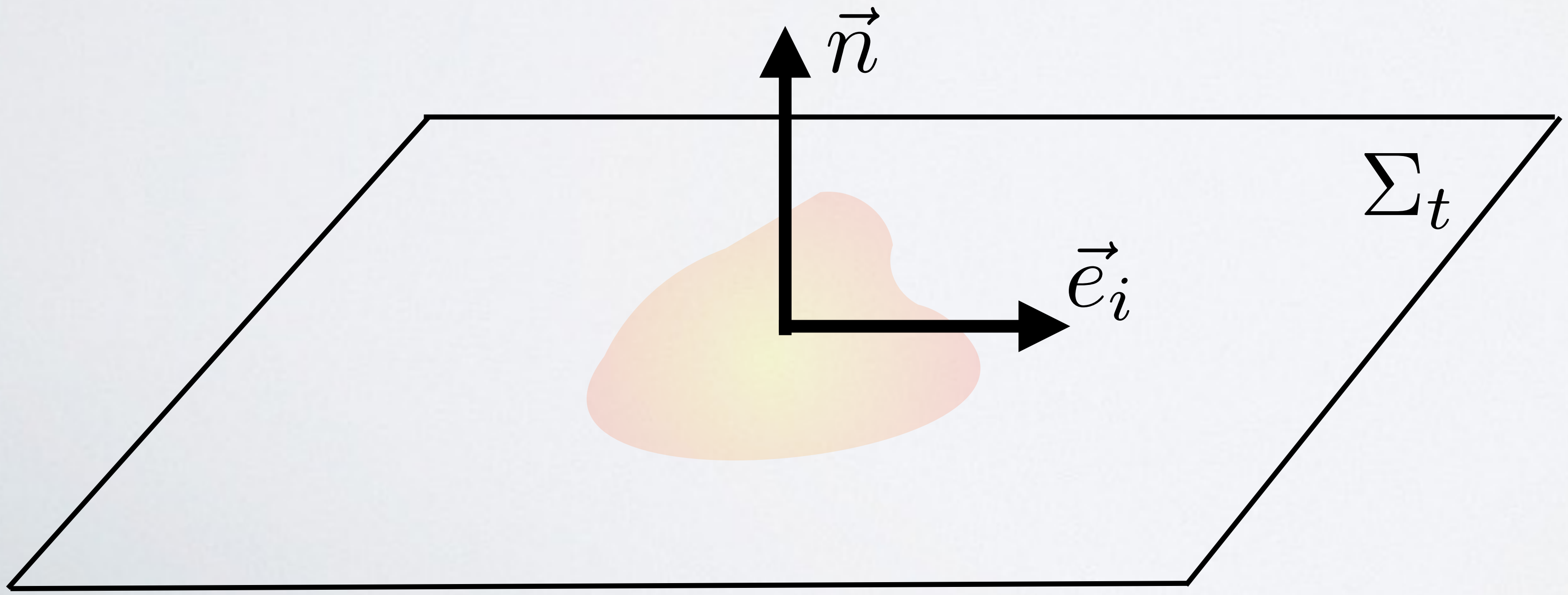
- **Elliptic Equations : Boundary value problem**



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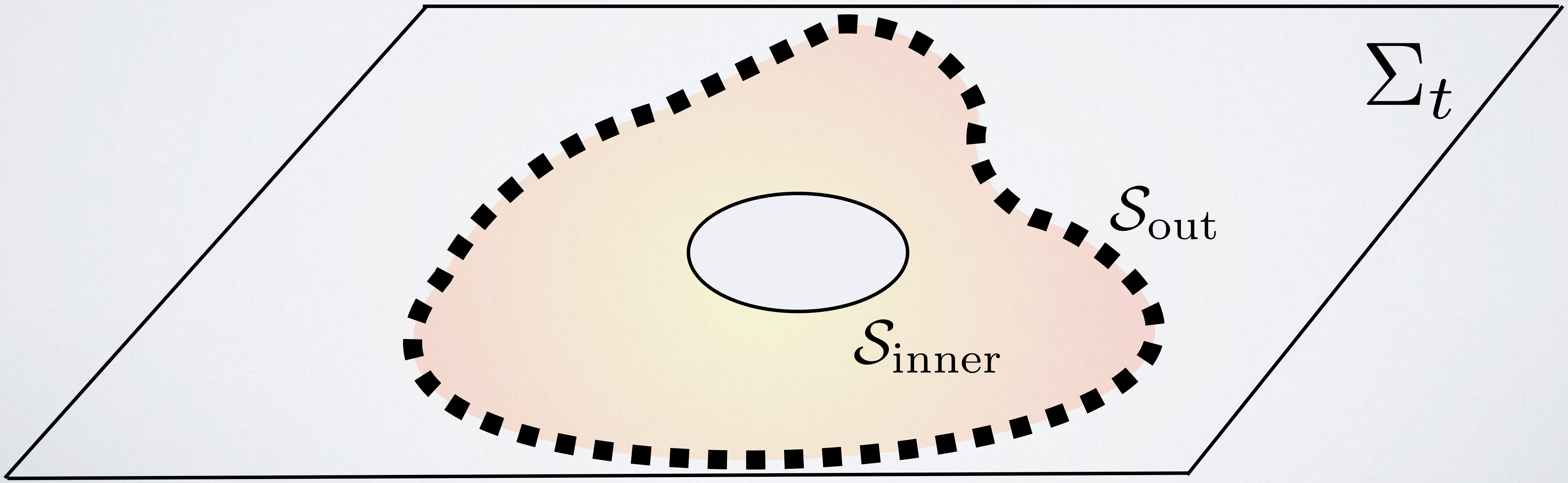
$\mathcal{S}_{\mathrm{inner}}$: Black-hole horizon $\mathcal{S}_{\mathrm{out}}$: infinity ("$r \to \infty$")



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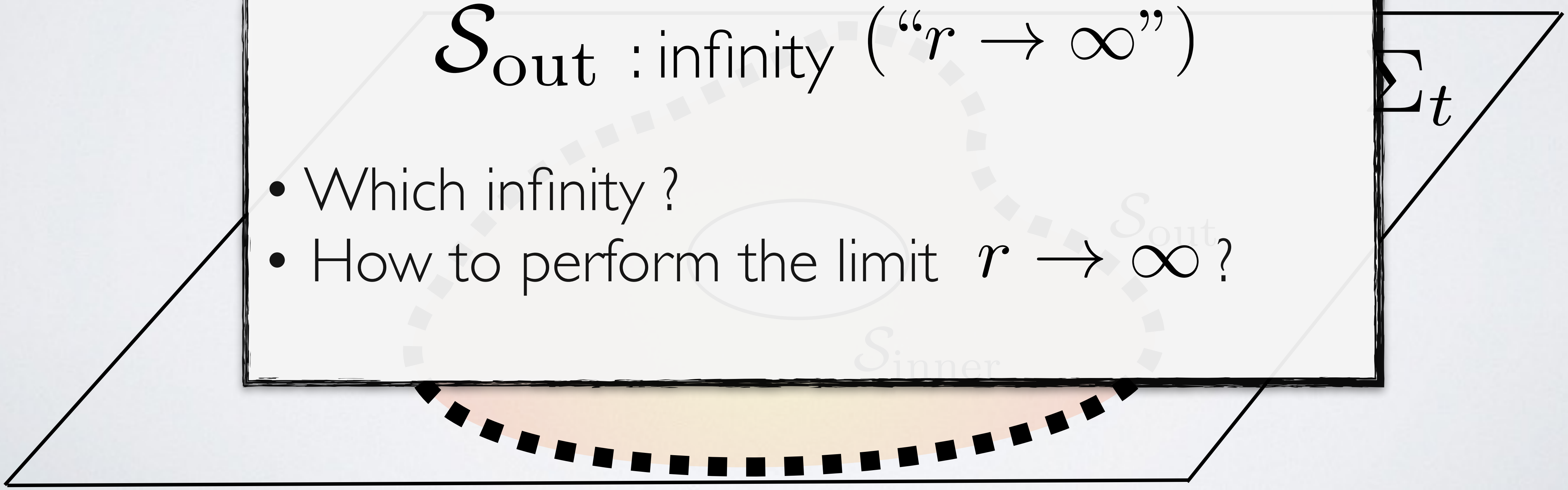


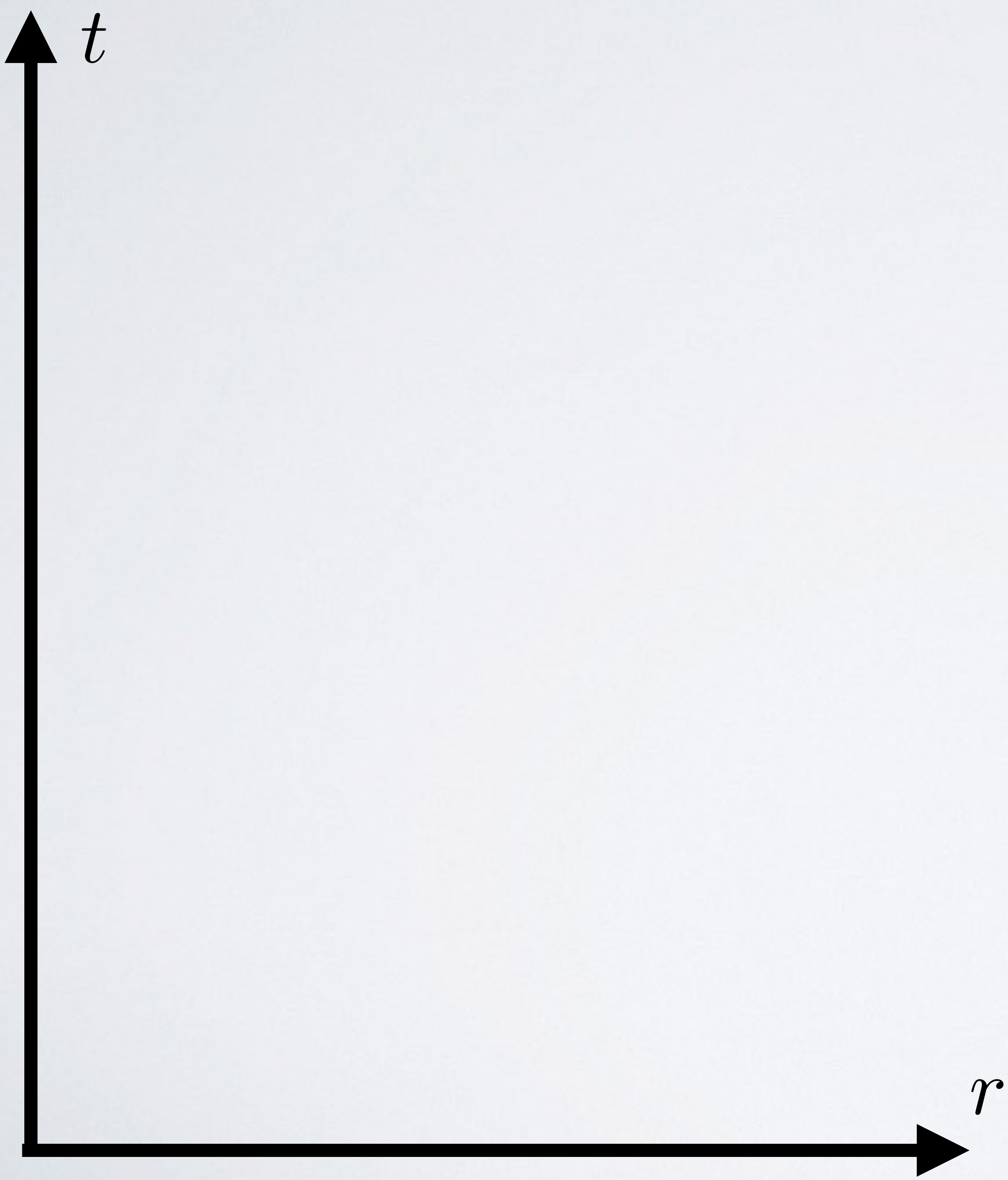
Outer Boundary

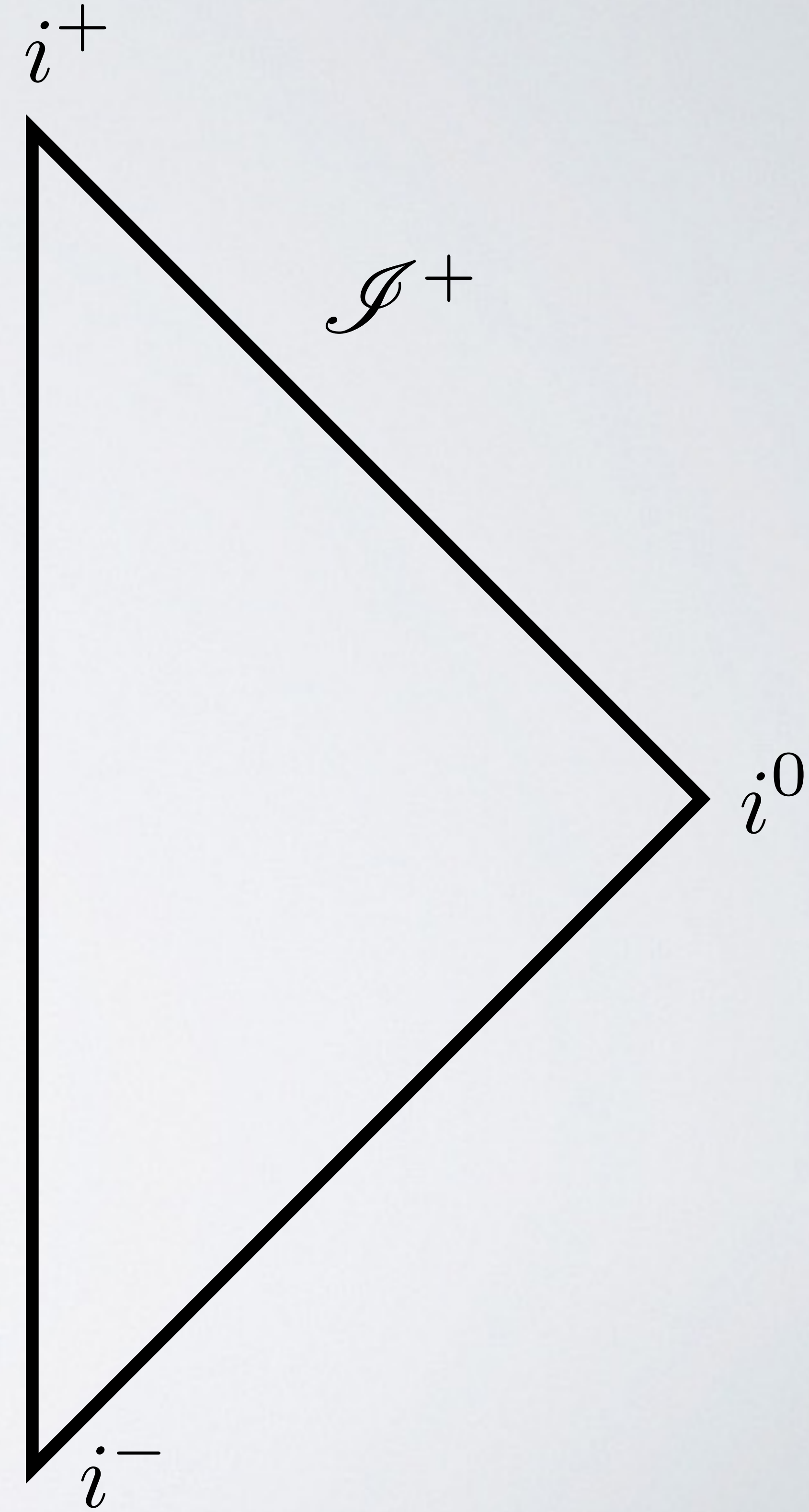
$\mathcal{S}_{\rm out}$: infinity (“$r \to \infty$”)

- Which infinity ?
- How to perform the limit $r \to \infty$?

HYPERBOLOIDAL SLICES

- Minkowski Spacetime:





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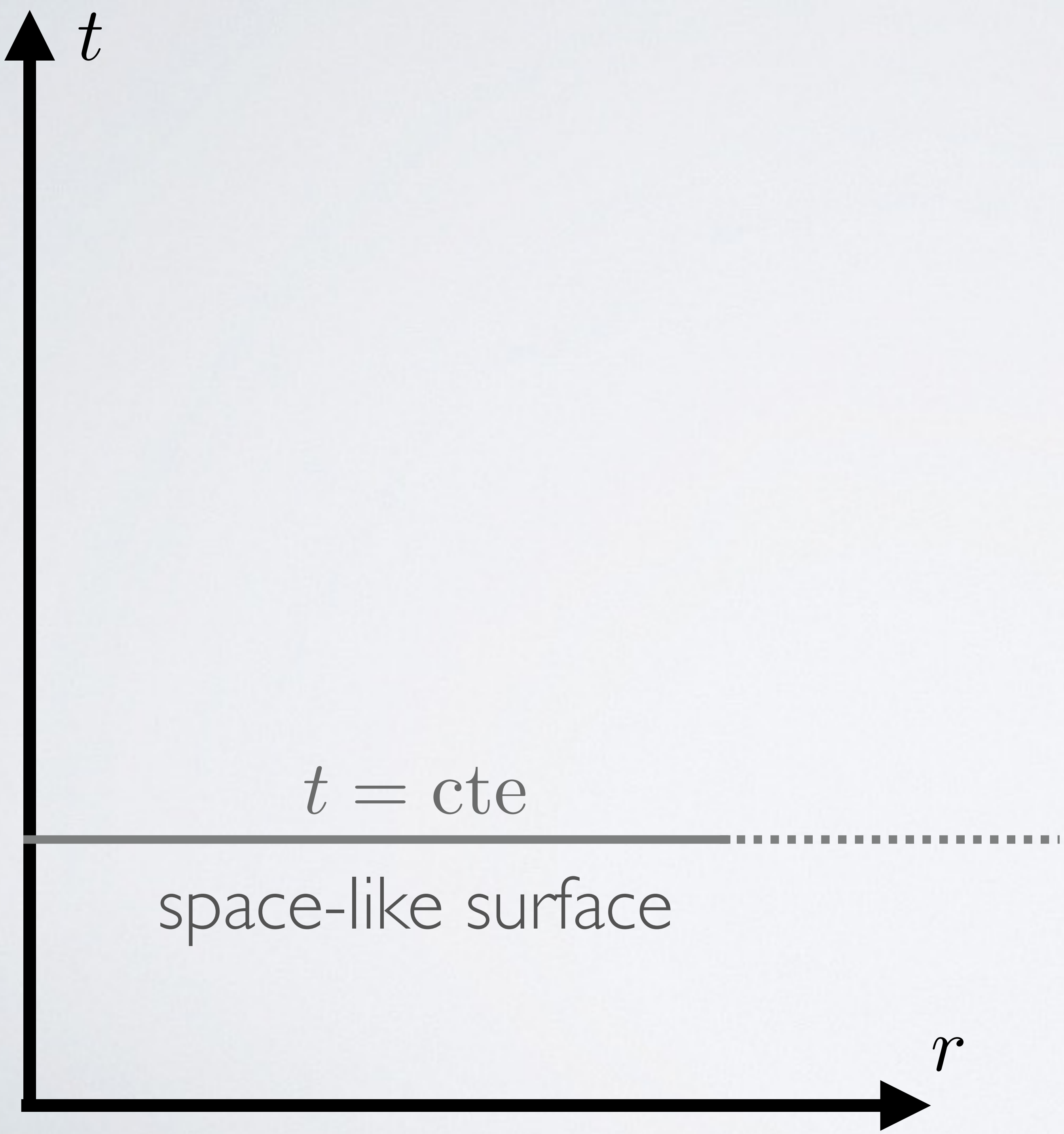

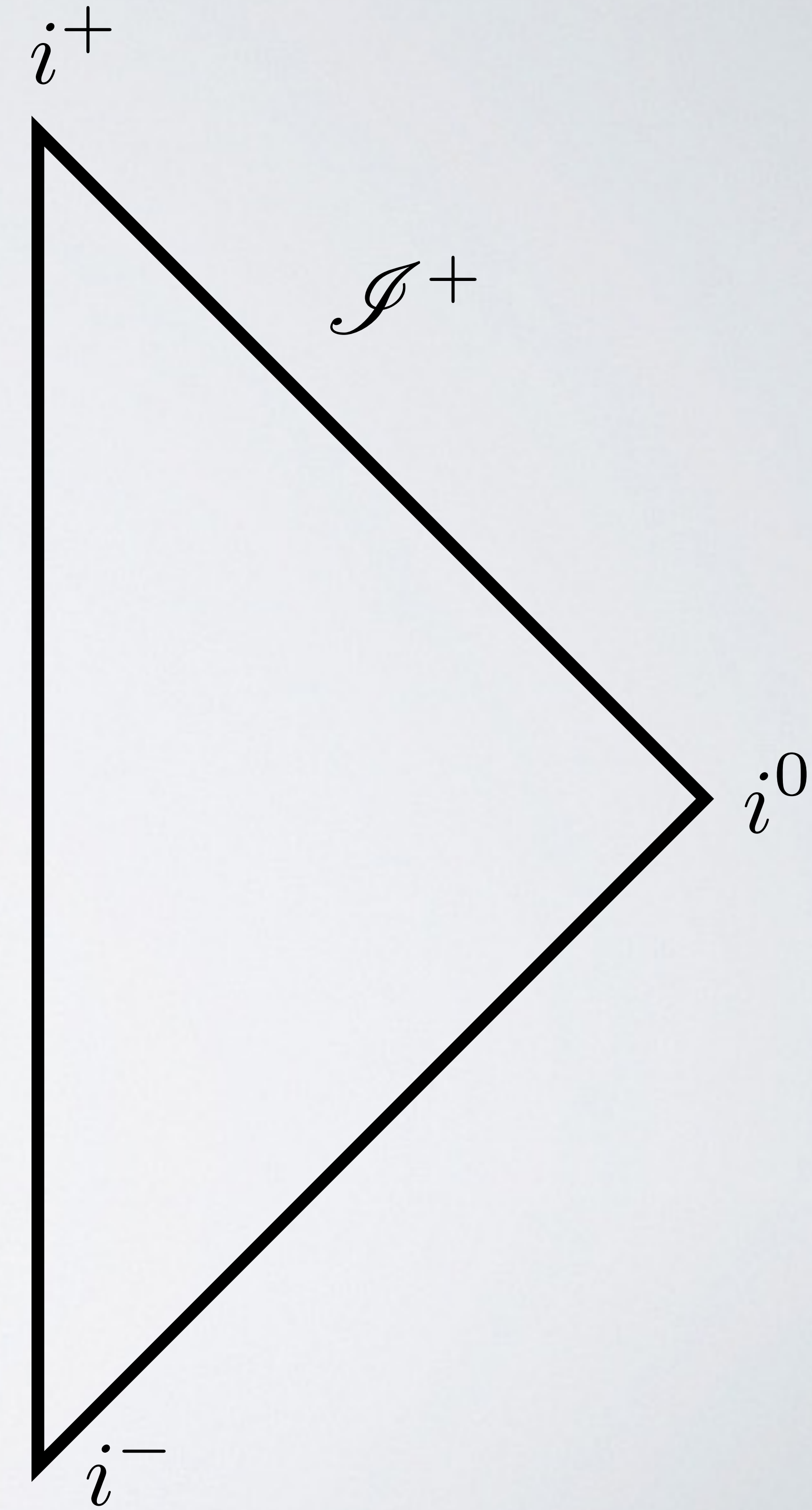

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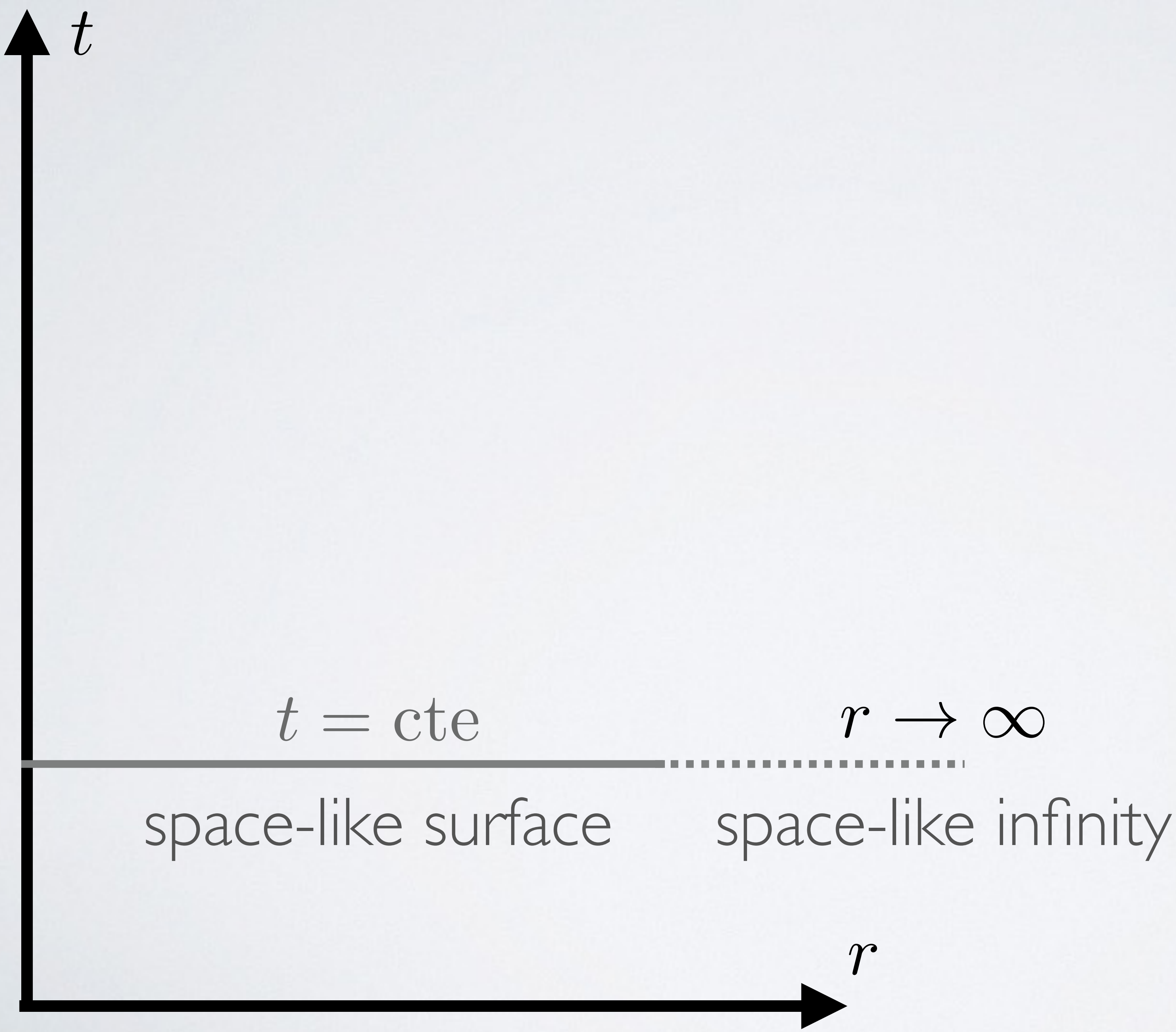

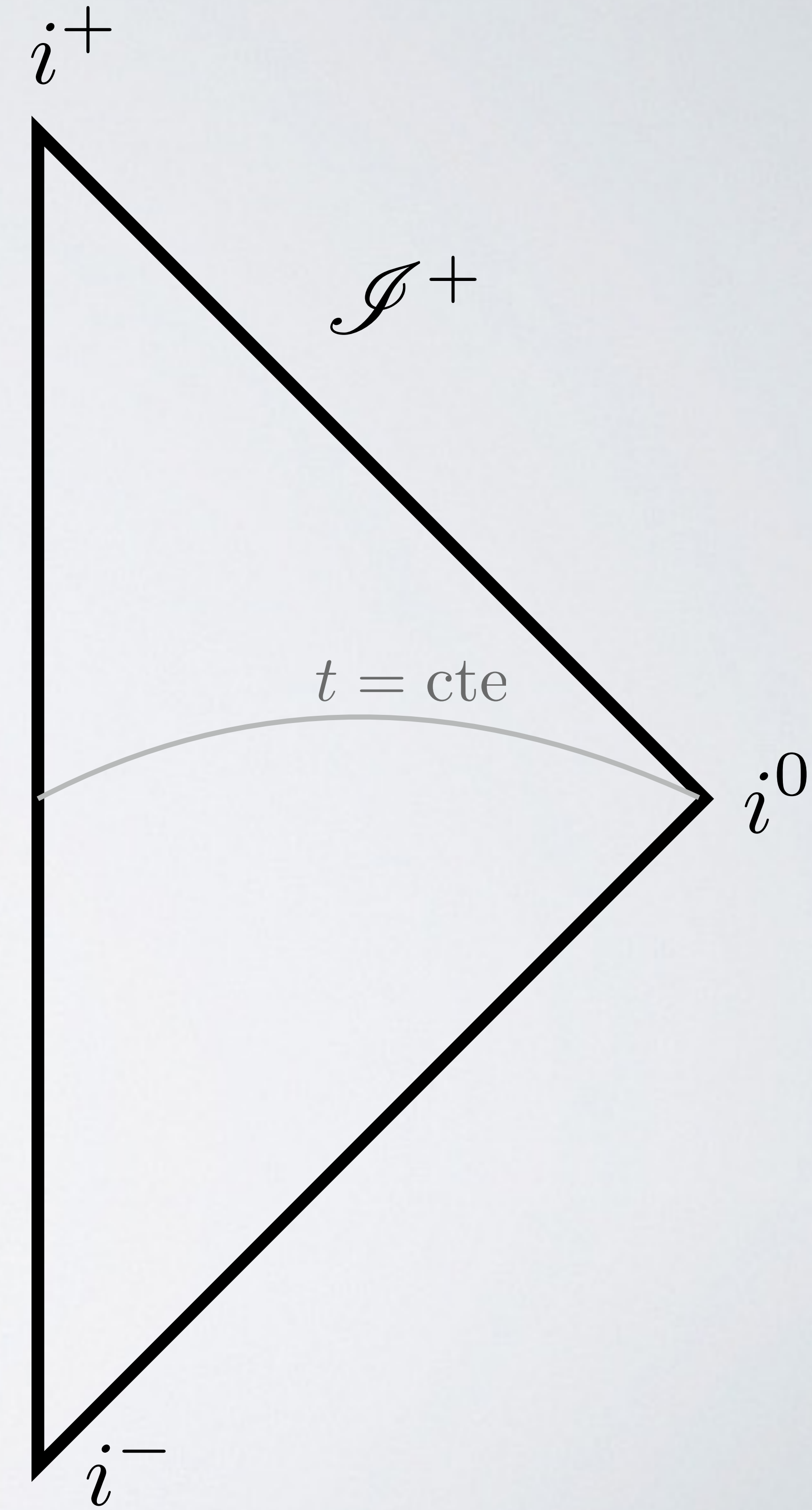

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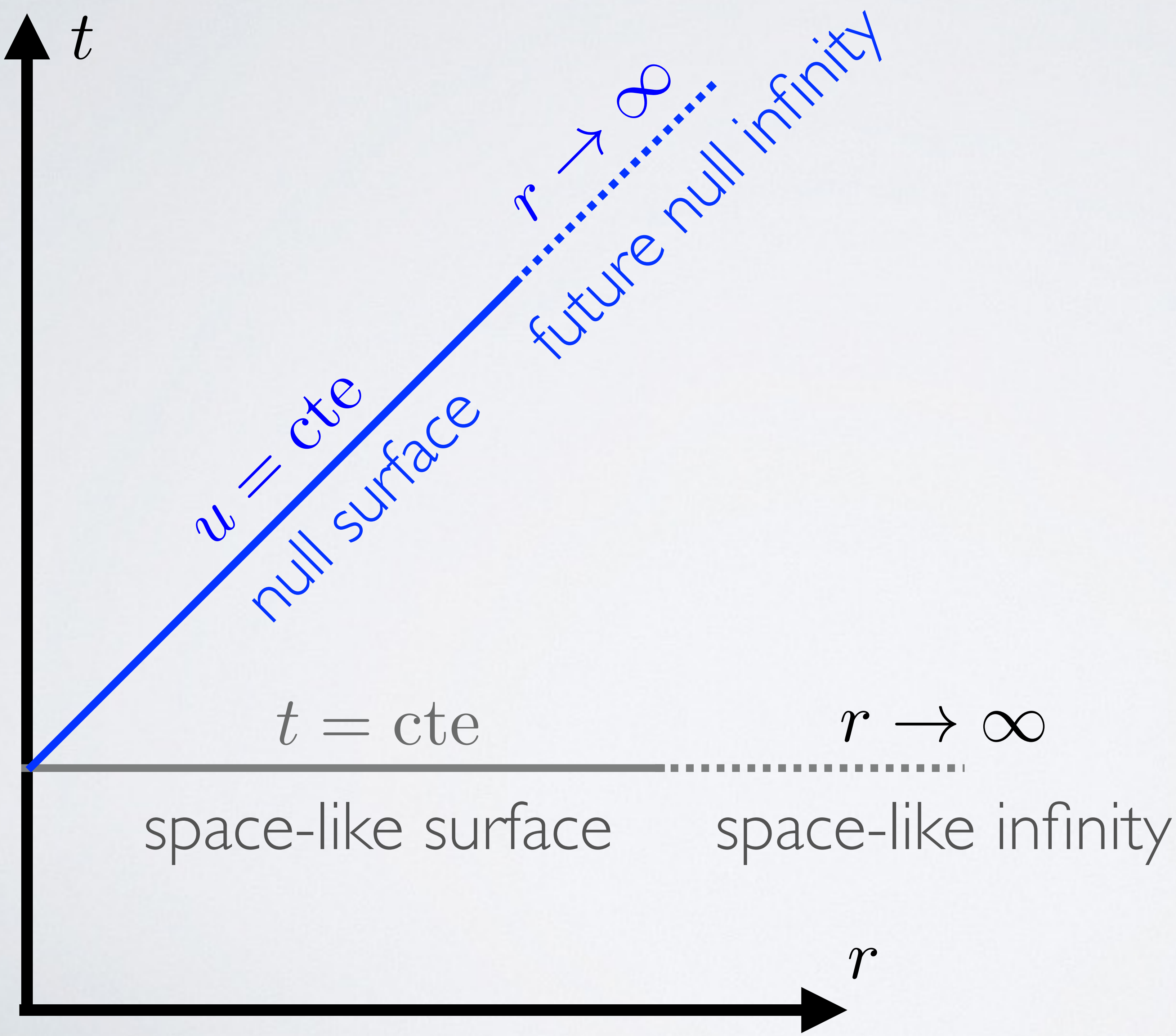

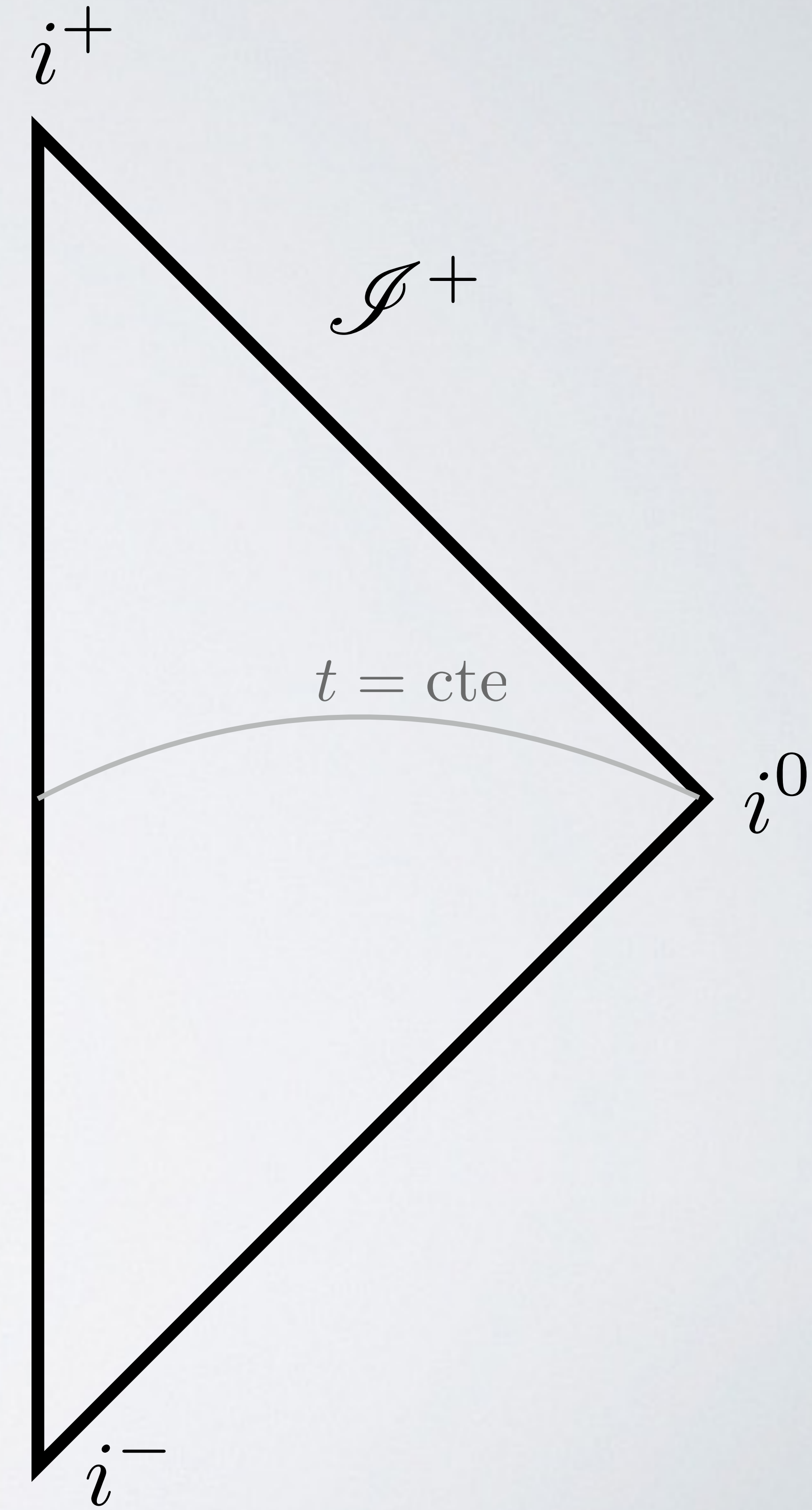

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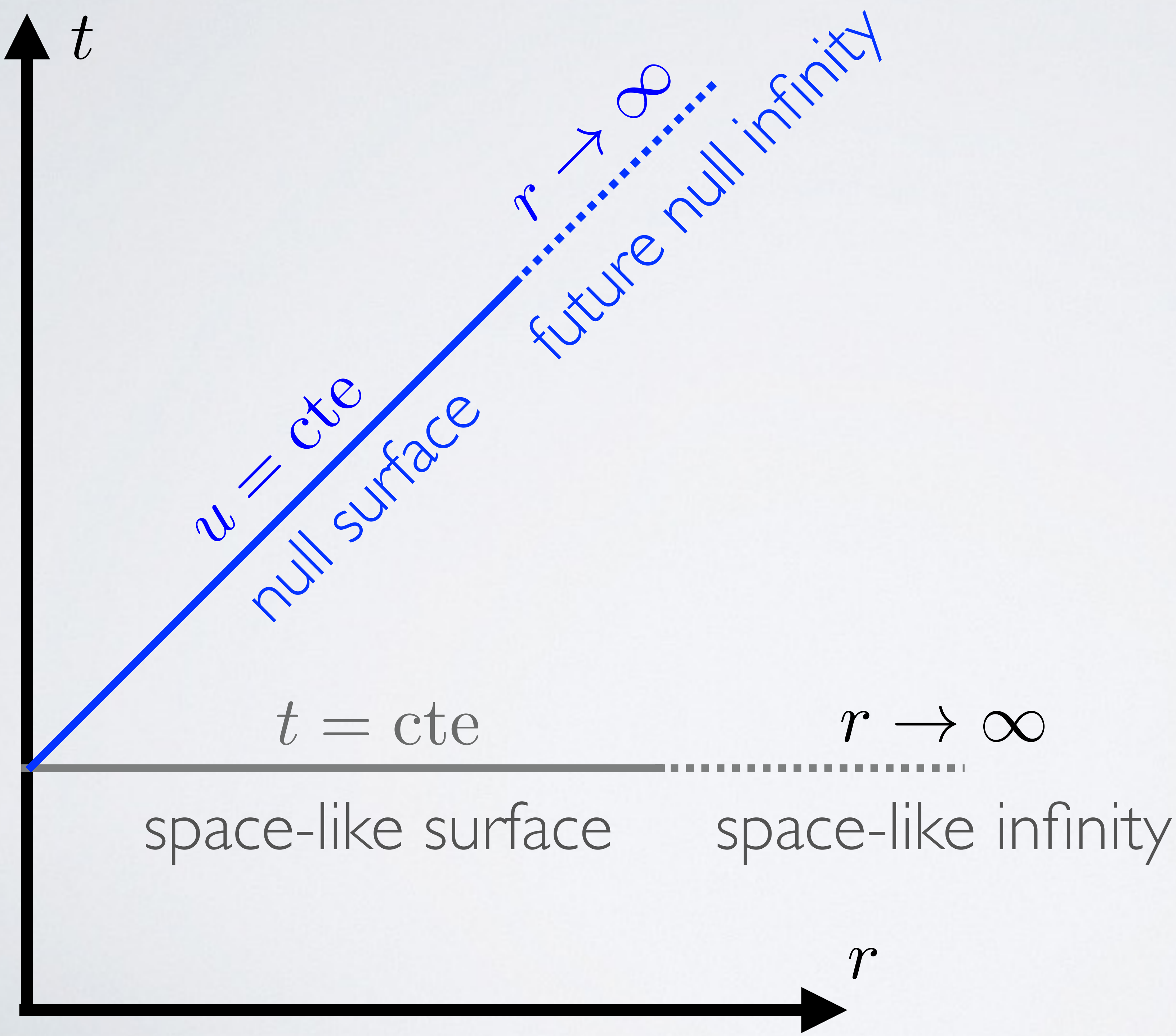

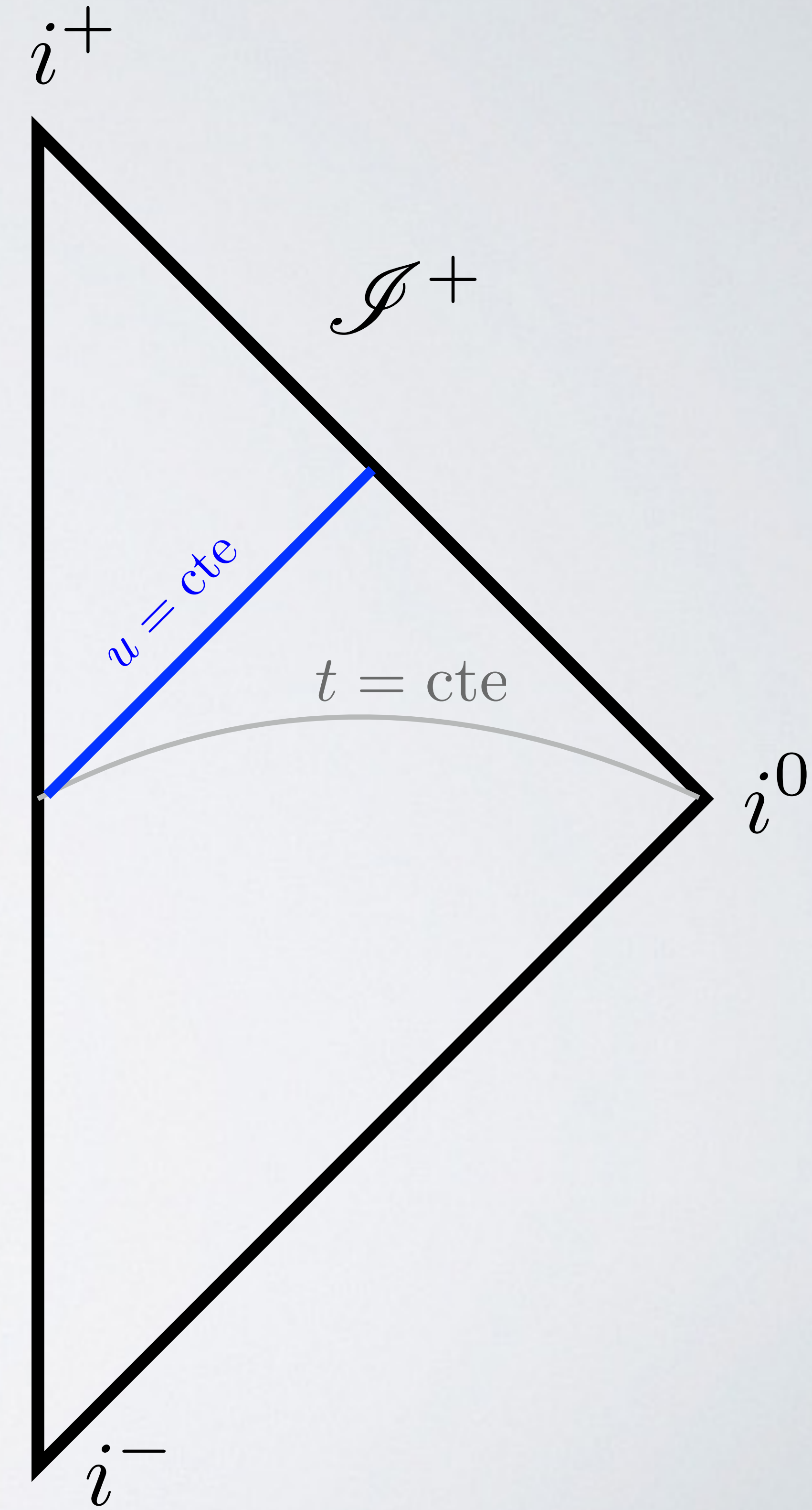

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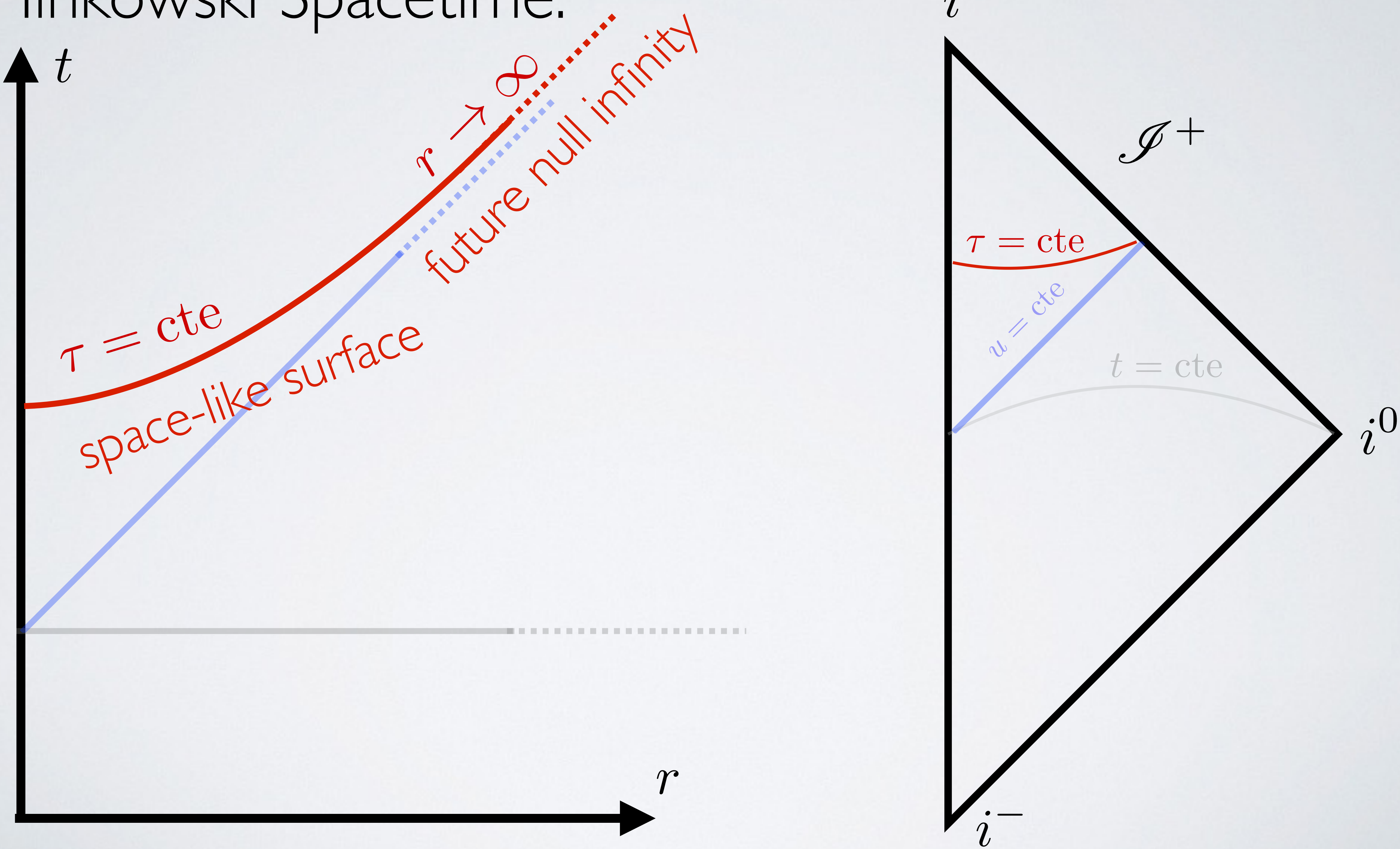

HYPERBOLOIDAL SLICES

- Schwarzschild Spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\omega^2$$

$$\{\tau, \sigma\}$$

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$$ds^2 = \Omega^{-2}\left[-\sigma^2(1-\sigma)d\tau^2 + (1+\sigma)d\sigma^2 + (1-2\sigma)d\tau d\sigma + \frac{1}{4}d\omega^2\right]$$

- Conformal factor: $\Omega = \dfrac{\sigma}{4M}$

HYPERBOLOIDAL SLICES

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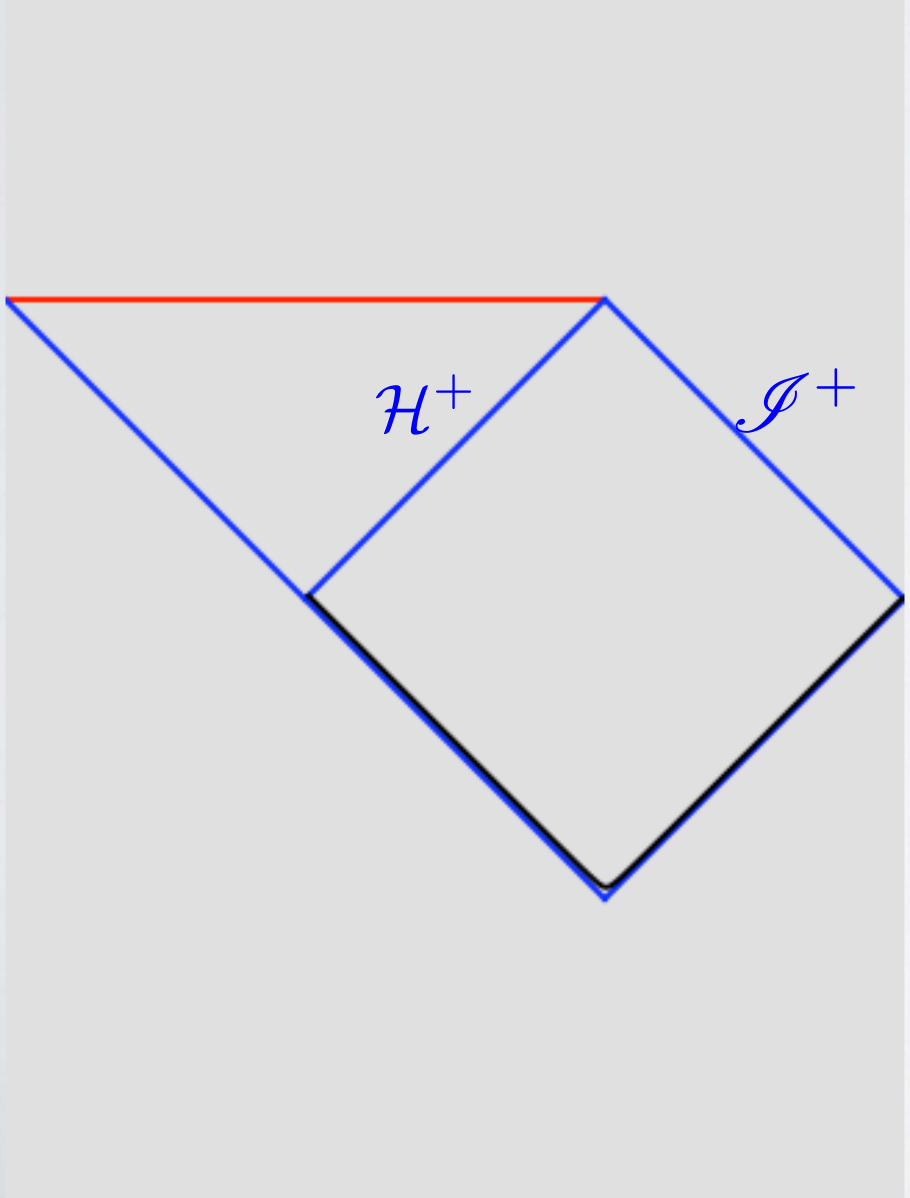
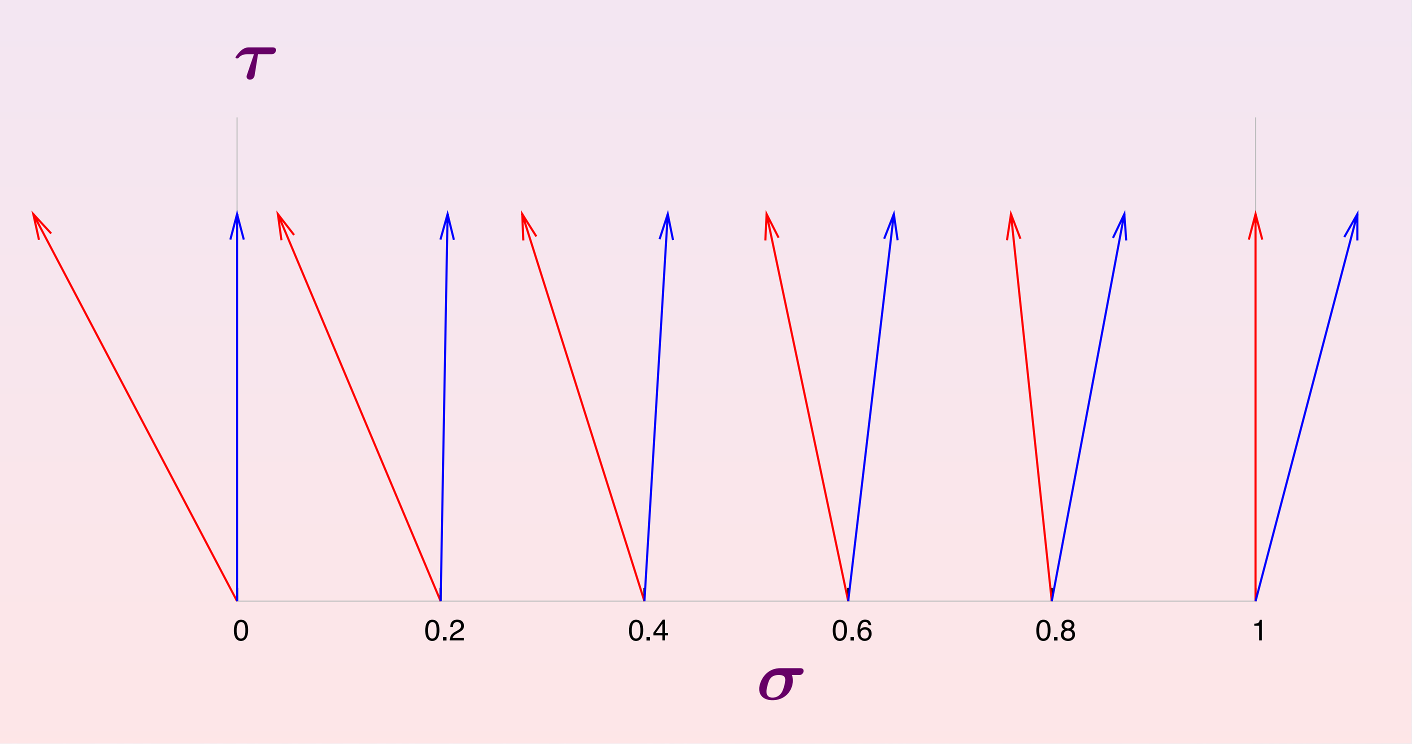
$$\sigma = \frac{2M}{r}$$

- Location of $\mathcal{H}^+$: $\sigma = 1$
- Location of $\mathscr{I}^+$: $\sigma = 0$





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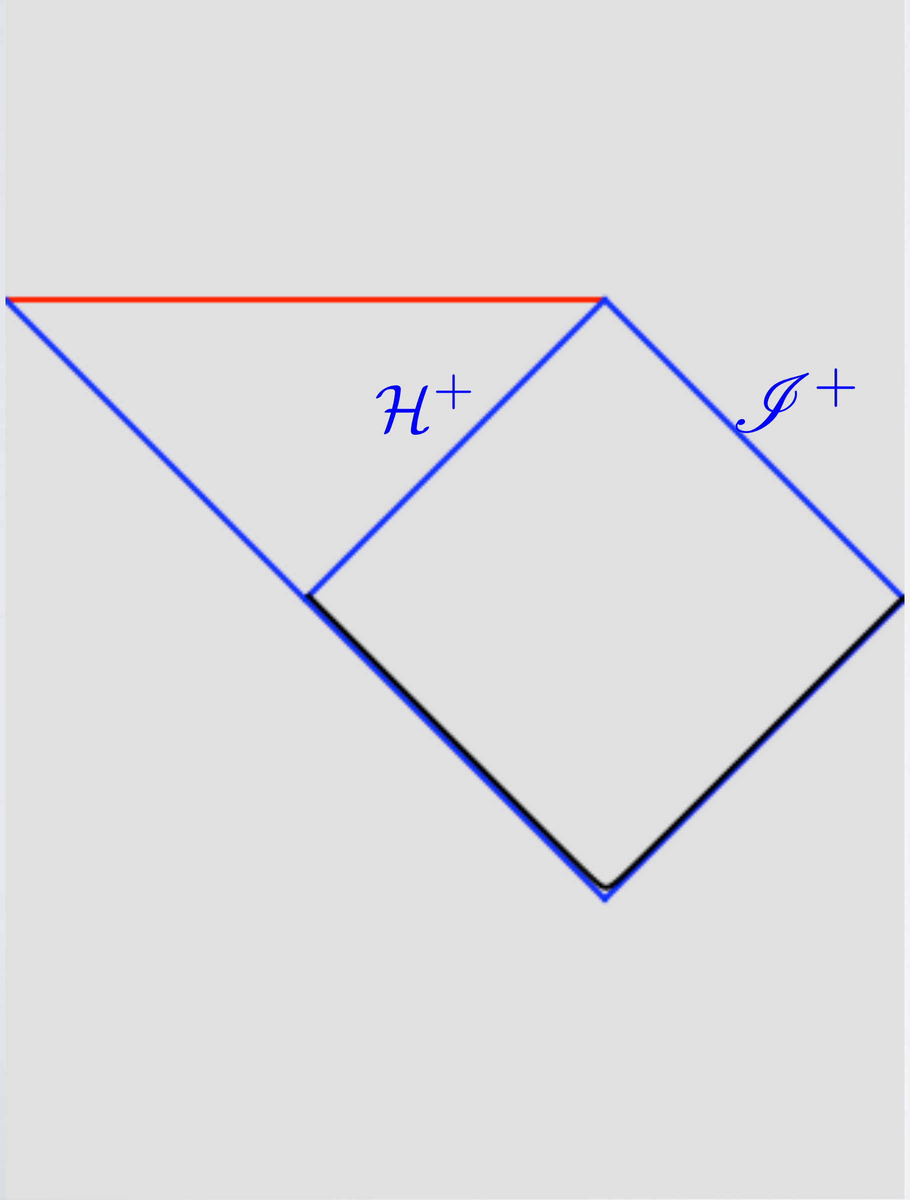

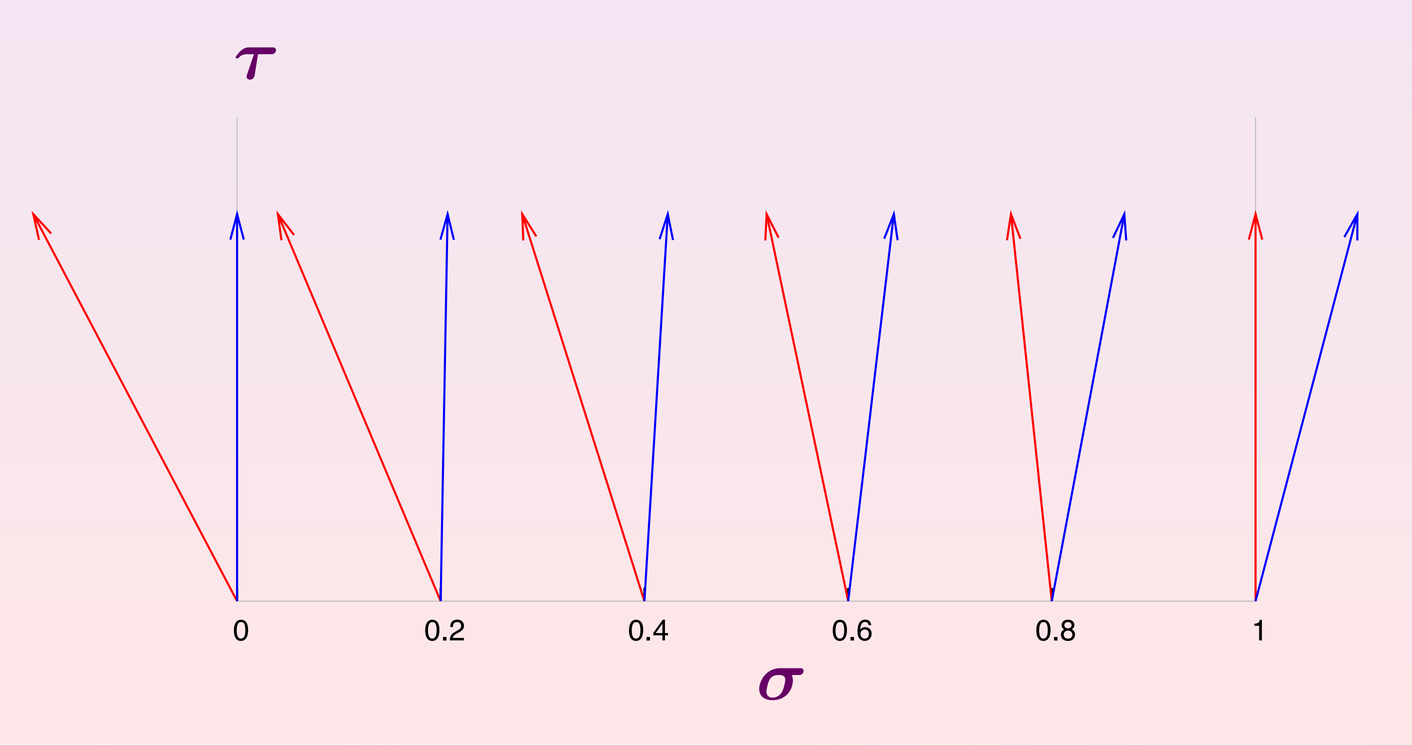

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- Generic Properties:

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✓ $\mathscr{I}^+$ is casually disconnected from numerical domain: no need of artificial boundary condition for time evolution

✓ $\mathscr{I}^+$ formal mathematical definition of the radiation zone: clean, well-defined notion of gravitational waves

HYPERBOLOIDAL SLICES

- Generic Properties:

$$g_{ab} = \Omega^{-2} \tilde{g}_{ab}$$

Physical metric
Irregular at $\mathscr{I}^+$

“non-physical” metric
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$\mathscr{I}^+$ located at $\Omega = 0$

✳ Einstein’s equation pick up Ω^{-2} terms, thus irregular at $\mathscr{I}^+$

✳ One needs robust numerical methods to deal with them

INITIAL-DATA PROBLEM

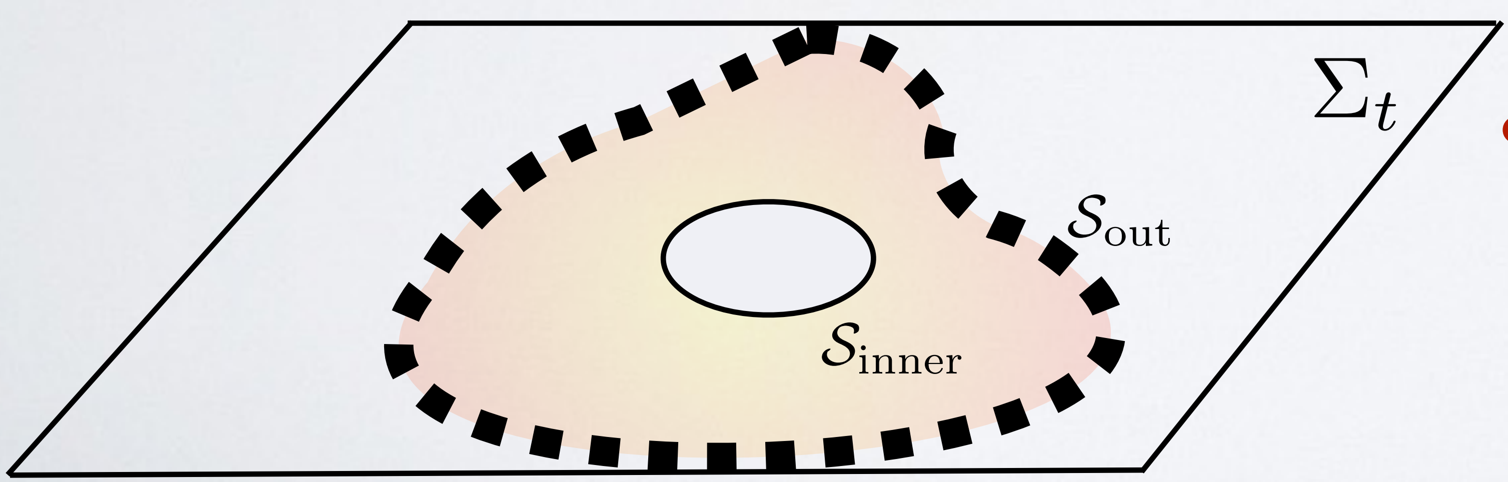
Physical meaning

- Constraint Equations : $\mathbb{H} = 0$ (Hamiltonian Constraint)
  $\mathbb{P}_i = 0$ (Momentum Constraint)
- **Elliptic Equations : Boundary value problem**

$\mathcal{S}_{\rm out}$: infinity $(\text{“}r \to \infty\text{”})$

$$\gamma_{ij} = \Omega^{-2}\tilde{\gamma}_{ij}$$

$$\mathcal{S}_{\rm out}: \ \Omega = 0$$



- **Boundary Conditions:**

Regularity conditions of
$\mathbb{H} = 0 \quad \mathbb{P}_i = 0$

(L. Andersson, P. Chrusciel and H. Friedrich '92,
L. Andersson and P. Chrusciel '93 '94)

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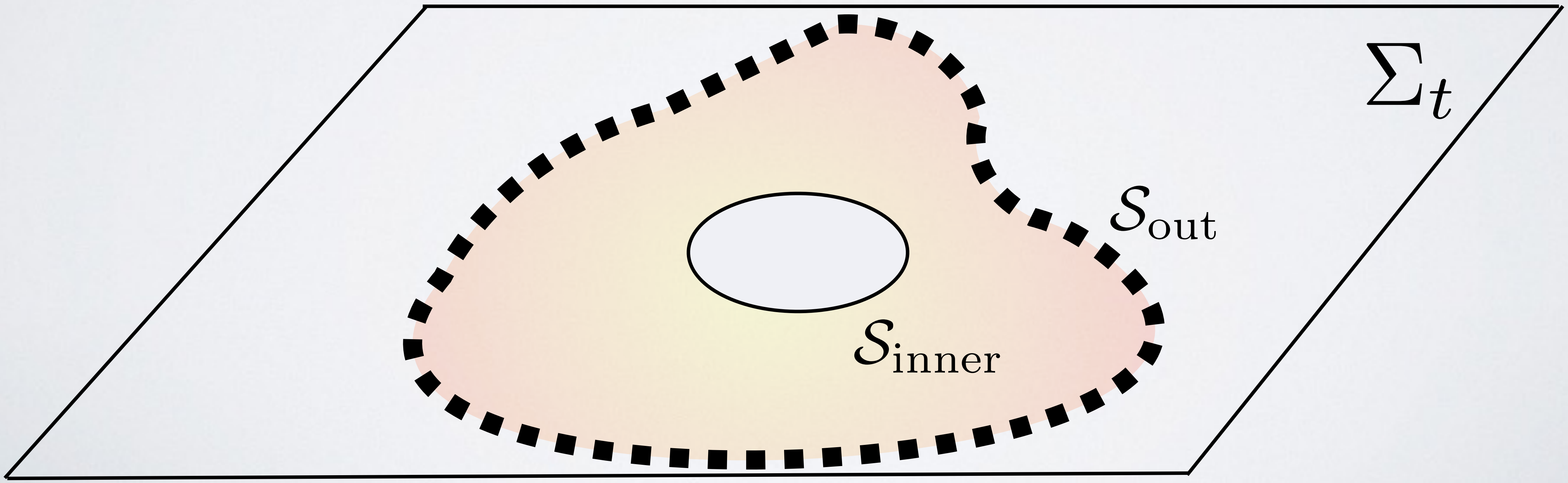

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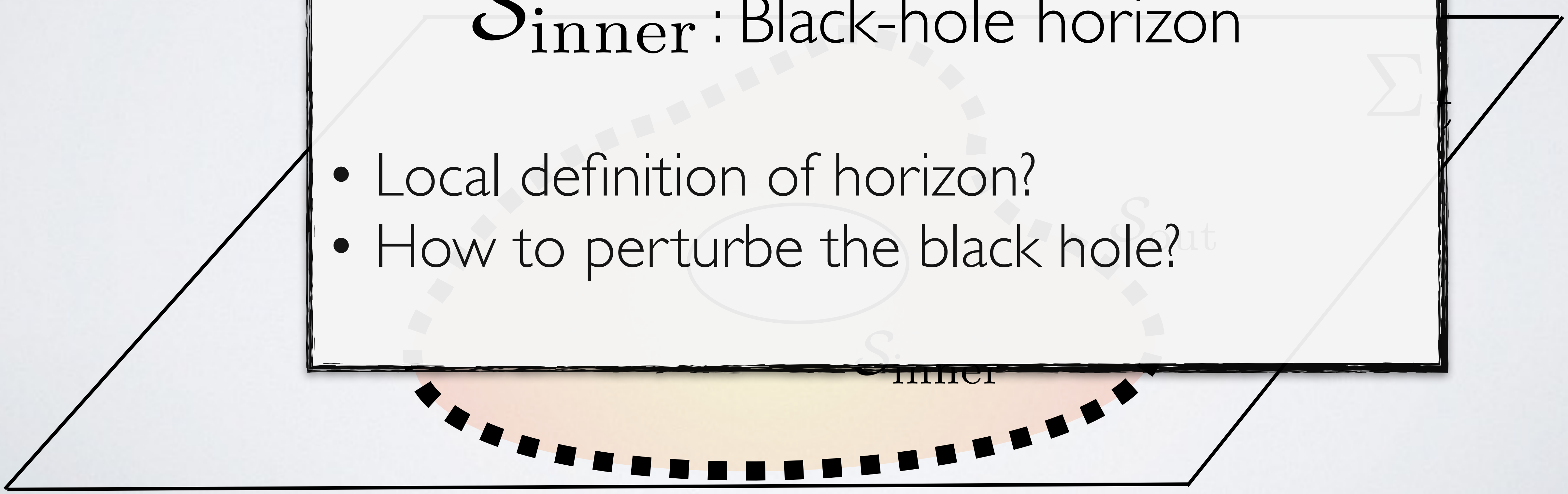
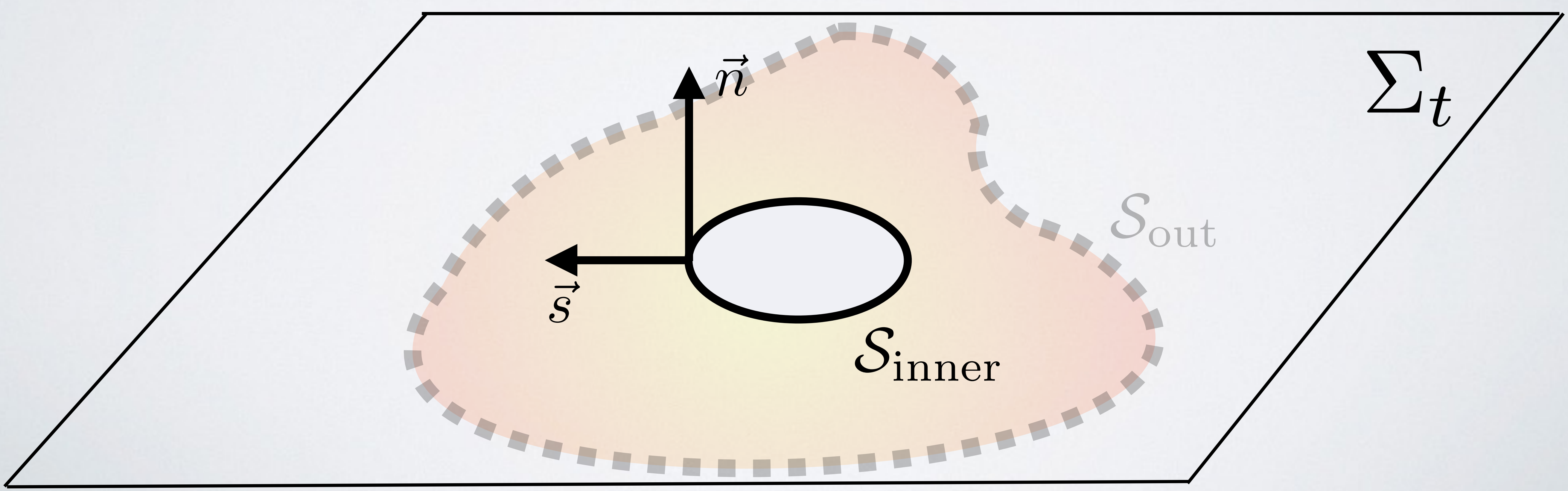


Inner Boundary

$\mathcal{S}_{\rm inner}$: Black-hole horizon

- Local definition of horizon?
- How to perturbe the black hole?

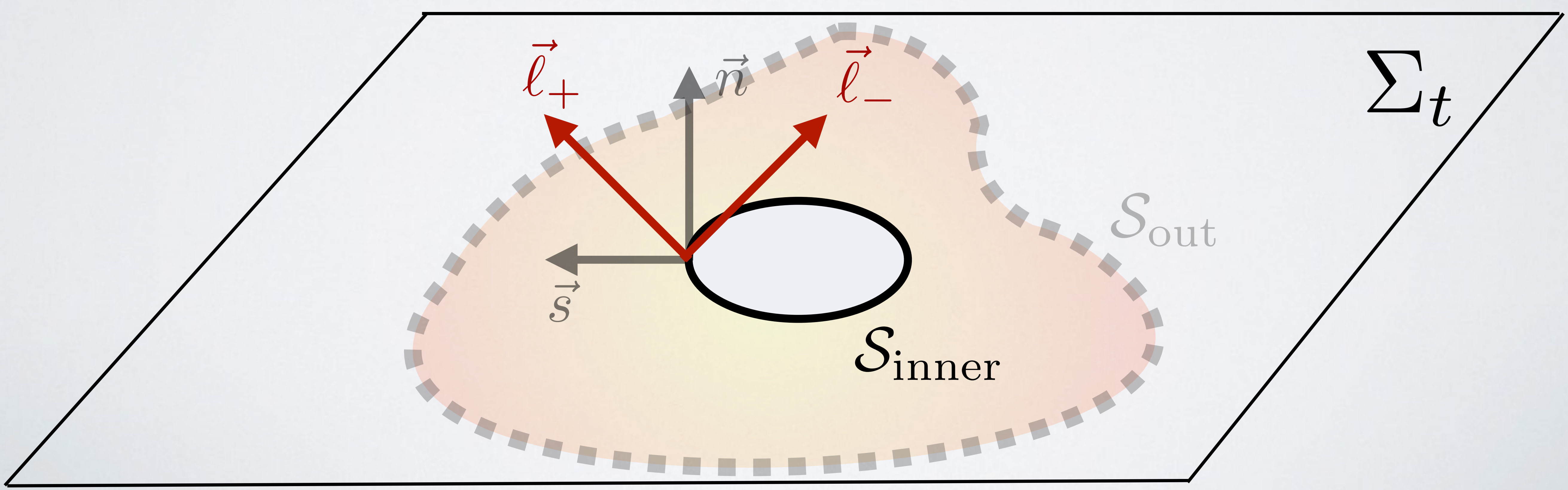
PERTURBED BLACK HOLES



PERTURBED BLACK HOLES

- Outgoing / Ingoing light-rays directions: $\vec{\ell}_+ \; / \; \vec{\ell}_-$
- Expansion of outgoing / ingoing light-rays:

$$\Theta_\pm = q_{ij} \nabla^i \ell^j_\pm \quad \text{with} \quad q_{ij} = \gamma_{ij}|_{\mathcal{S}_{\rm inner}}$$



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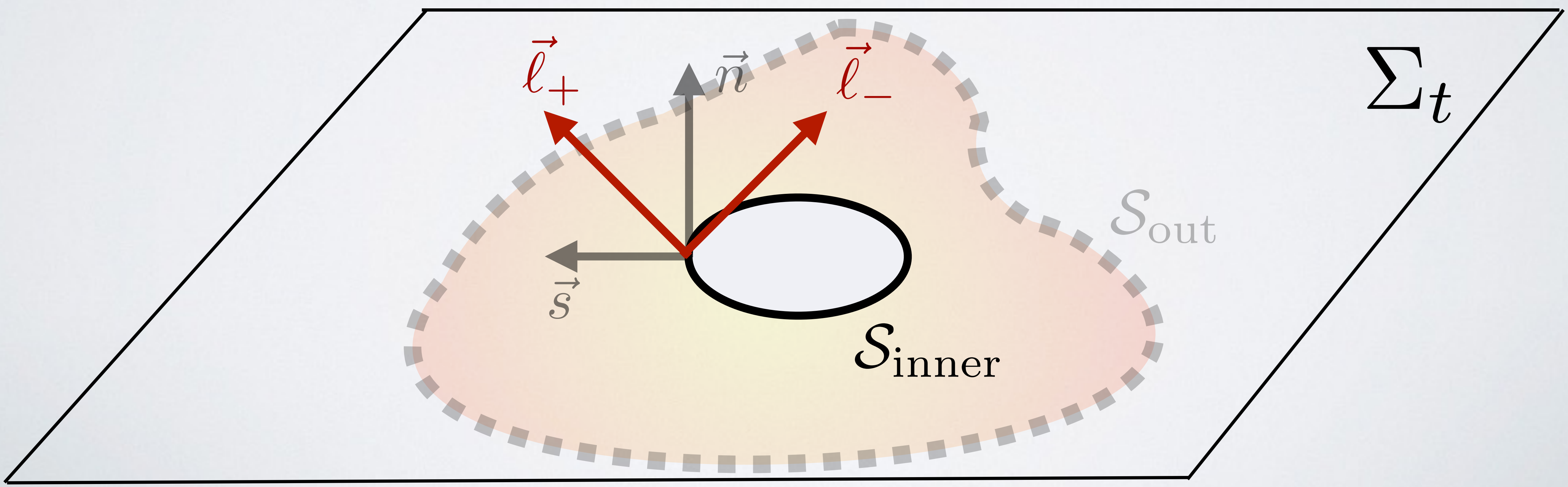
$$\Theta_\pm = K - s^i s^j \pm {}^{(3)}\nabla_i s^i$$

- **Black Hole apparent Horizont:** $\Theta_+ = 0$

  (outgoing light-rays stop expanding outwards)



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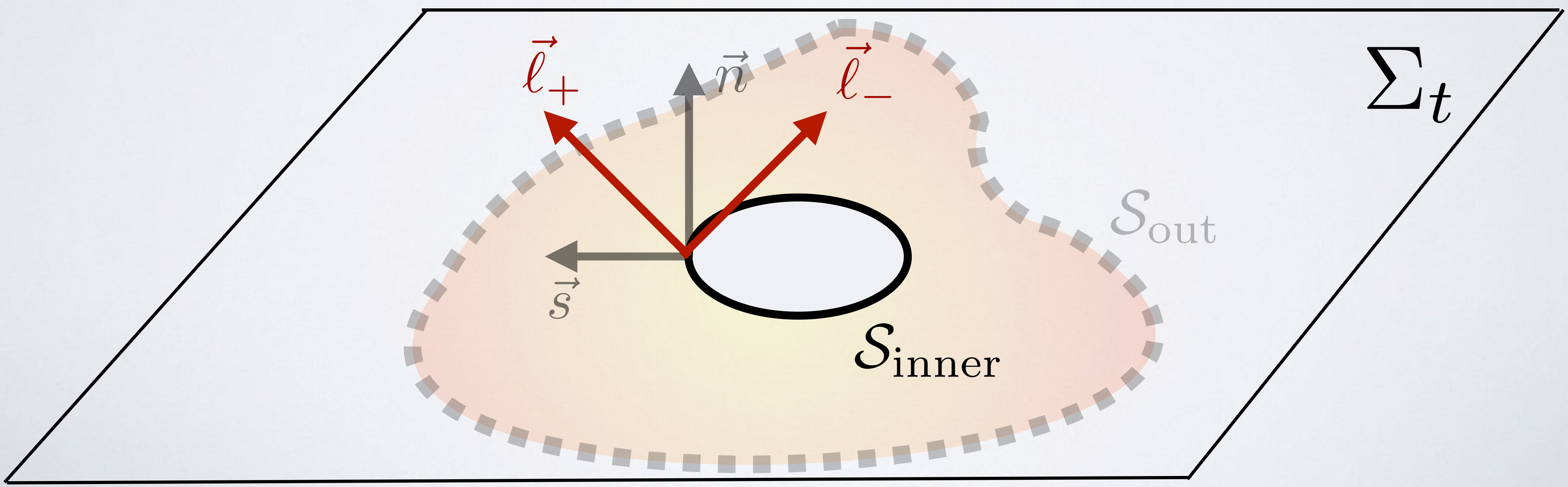
- **White Hole apparent Horizont: $\Theta_- = 0$**
(ingoing light-rays stop "expanding inwards" - contracting)



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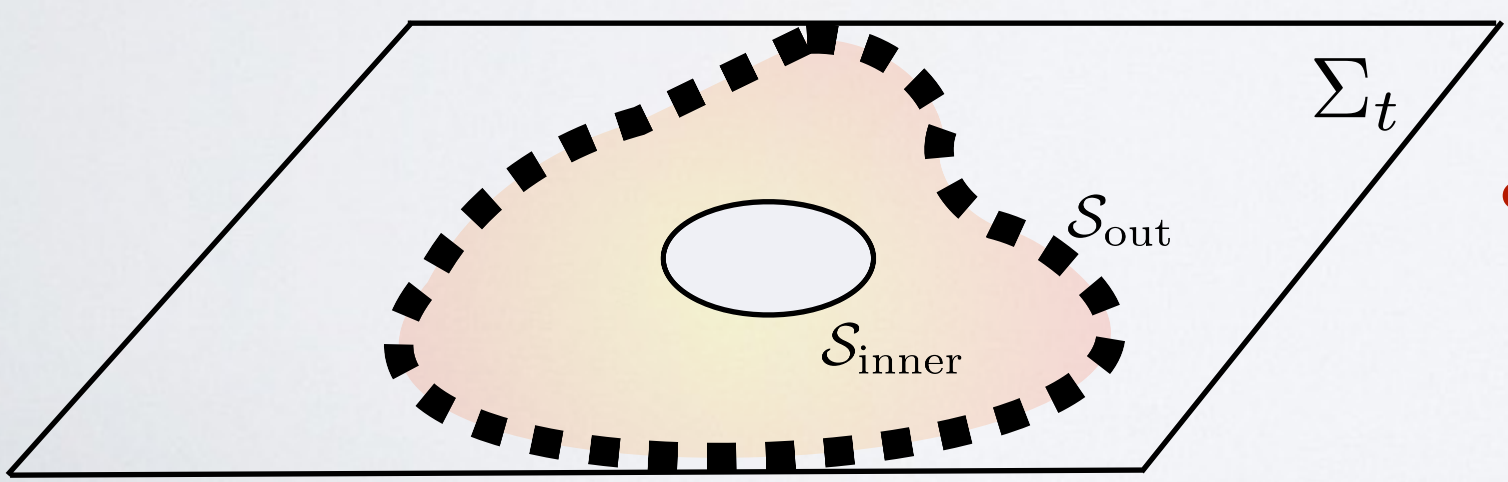
$$\mathcal{S}_{\rm inner} : \Theta_+ = 0$$



- **Boundary Condition:**

$$K - s^i s^j \pm {}^{(3)}\nabla_i s^i = 0$$

PERTURBED BLACK HOLES

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- Constraint equations (with boundary conditions) fix the other 4 variables

PERTURBED BLACK HOLES

- Variables: γ_{ij} K_{ij} (2 , 3X3 symmetric tensors)

  ➡Total of: 2 x 6 = 12 unknown variables

- Constraint Equations : $\mathbb{H} = 0$ (Hamiltonian Constraint)

  $\mathbb{P}_i = 0$ (Momentum Constraint)

  ➡Total of: 4 equations

- We can **freely** specify 8 quantities

- Constraint equations (with boundary conditions) fix the other 4 variables

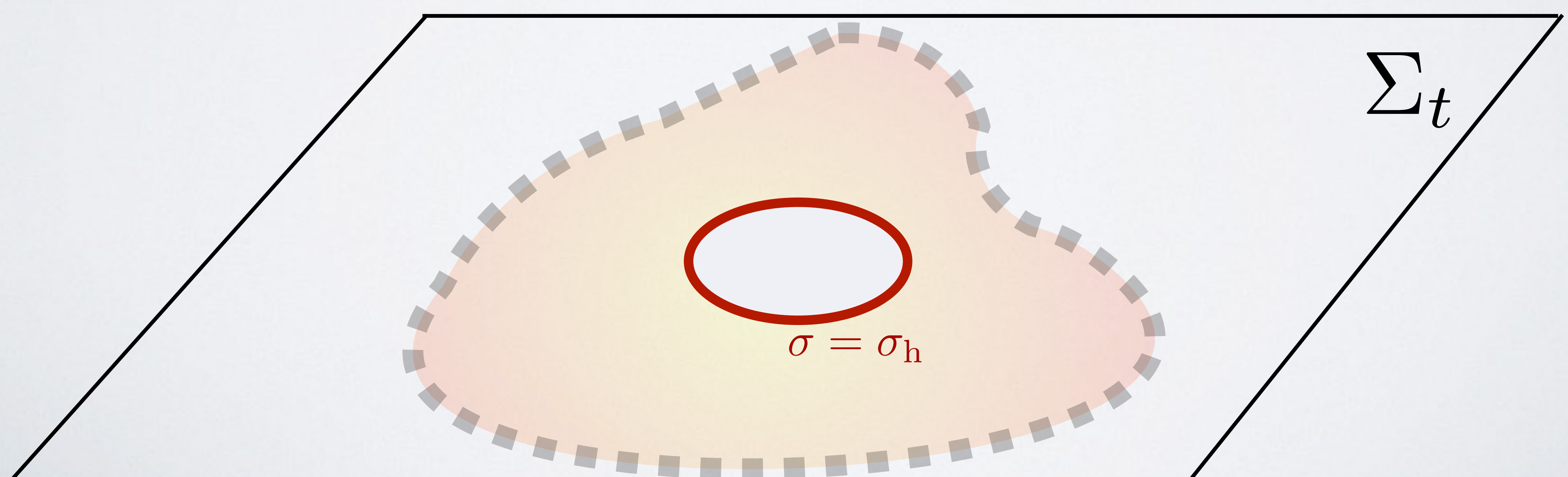
"Conformal transverse traceless decomposition"

PERTURBED BLACK HOLES

- We can **freely** specify 8 quantities :

Read from Kerr space-time ($\mathrm{j} = \frac{\mathbf{a}}{\mathrm{M}}$)
in hyperboloidal coordinates

Black Hole
$\Rightarrow \sigma = \sigma_{\mathrm{h}}$



PERTURBED BLACK HOLES

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Read from Kerr space-time ($j = \frac{a}{M}$) Black Hole in hyperboloidal coordinates $\Rightarrow \sigma = \sigma_{\mathrm{h}}$

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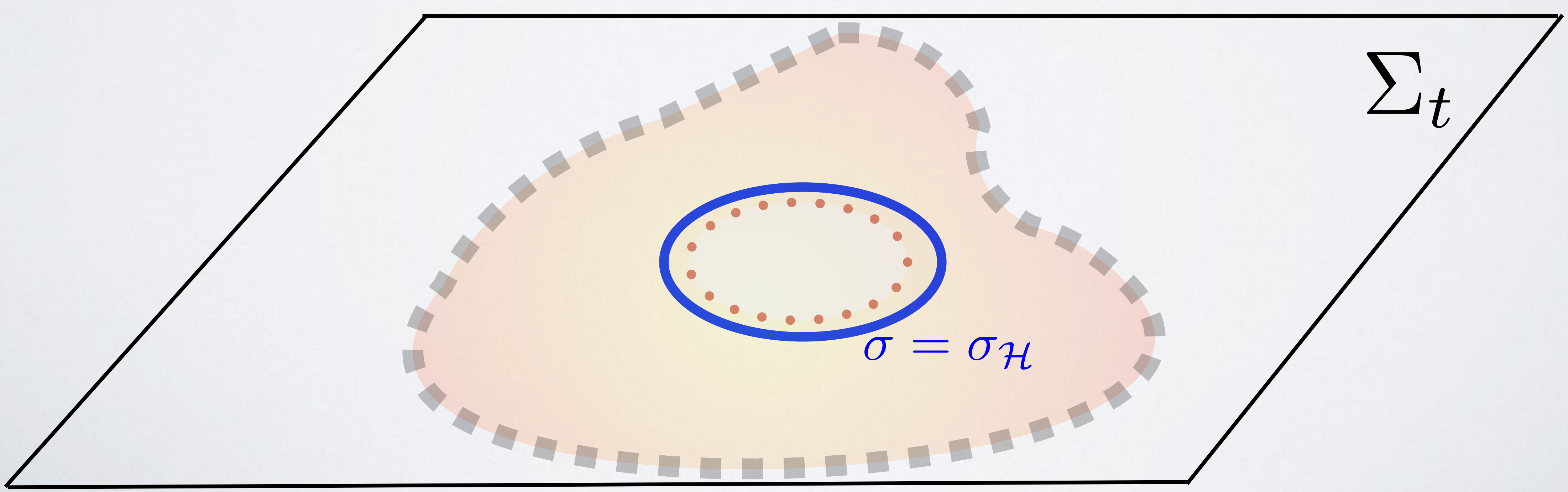
$\mathcal{S}_{\mathrm{inner}} : \Theta_+ = 0$ at $\sigma = \sigma_{\mathcal{H}} \neq \sigma_{\mathrm{h}}$



PERTURBED BLACK HOLES

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Read from Kerr space-time ($\mathrm{j} = \frac{\mathbf{a}}{\mathrm{M}}$) Black Hole in hyperboloidal coordinates $\Rightarrow \sigma = \sigma_{\mathrm{h}}$

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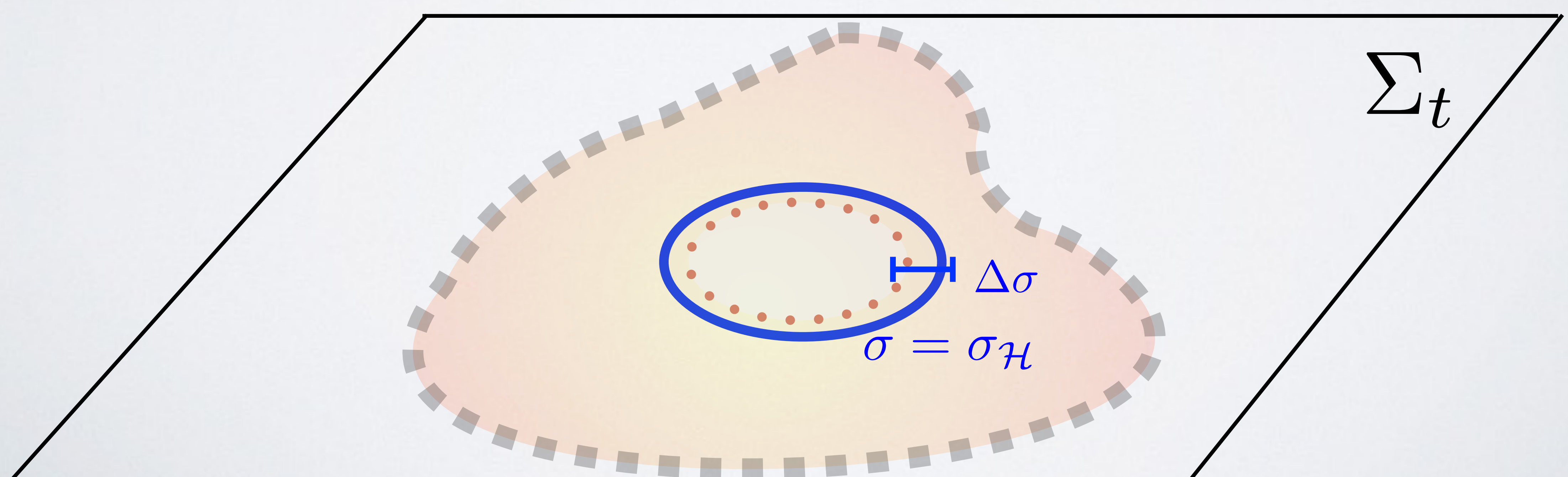
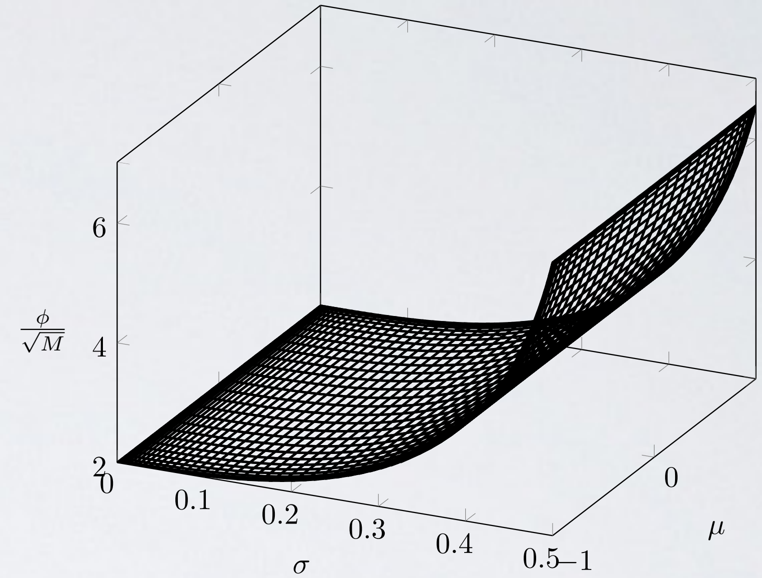
$\mathcal{S}_{\mathrm{inner}} : \Theta_+ = 0$ at $\sigma = \sigma_{\mathcal{H}} \neq \sigma_{\mathrm{h}}$

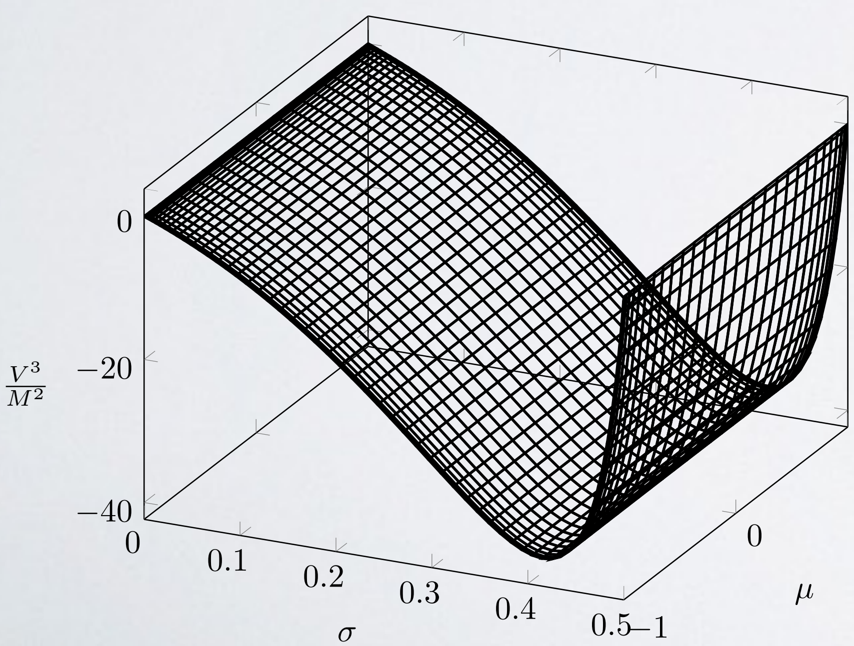
Perturbation parameter: $\Delta\sigma = \sigma_{\mathrm{h}} - \sigma_{\mathcal{H}}$

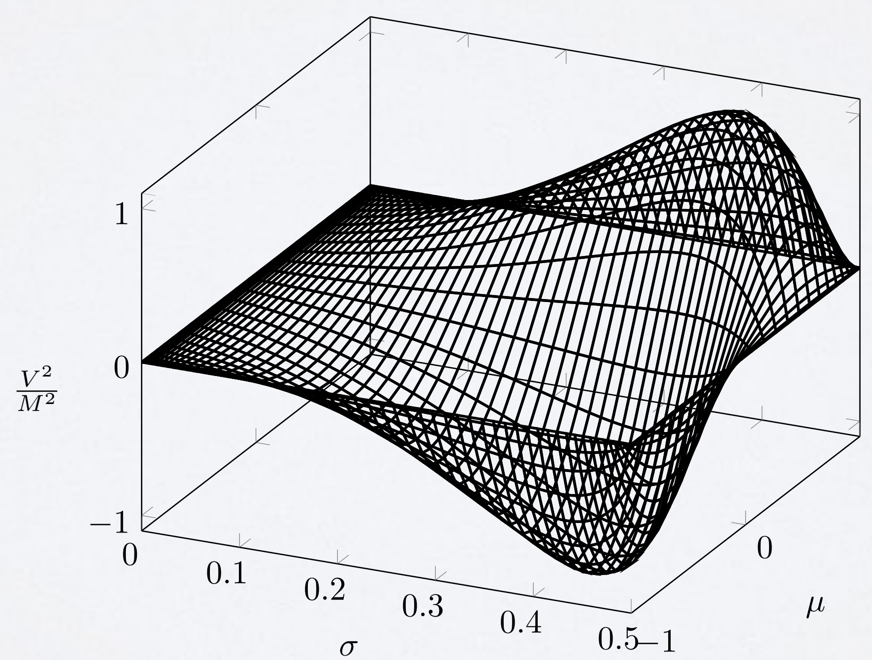


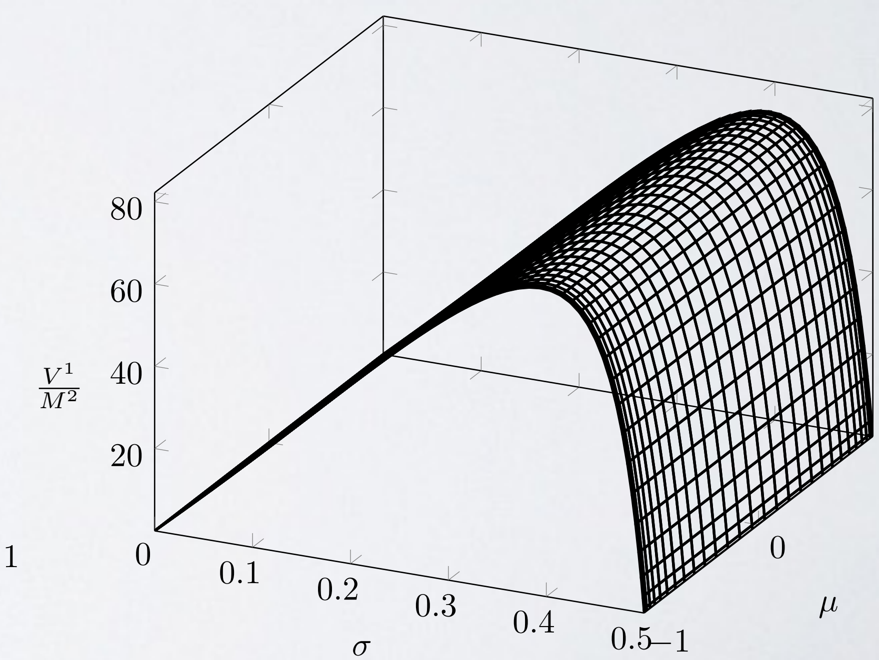
NUMERICAL SOLUTION

- Axisymmetry
- Pseudo-Spectral methods
- Gauß-Lobatto grid points
- Newton-Raphson method
- BicCGStab with finite-difference pre-conditioner









NUMERICAL SOLUTION

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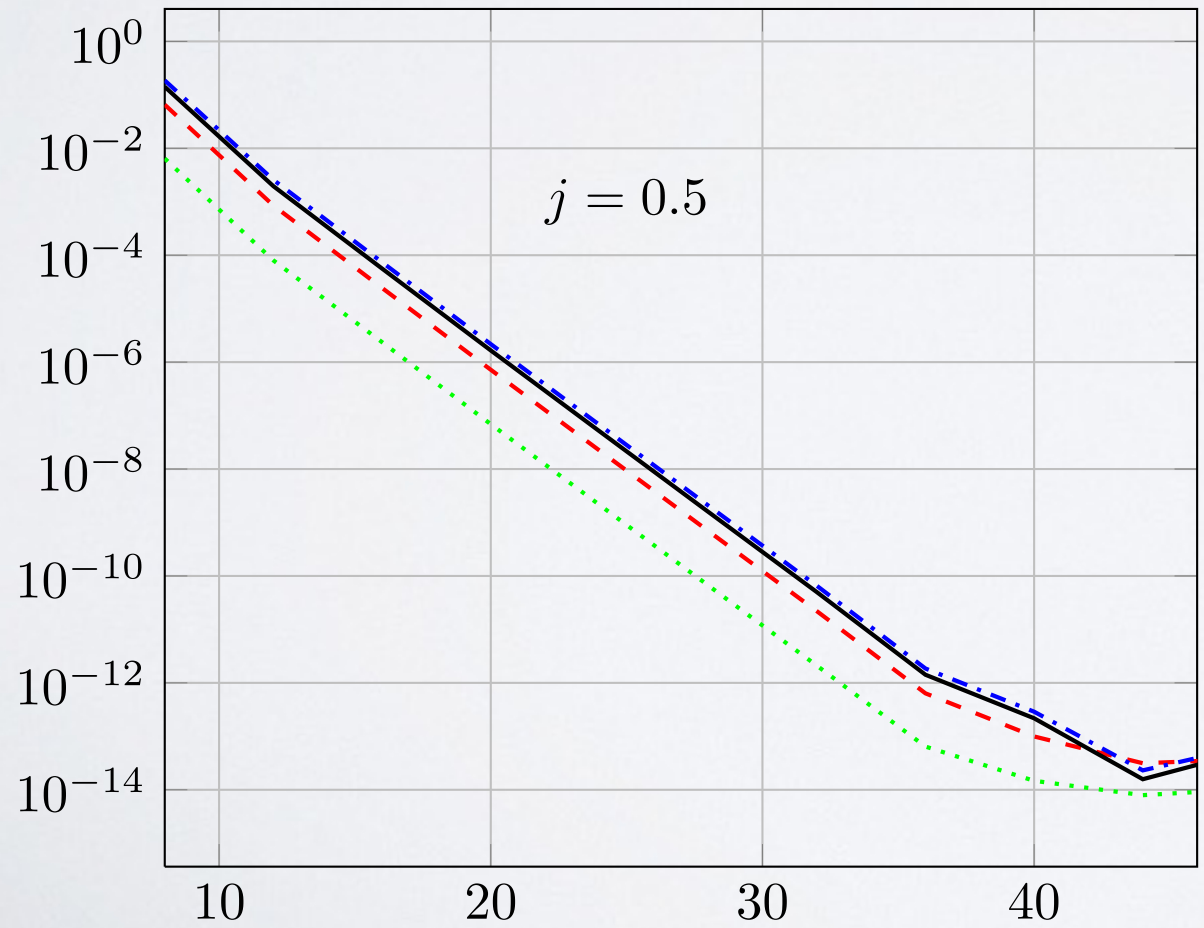
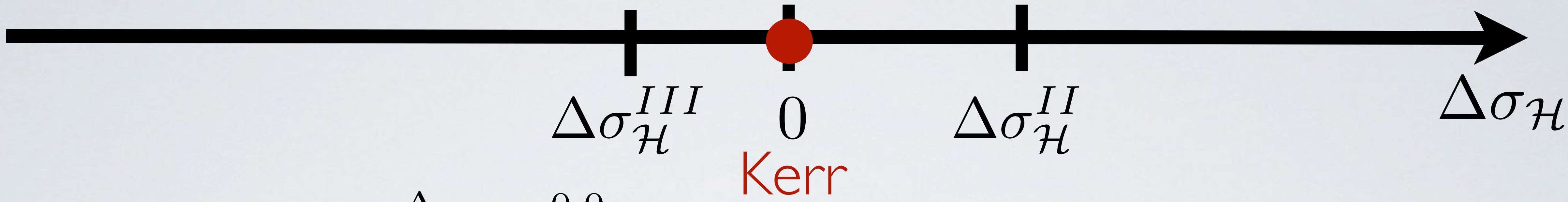
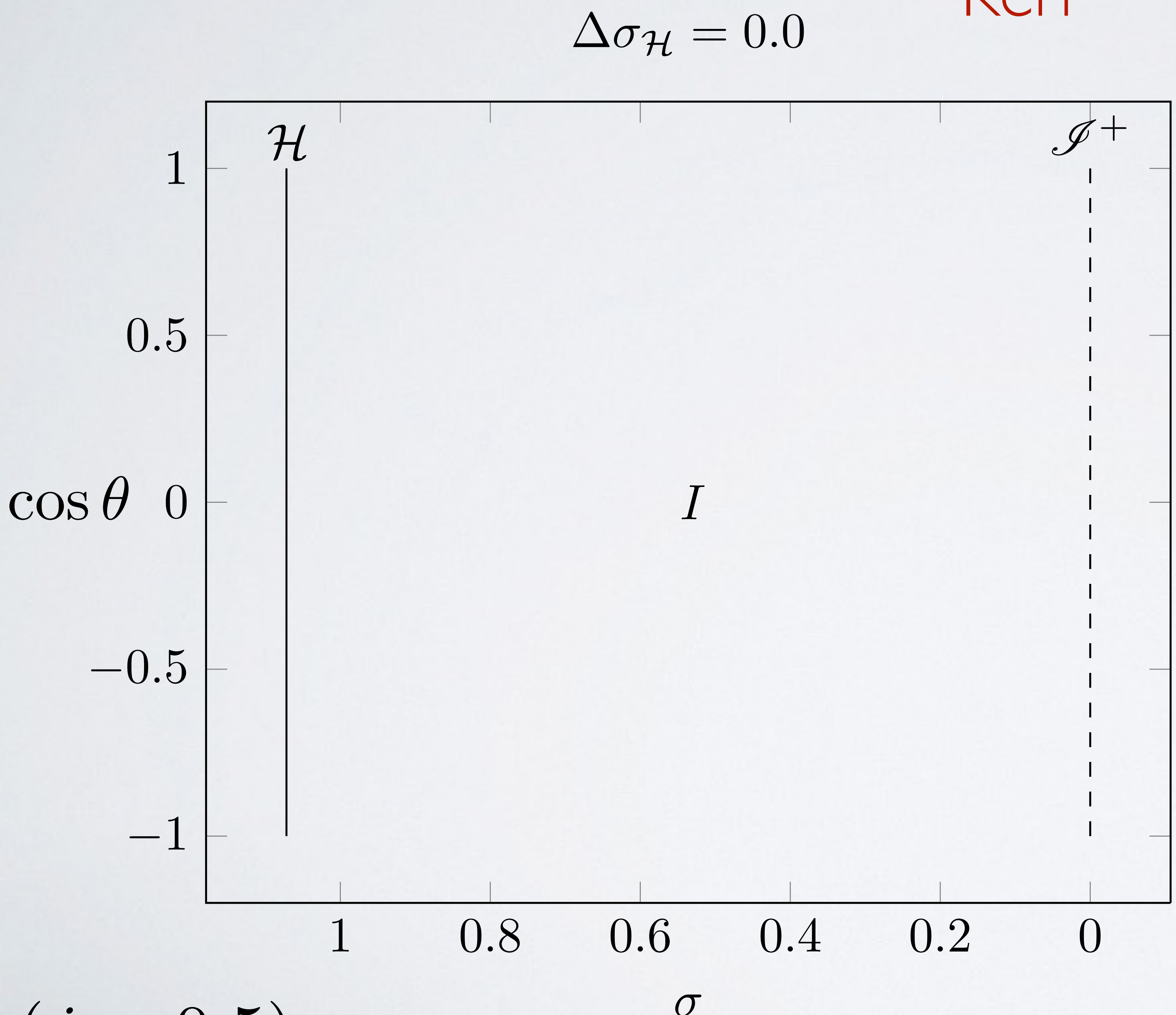
Error



Accuracy down to machine precision: 11~12 decimal digits

Trapped Regions





$(j = 0.5)$

- Numerical Boundaries

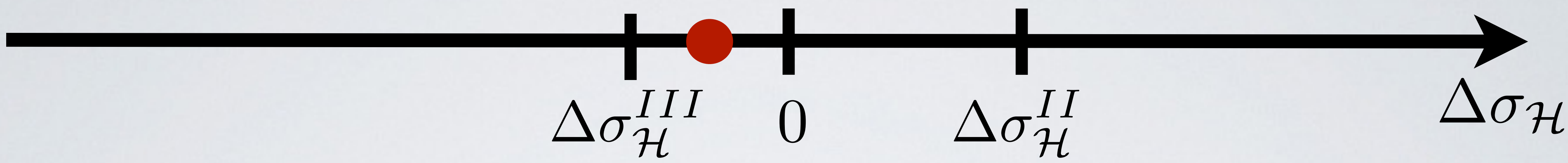
$$\mathscr{I}^+ \qquad \sigma = \sigma_{\mathcal{H}}$$

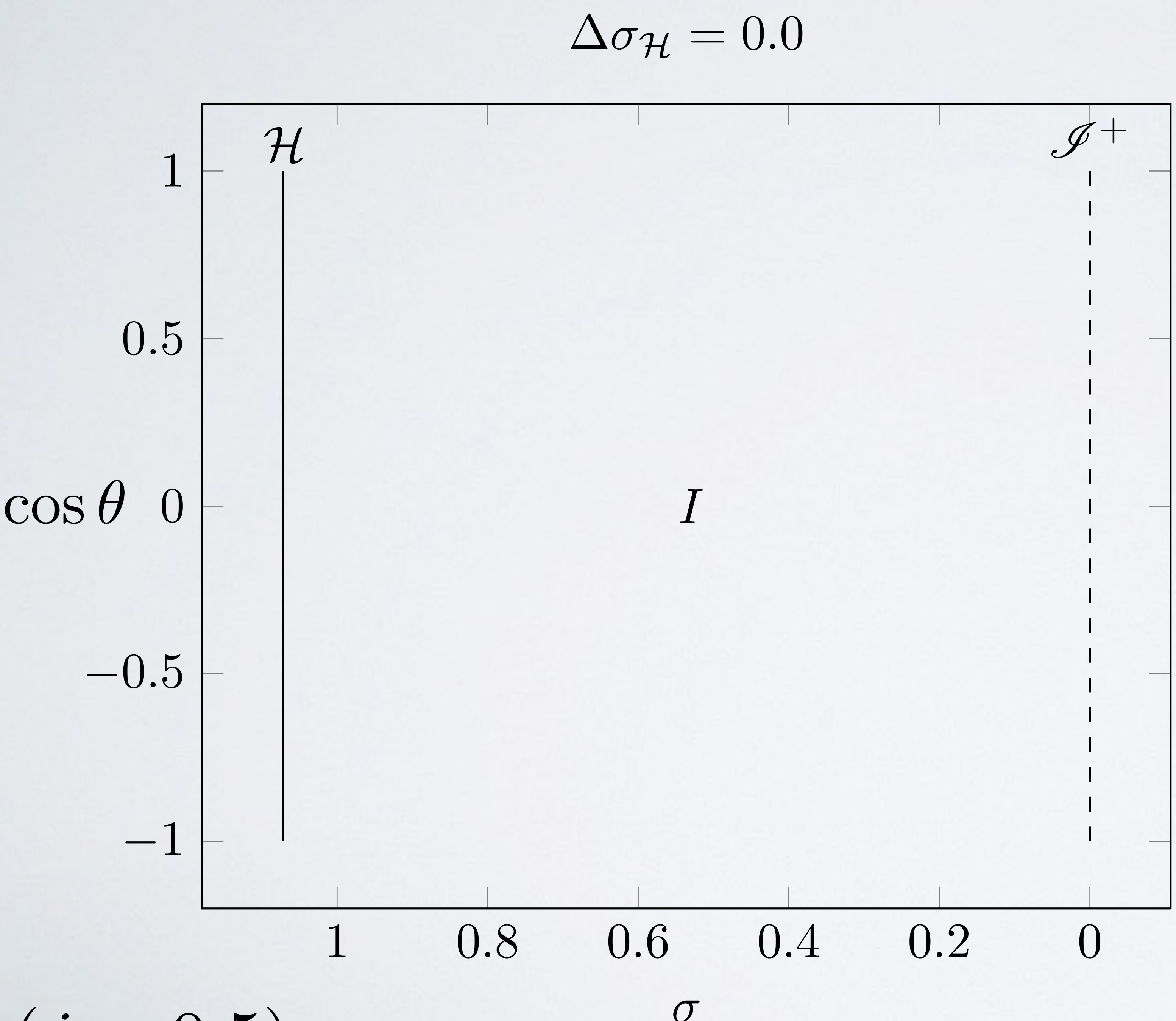
- Region I:

$$\Theta_+ > 0 \quad \Theta_- < 0$$

Exterior solution

Trapped Regions





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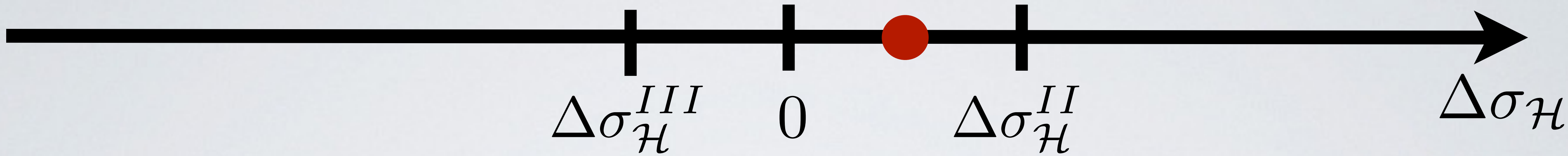

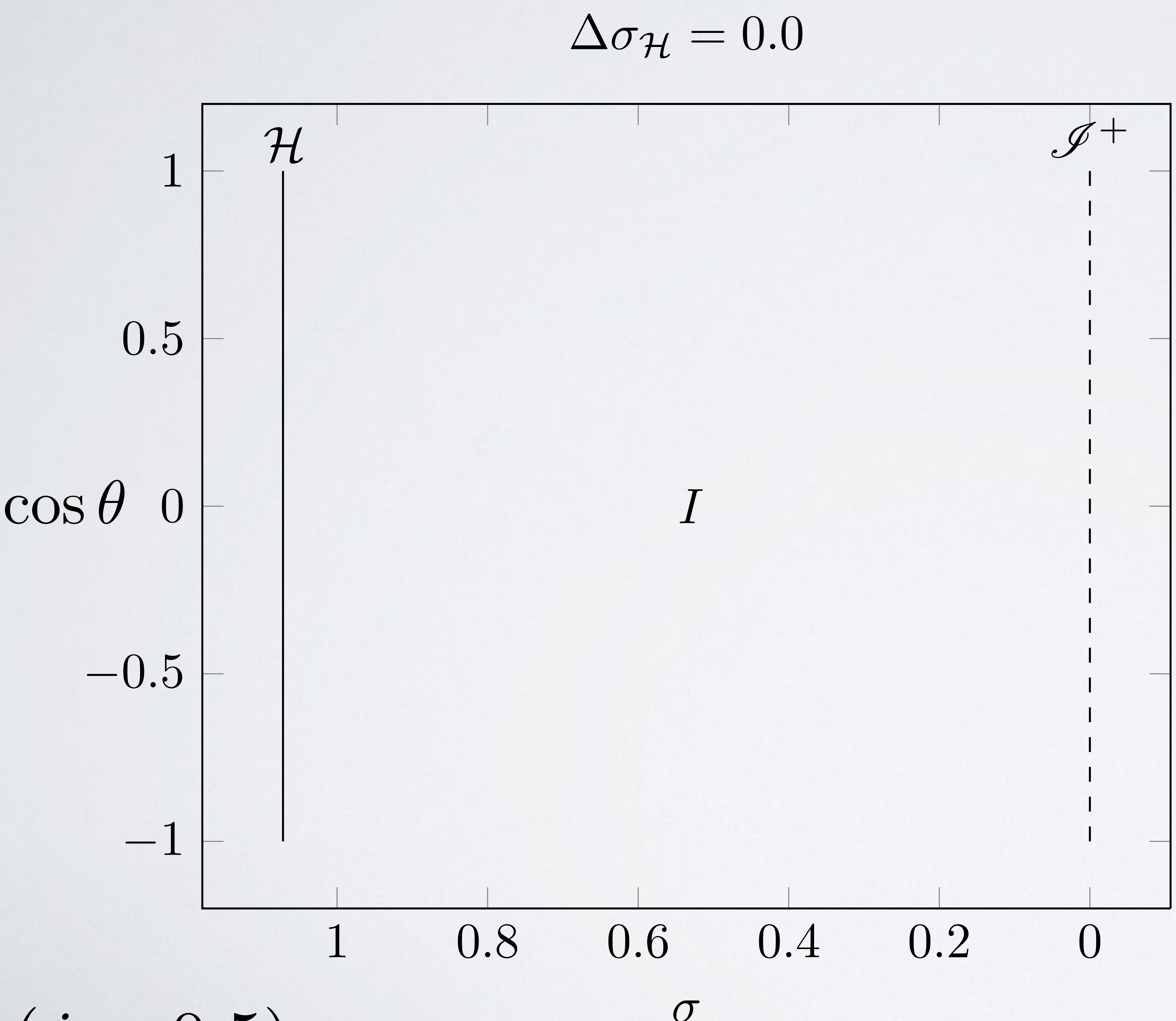

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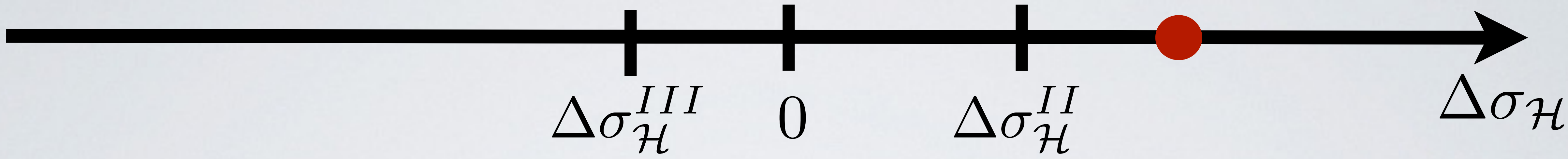

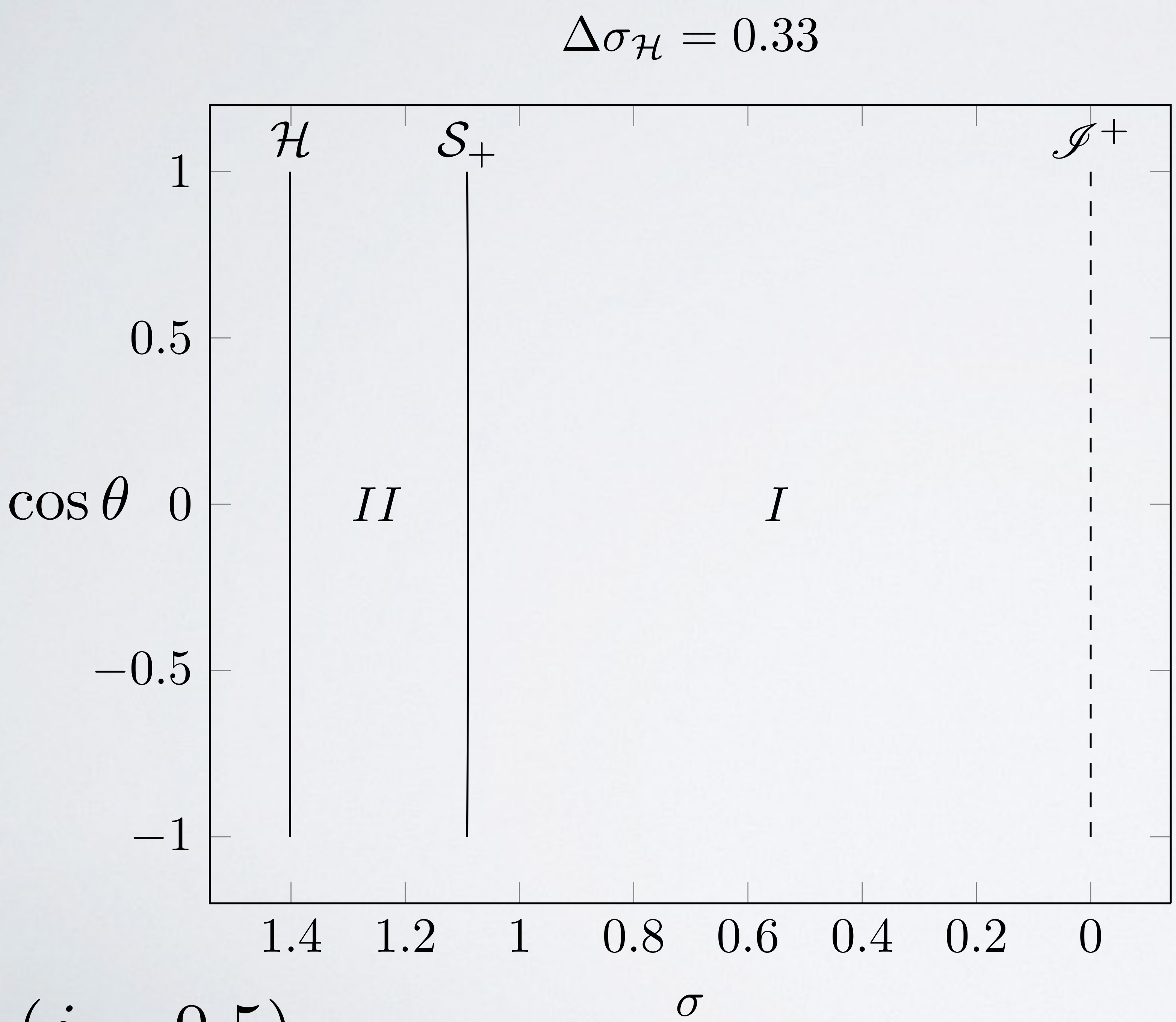

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- New surface $\mathcal{S}_+$

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  Exterior solution

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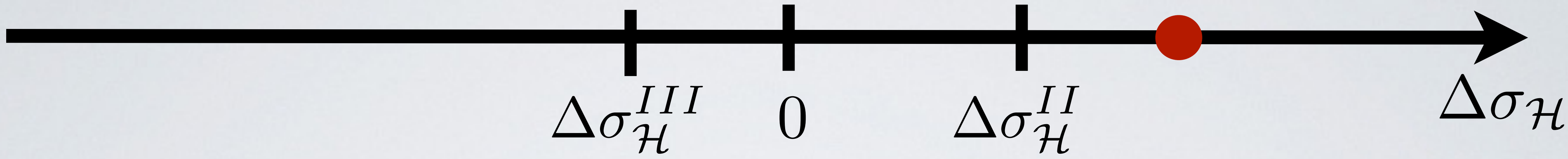
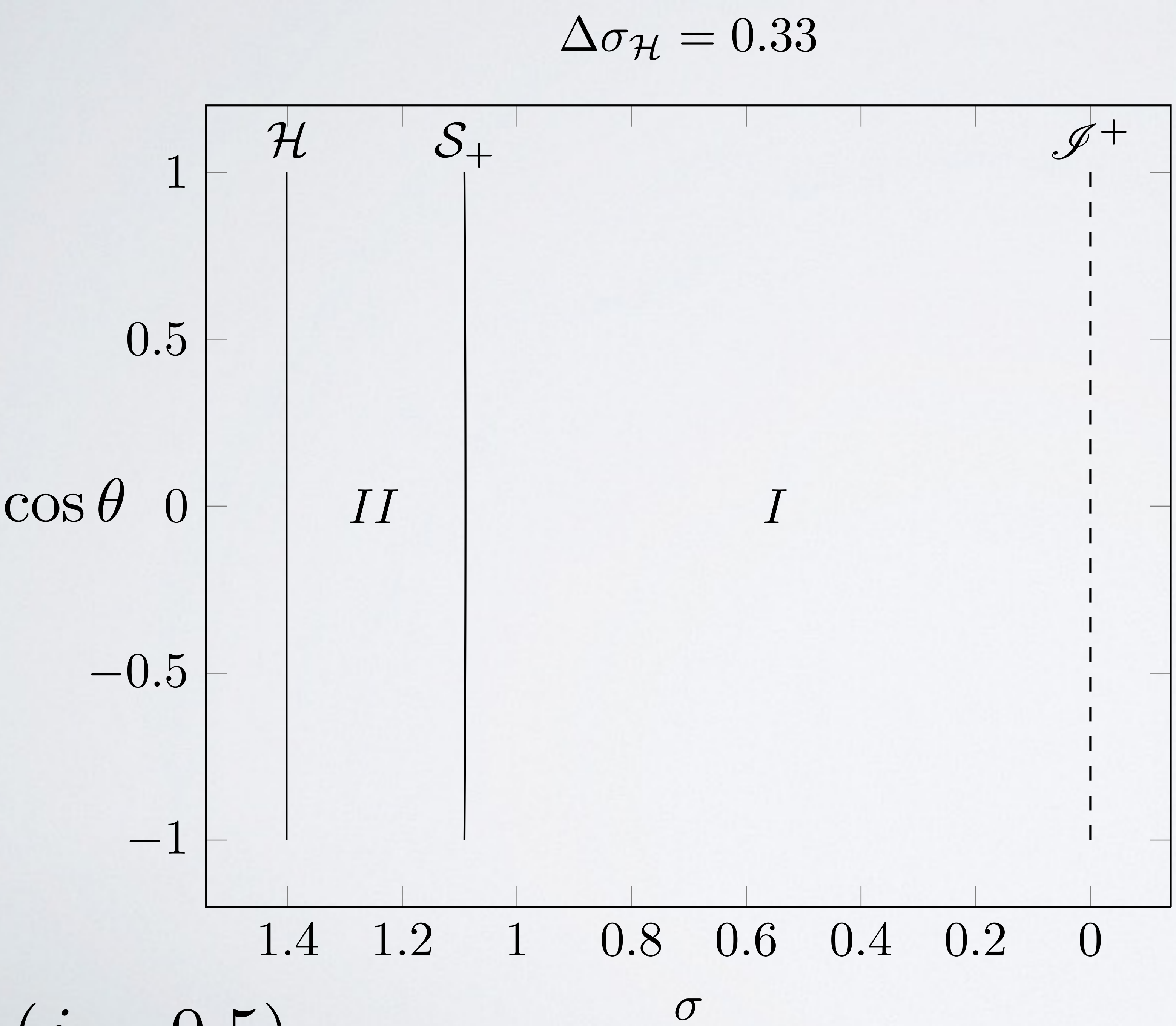
Trapped Regions





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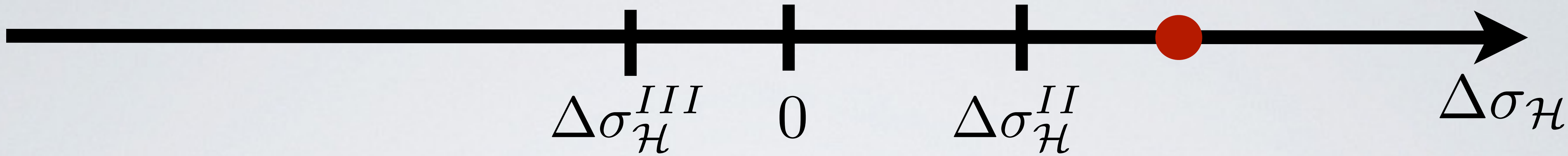
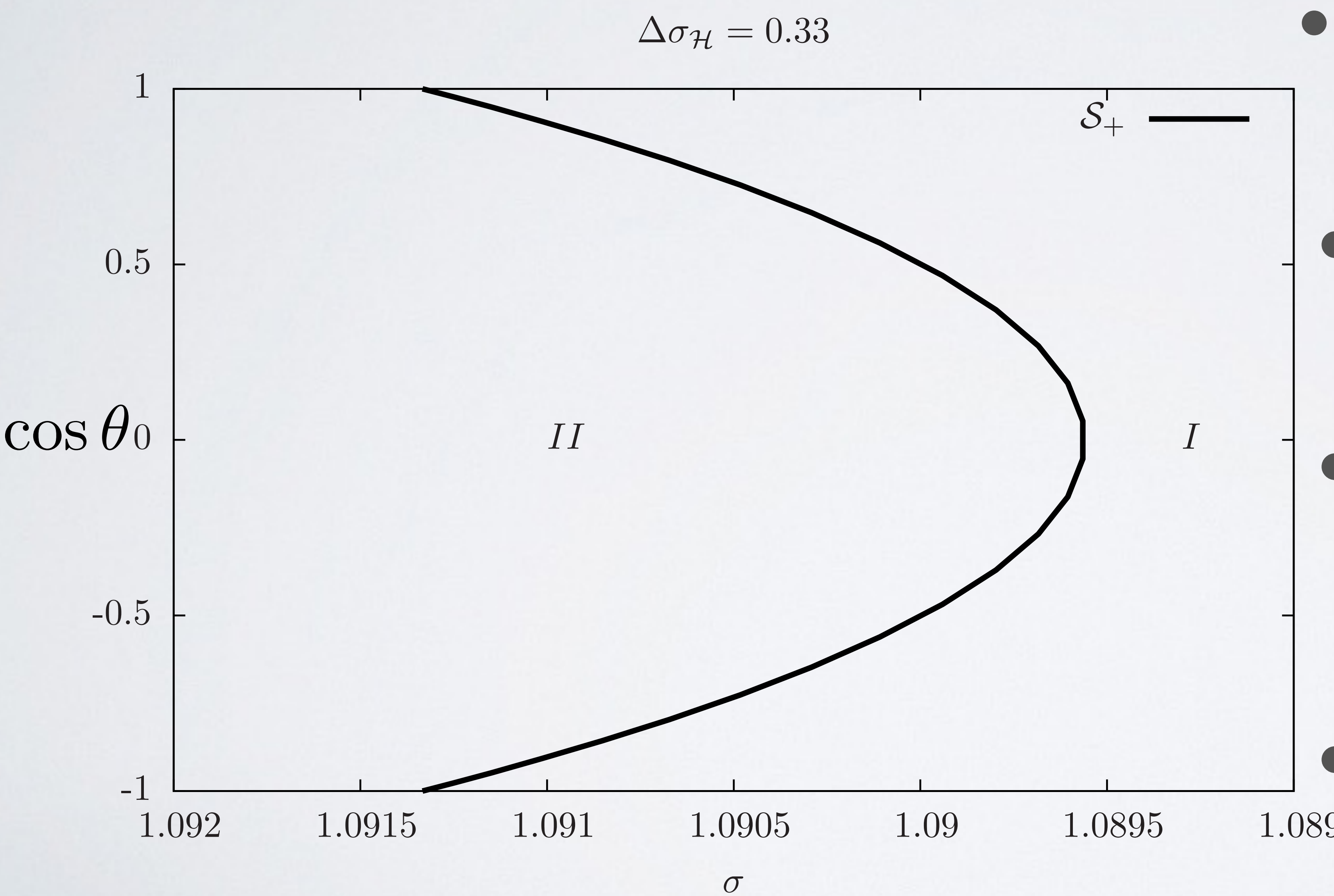
  Black Hole

Trapped Regions





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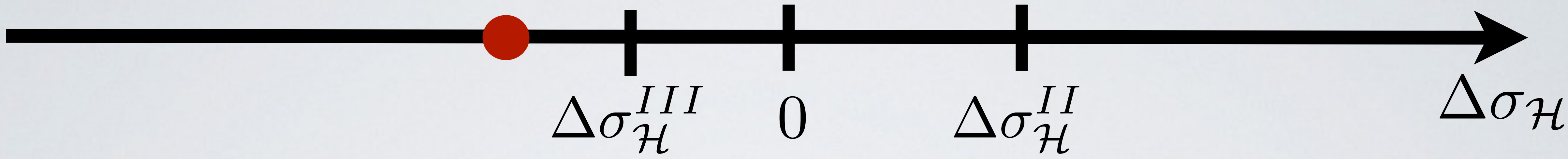

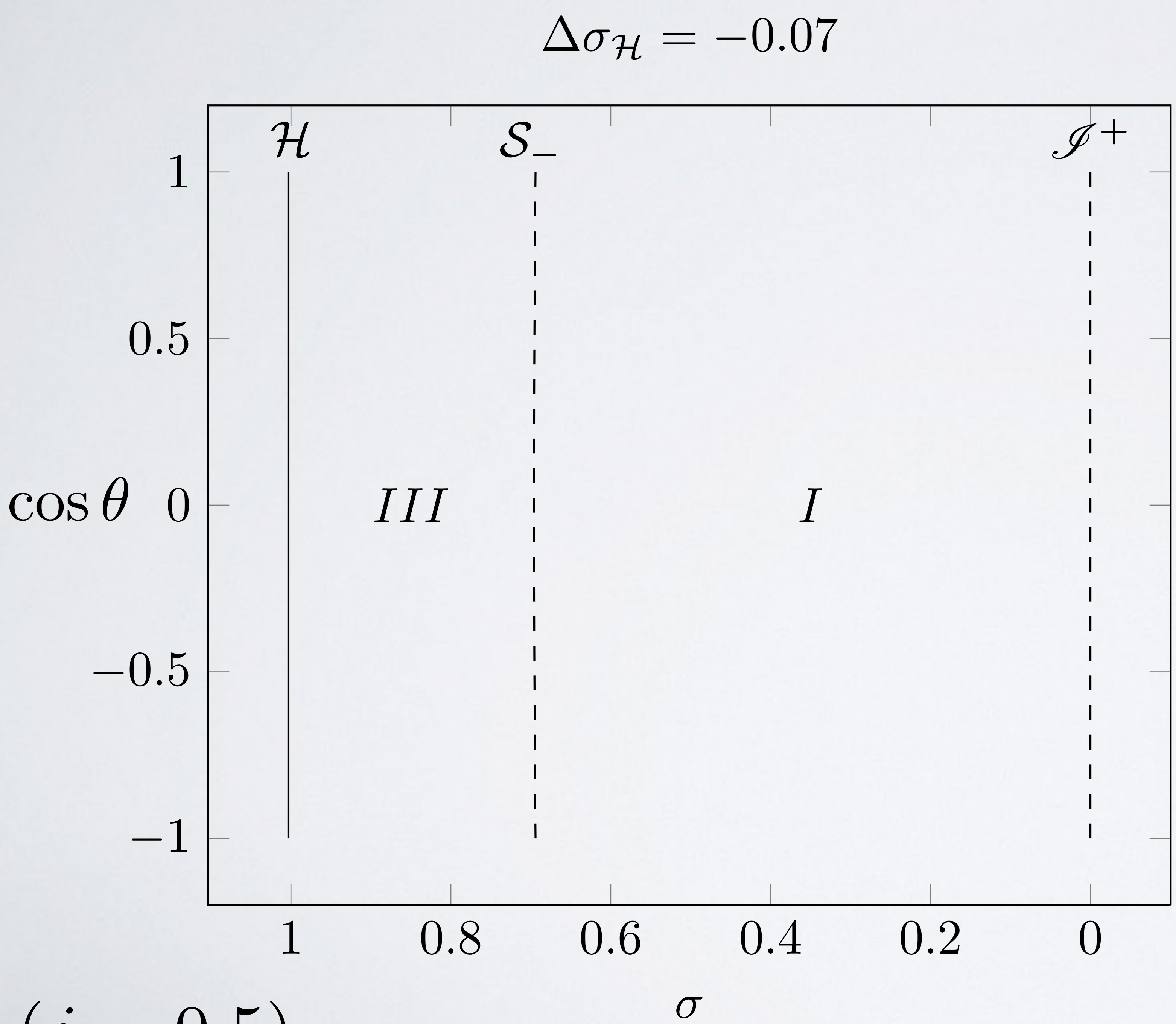

$(j = 0.5)$

- Numerical Boundaries

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- New surface $\mathcal{S}_-$

  $\Theta_- = 0$

- Region I:

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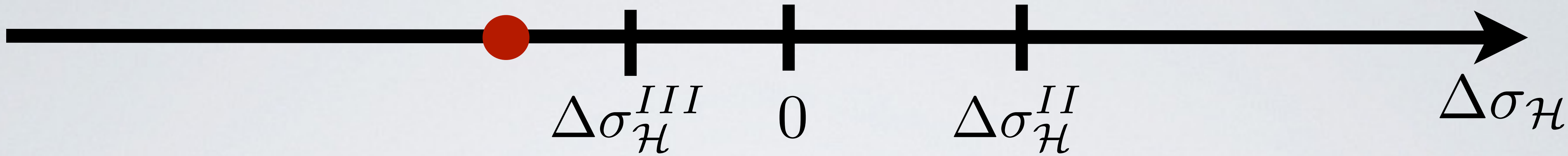
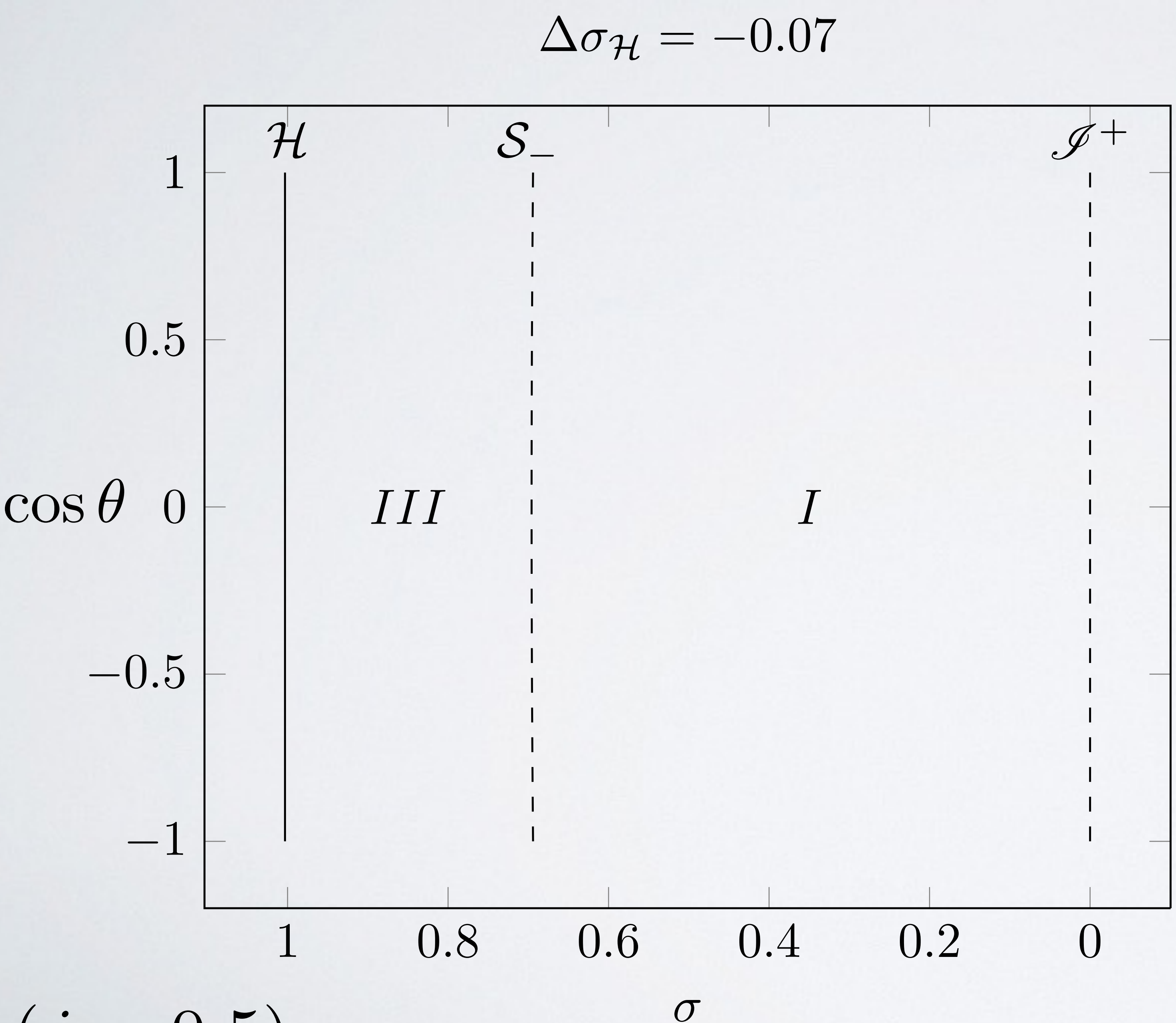
  Exterior solution

- Region III:

  $\Theta_+ > 0 \quad \Theta_- > 0$

Trapped Regions





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  Exterior solution

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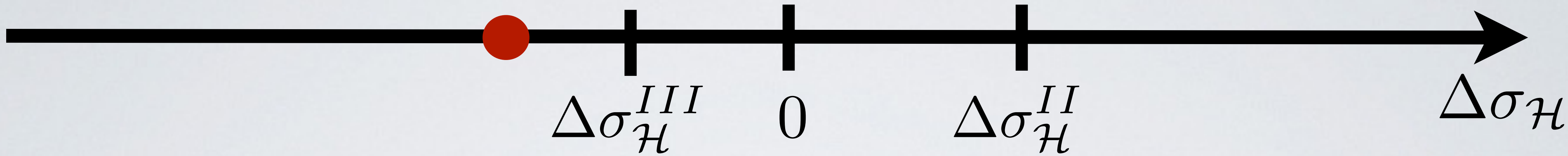
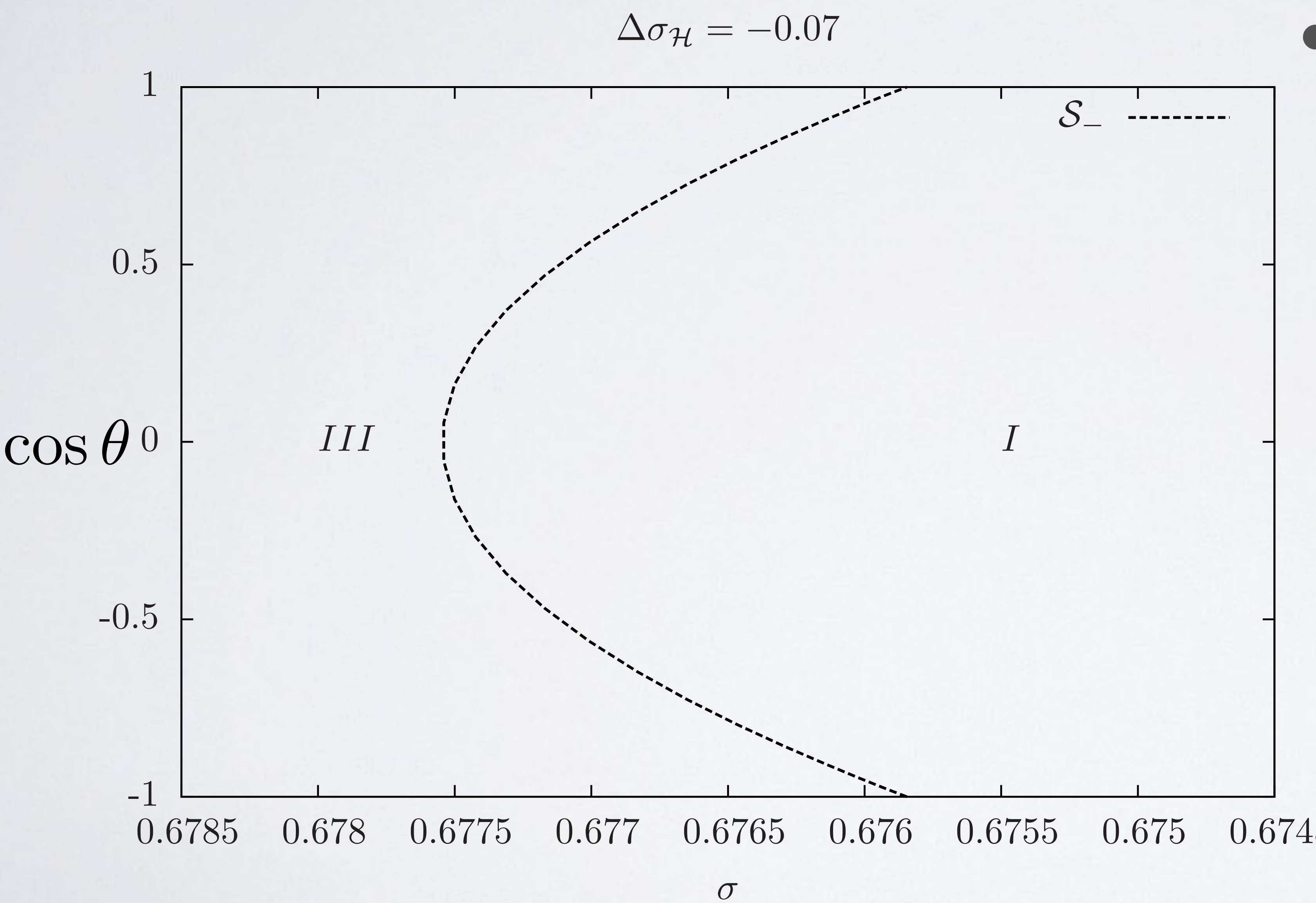
White Hole

Trapped Regions





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  Exterior solution

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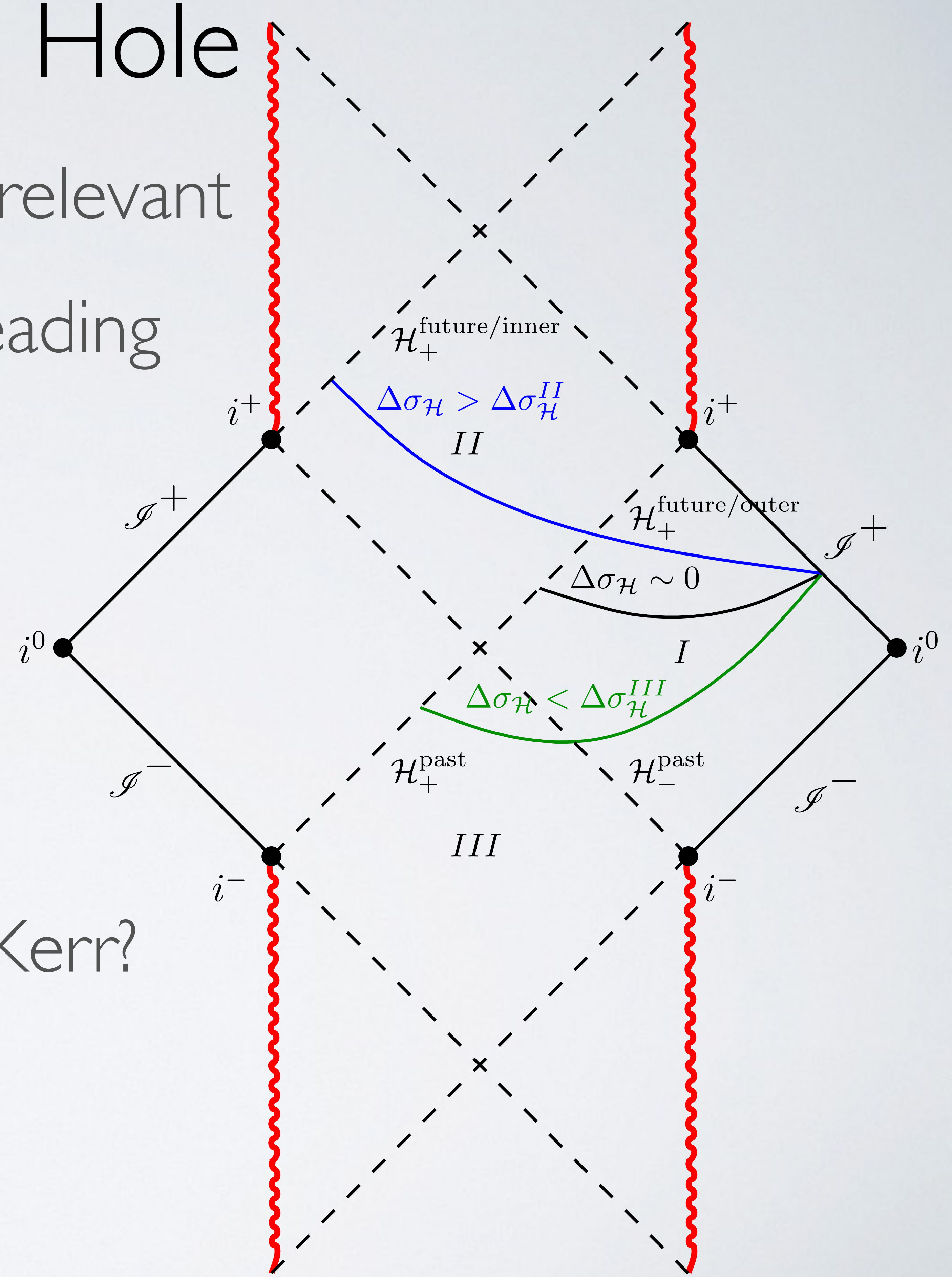
$(j = 0.5)$

White Hole

White Hole

- Astrophysical: solution not relevant
  - No scenario in nature leading to white holes

- Mathematic perspective:
  - Any role in stability of Kerr?
  - Bifurcation point?



Bondi Mass and Angular Momentum

- Bondi mass: Hawking mass evaluated at $\mathscr{I}^+$

$$M_B = \lim_{S \to \mathscr{I}^+} \left\{ \sqrt{\frac{\mathcal{A}_S}{16\pi}} \left[1 + \frac{1}{8\pi} \oint_S \Theta_+ \Theta_- dA \right] \right\},$$

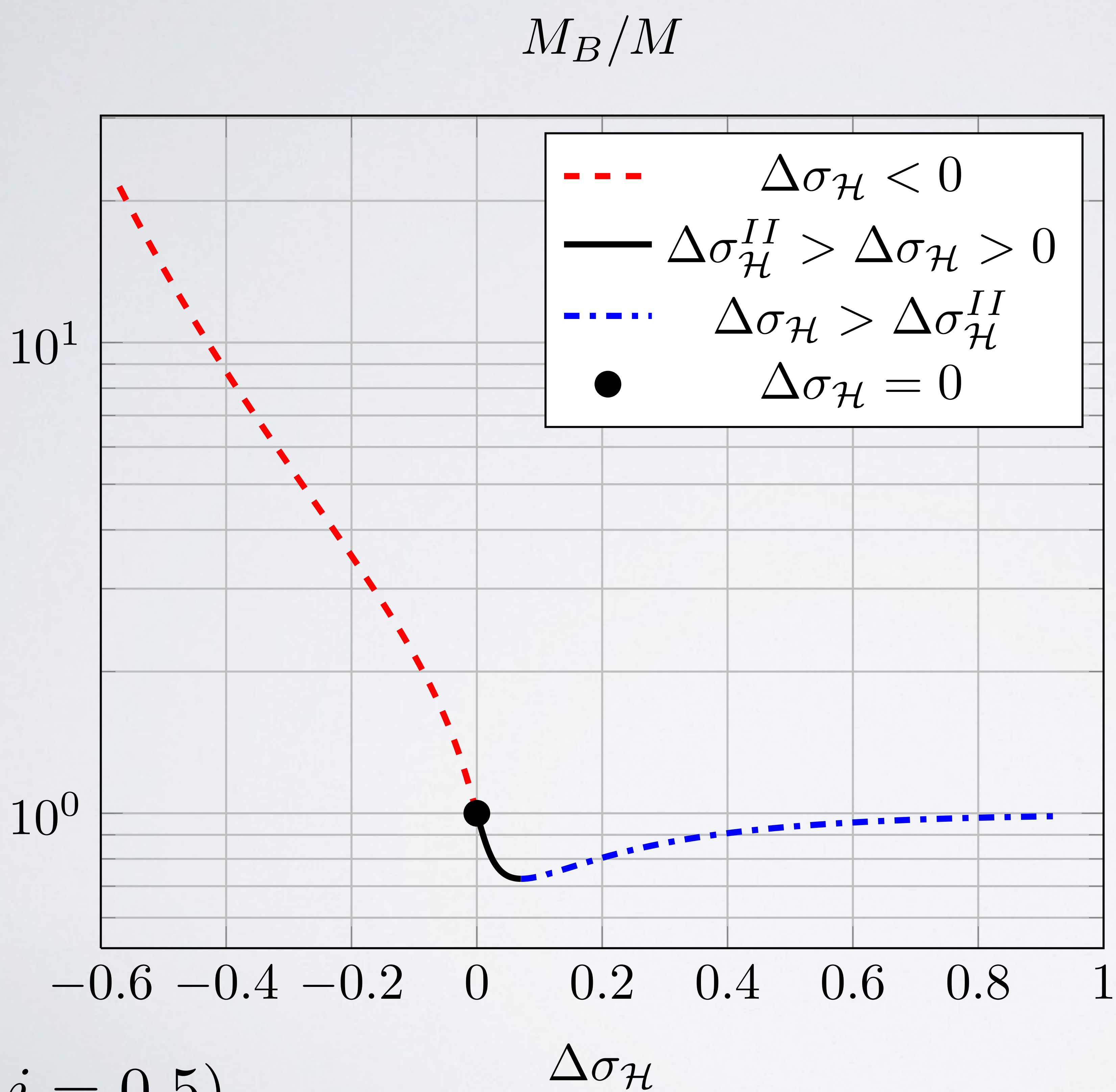
- Angular momentum: Komar angular momentum (or Asthekar’s first angular multipole)

$$J = \frac{1}{8\pi} \oint_{\mathcal{H}} K_{ij} \phi^i s^j dA$$

- Dimensionless angular momentum:

$$j_B = \frac{J}{M_B^2}$$

Bondi Mass and Angular Momentum

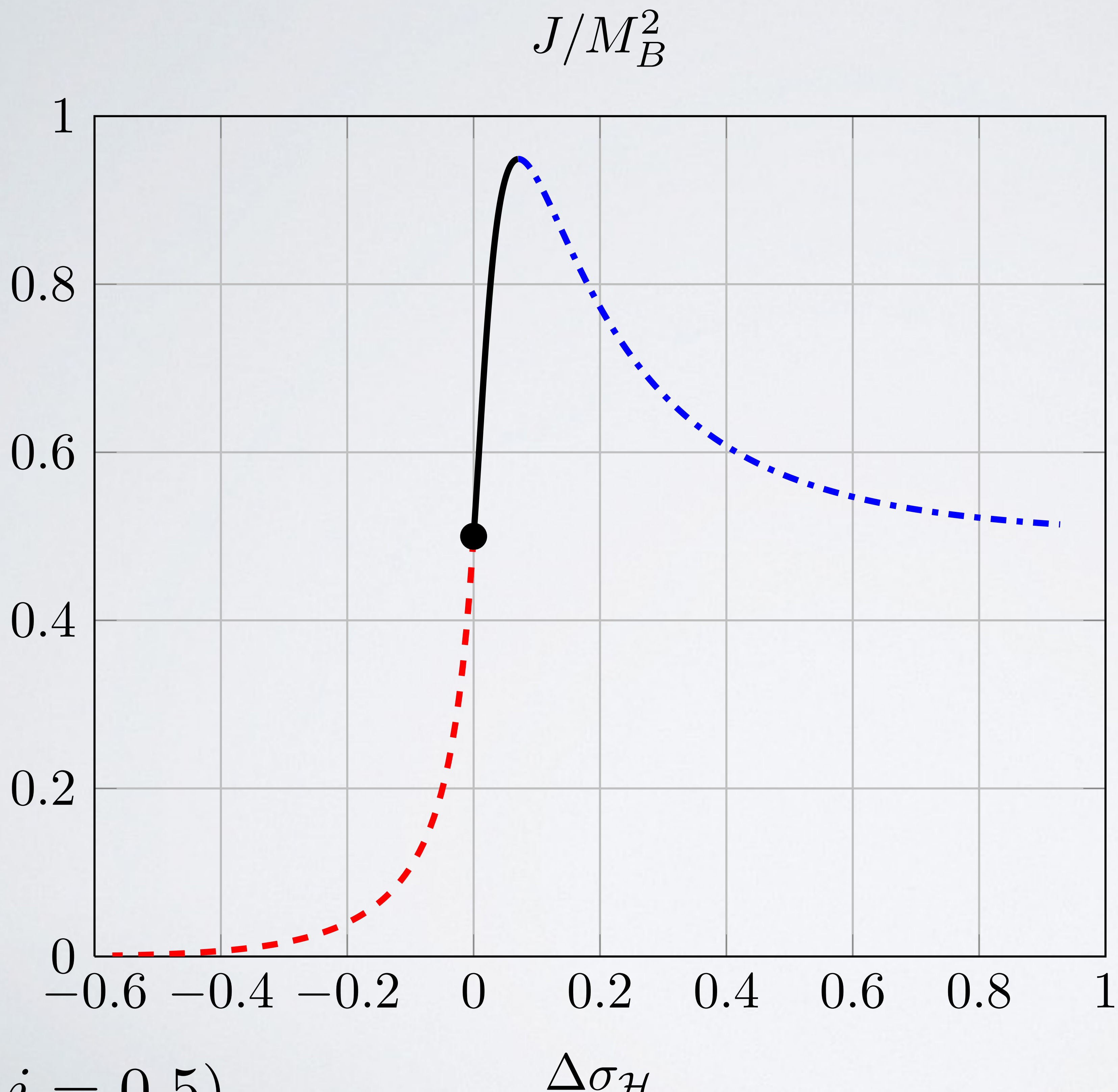


$(j = 0.5)$

- For $\Delta\sigma_{\mathcal{H}} >> \Delta\sigma_{\mathcal{H}}^{II}$

$$M_B \to M$$

Bondi Mass and Angular Momentum



$(j = 0.5)$

- Maximum:

$$j_B \approx 0.95$$

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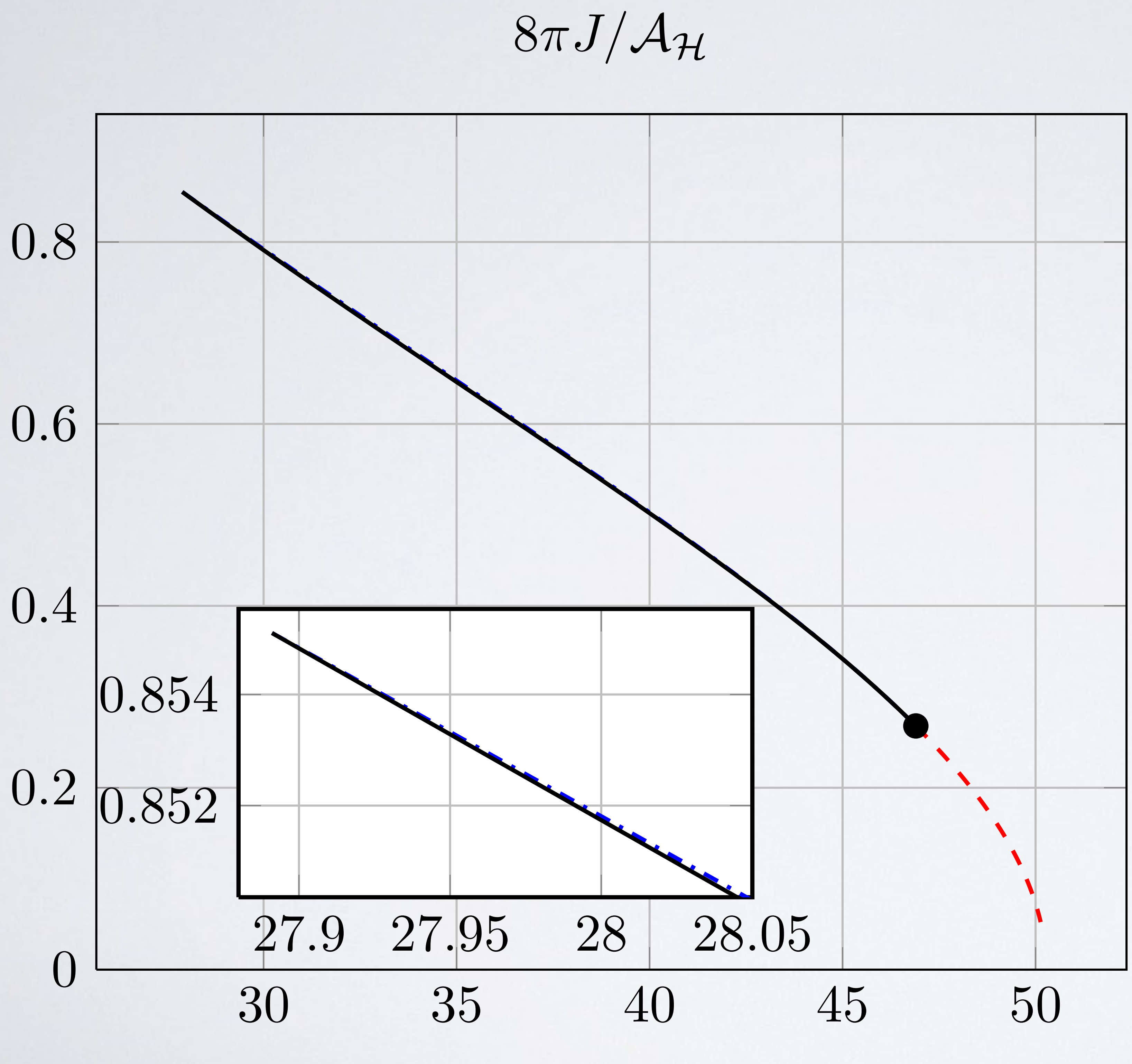
Bondi Mass and Angular Momentum



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- For $\Delta\sigma_{\mathcal{H}} >> \Delta\sigma_{\mathcal{H}}^{II}$

$$M_B \rightarrow M$$
$$j_B \rightarrow 0.5$$

Black hole inequalities

- Assuming weak cosmic censorship to hold, Penrose conjectured

$$\mathcal{A}_{\mathcal{H}} \leq 16\pi M_{\mathrm{ADM}}^2.$$

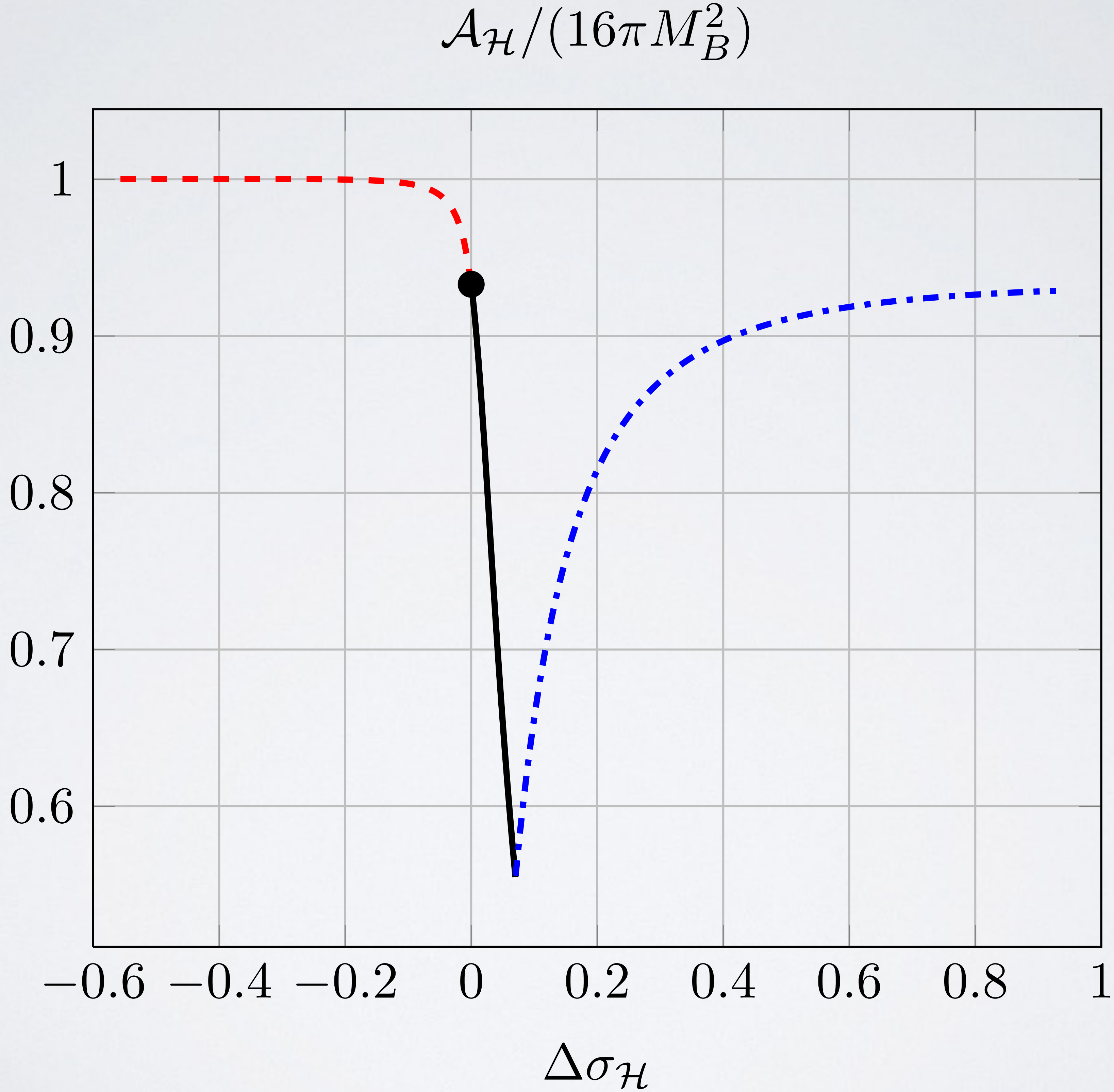
- Reasoning to the inequality passes through an intermediate stage

$$\mathcal{A}_{\mathcal{H}} \leq 16\pi M_{\mathrm{B}}^2,$$

- Stronger inequality since $M_B \leq M_{\mathrm{ADM}}$

- Rigidity: equality holds only for Schwarzschild

Black hole inequalities



$(j = 0.5)$

Black hole inequalities

- For Kerr, Dain *et. al.* ('02) introduced a refined inequality

$$\epsilon_A := \frac{\mathcal{A}_{\mathcal{H}}}{8\pi(M_{\mathrm{ADM}}^2 + \sqrt{M_{\mathrm{ADM}}^4 - J^2})} \leq 1,$$

- Following the arguments for the Penrose inequality, we (thanks to J.L. Jaramillo) propose a stronger relation in terms of a **Dain-Bondi number**

$$\epsilon_{AB} = \frac{\mathcal{A}_{\mathcal{H}}}{8\pi(M_{\mathrm{B}}^2 + \sqrt{M_{\mathrm{B}}^4 - J^2})} \leq 1$$

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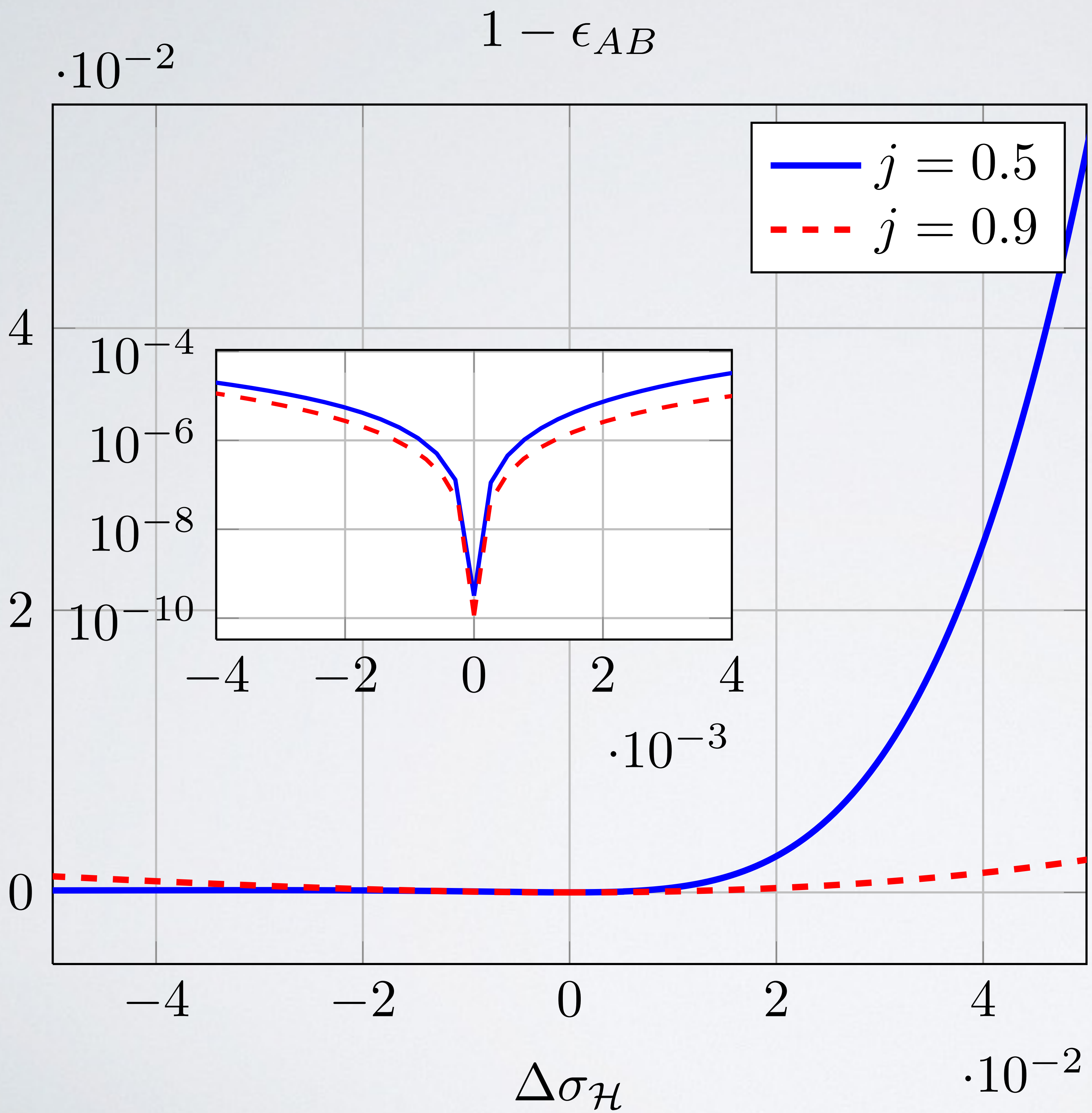
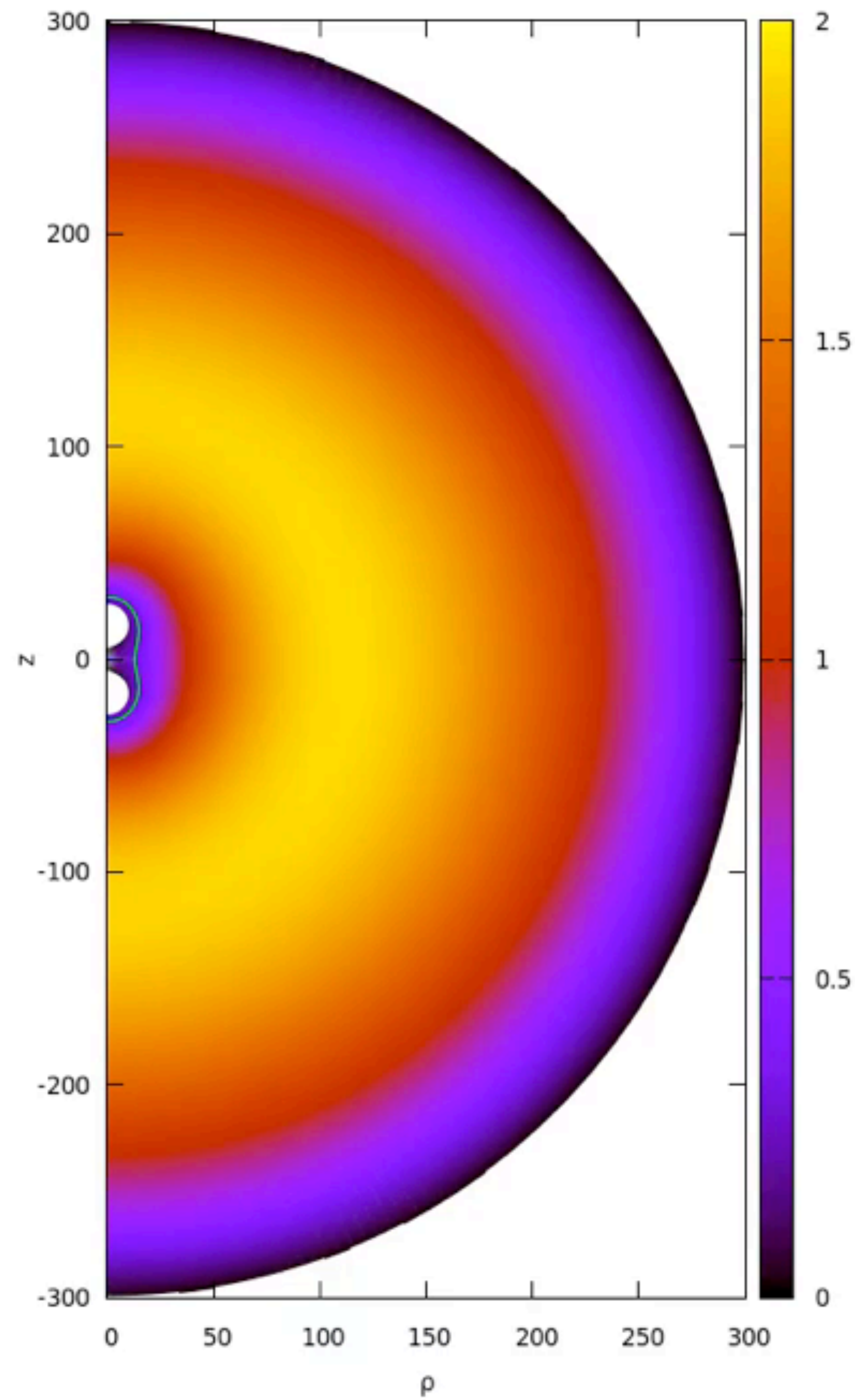
Black hole inequalities



- Results confirms the inequality (within numerical precision)
- Spectral methods are essential to attest the validity of such inequalities

Future Work

- Initial data for axis-symmetric binary black holes (head-on) in hyperboloidal slices



Credit: David Schinkel

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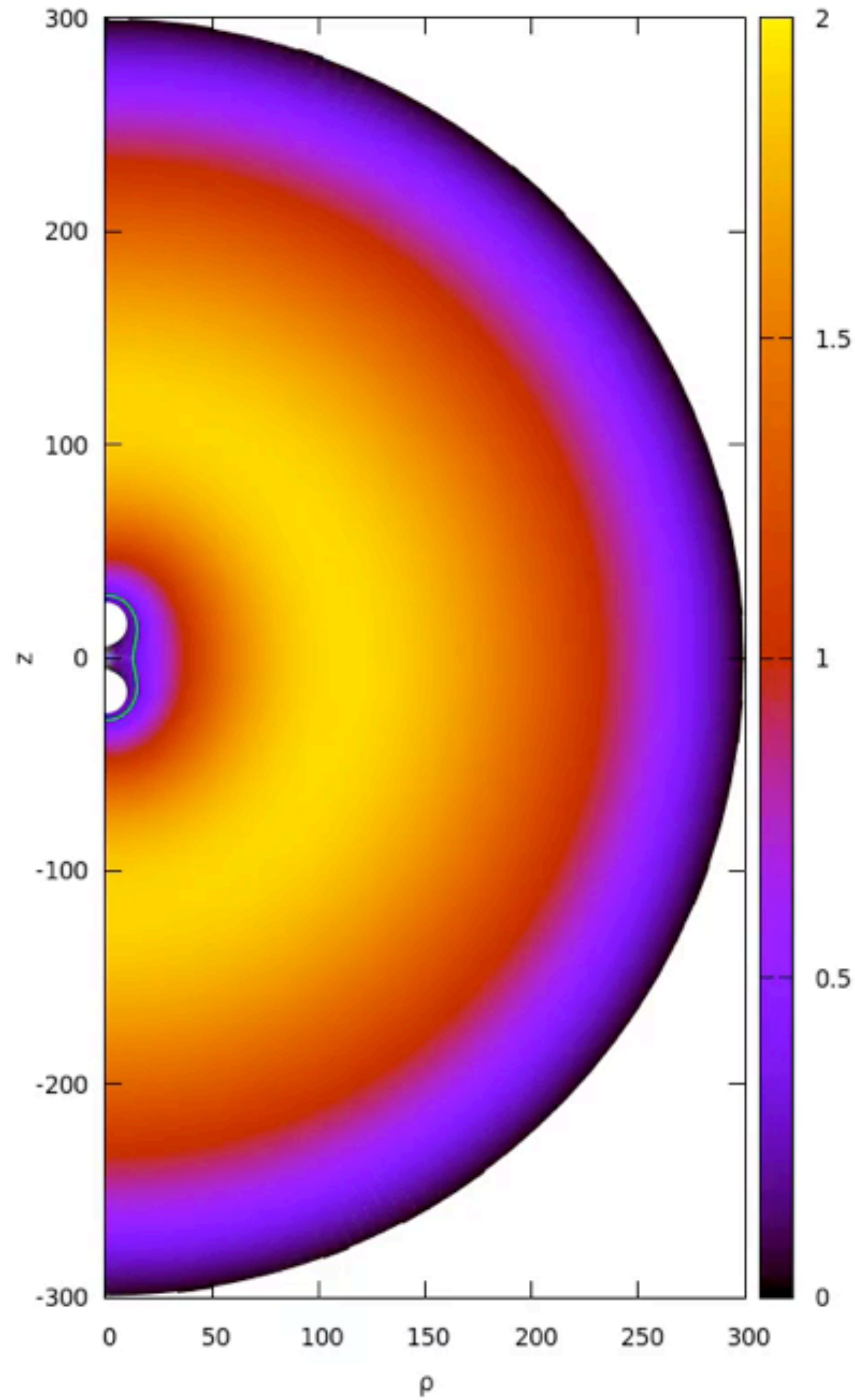
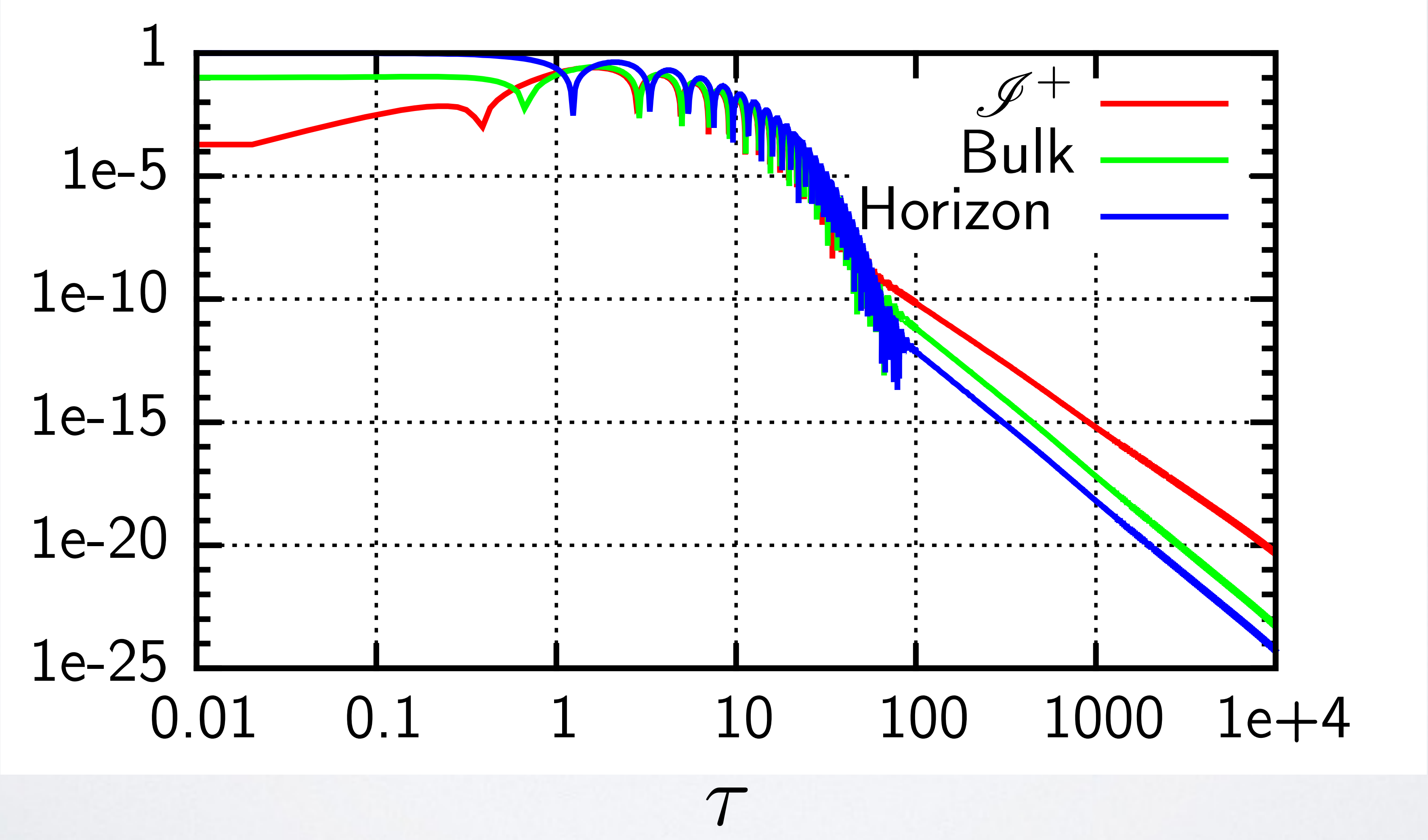


Credit: David Schinkel

Future Work

- Time evolution of Einstein's equations
- Fully-spectral Numerical Code: J. Comp. Phys **276** 357 (2014)



Linear Regime (Teukolsky Equation)

Conclusion

- Initial data for perturbed rotating black hole on hyperboloidal slices

- Trapped surfaces showed a rich structure for strong perturbations

  - New black and white hole regions

- Fast rotating perturbed black holes $j_B \approx 0.95$

  free parameters $(j = 0.5)$

- Dain-Bondi number as a relevant quantity in the standard picture of gravitational collapse

$$\epsilon_{AB} = \frac{\mathcal{A}_{\mathcal{H}}}{8\pi(M_{\mathrm{B}}^2 + \sqrt{M_{\mathrm{B}}^4 - J^2})}$$

Mass and angular multipoles

- Does the solution describes a perturbed black hole or is it just Kerr in a different slice (with new parameters)?

Mass and angular multipoles

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- Ashtekar's mass and angular multipoles:

$$M_n = M_{\mathcal{H}} \frac{R_{\mathcal{H}}^n}{8\pi} \oint_{\mathcal{H}} {}^{(2)}R(\mu) P_n(\hat{\mu}(\mu)) dA$$

$$J_n = \frac{R_{\mathcal{H}}^{n-1}}{8\pi} \oint_{\mathcal{H}} K_{\Phi s}(\mu) P_n'\left(\hat{\mu}(\mu)\right) dA$$

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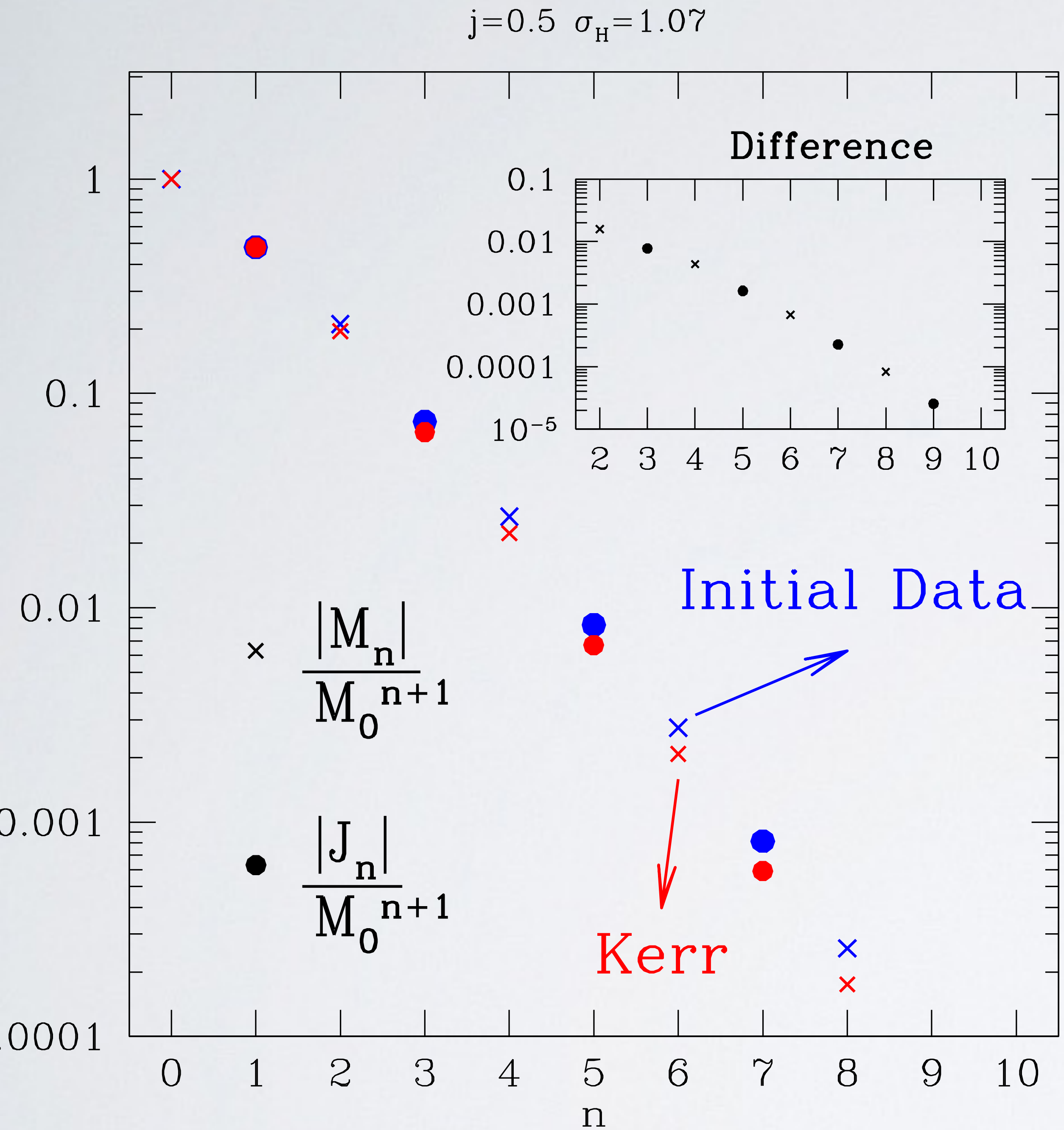
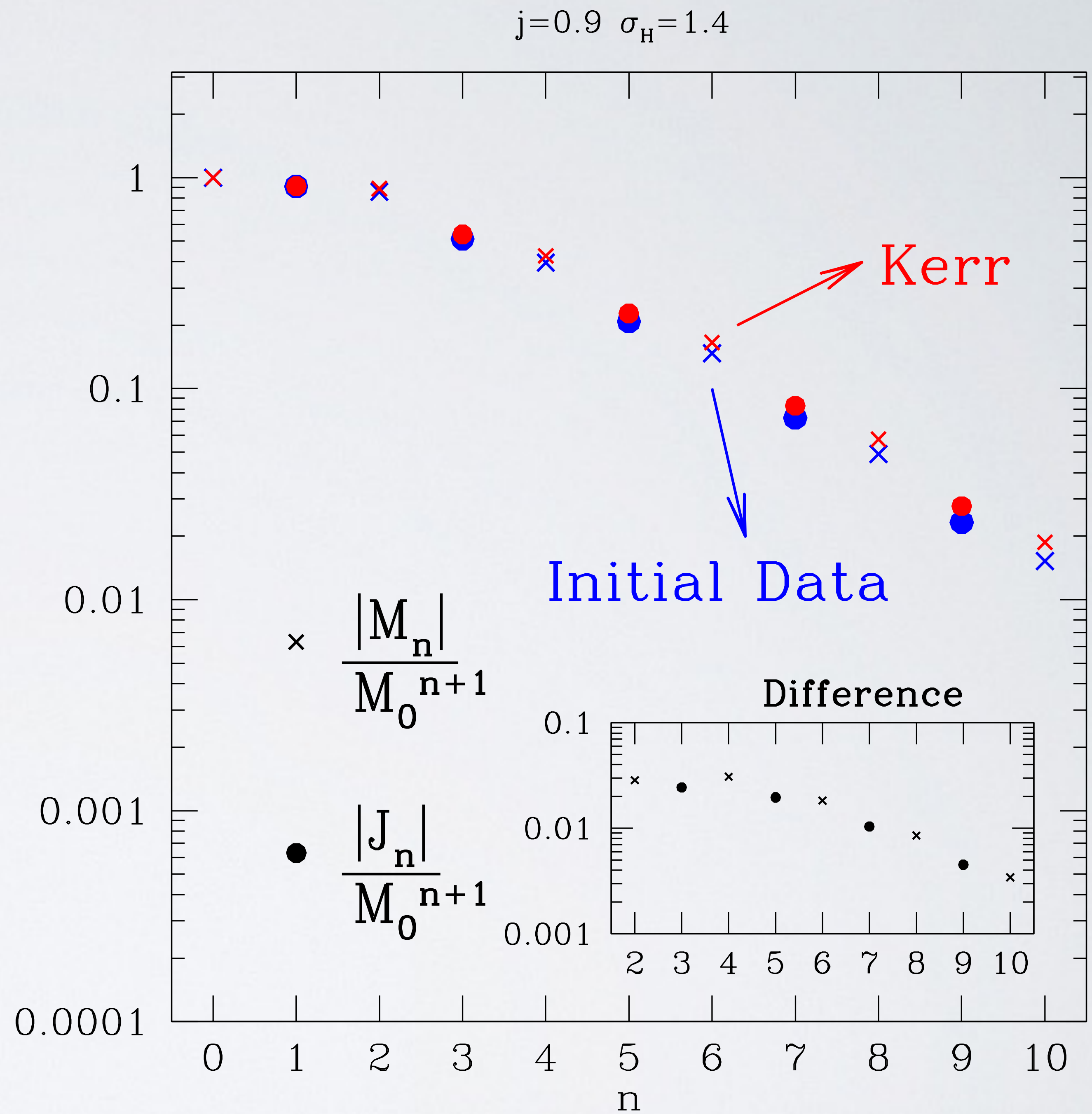
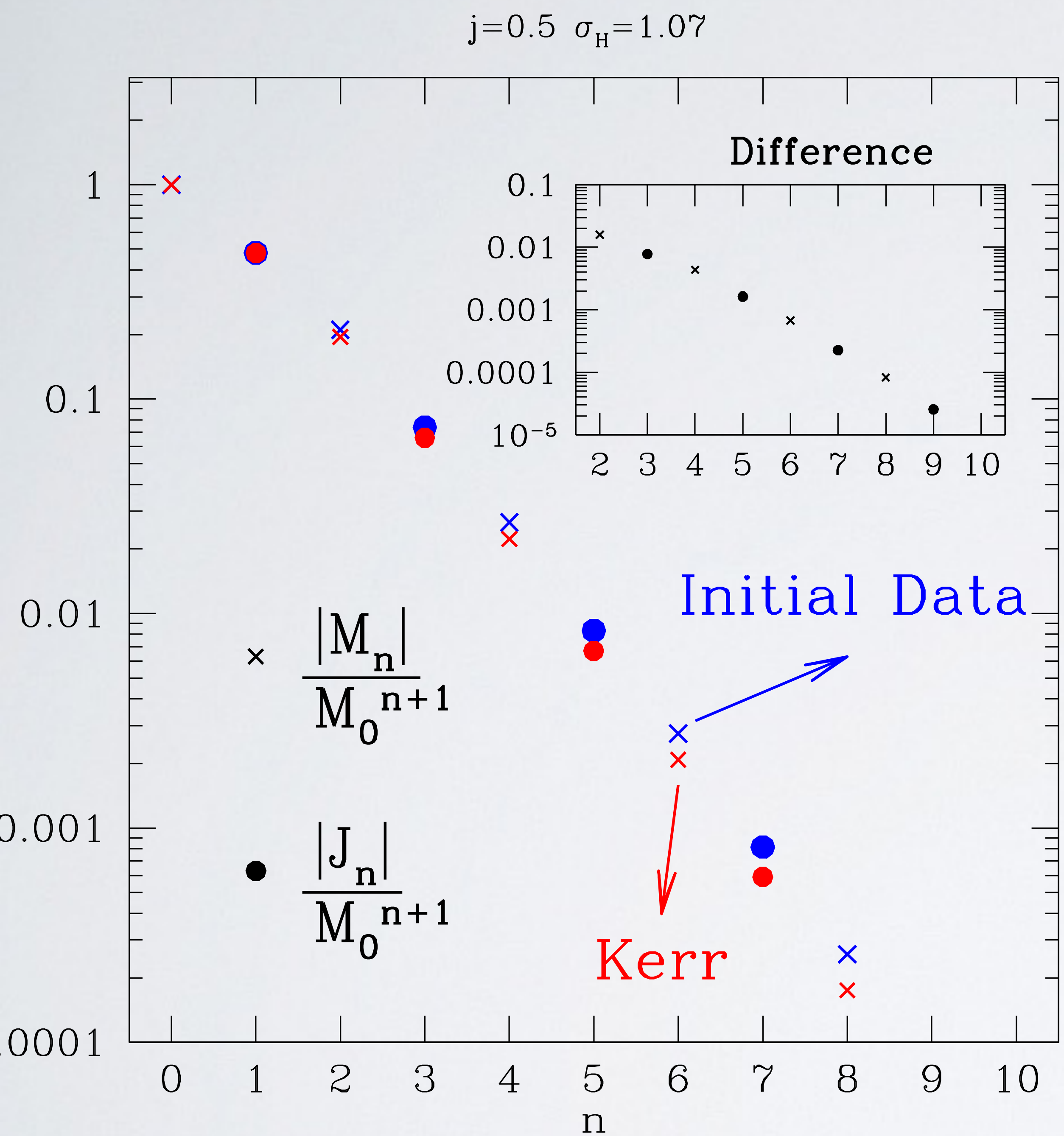
- Compute M_0 and J_1 from the solution
- Assume Kerr space-time with $M = M_0$ and $a = \dfrac{J_1}{M_0}$
- Compare higher multipoles

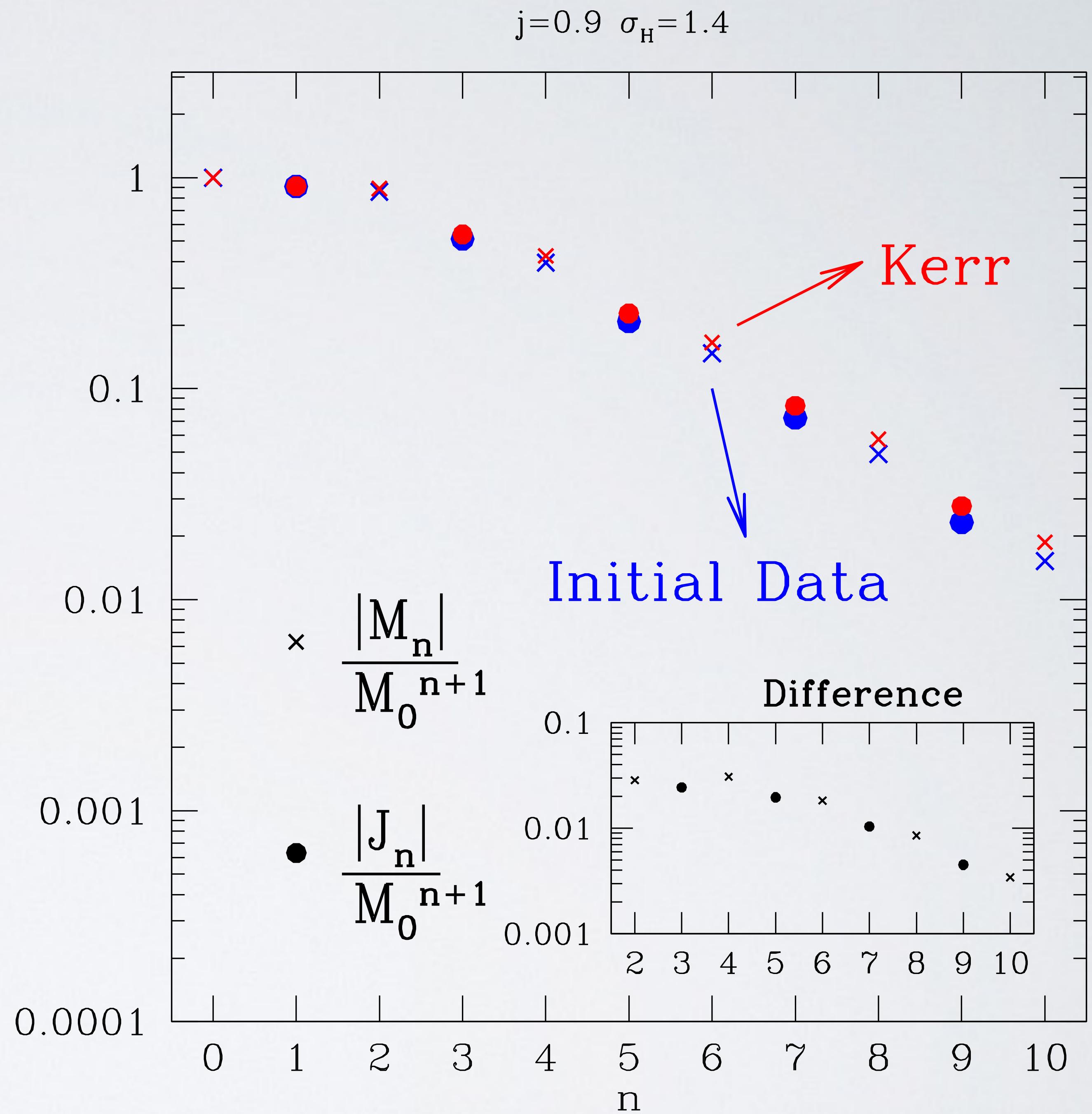
Mass and angular multipoles





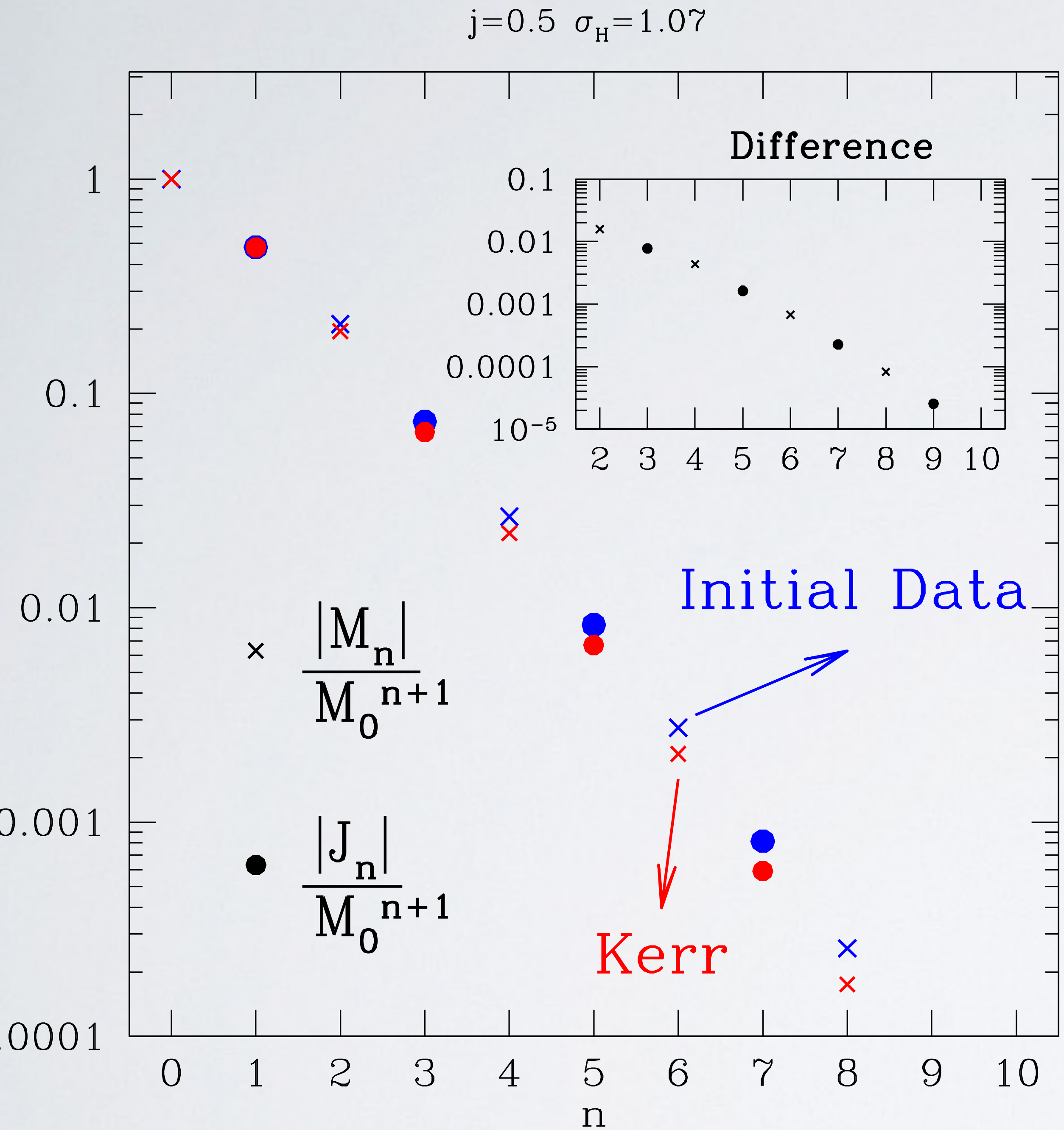
Mass and angular multipoles

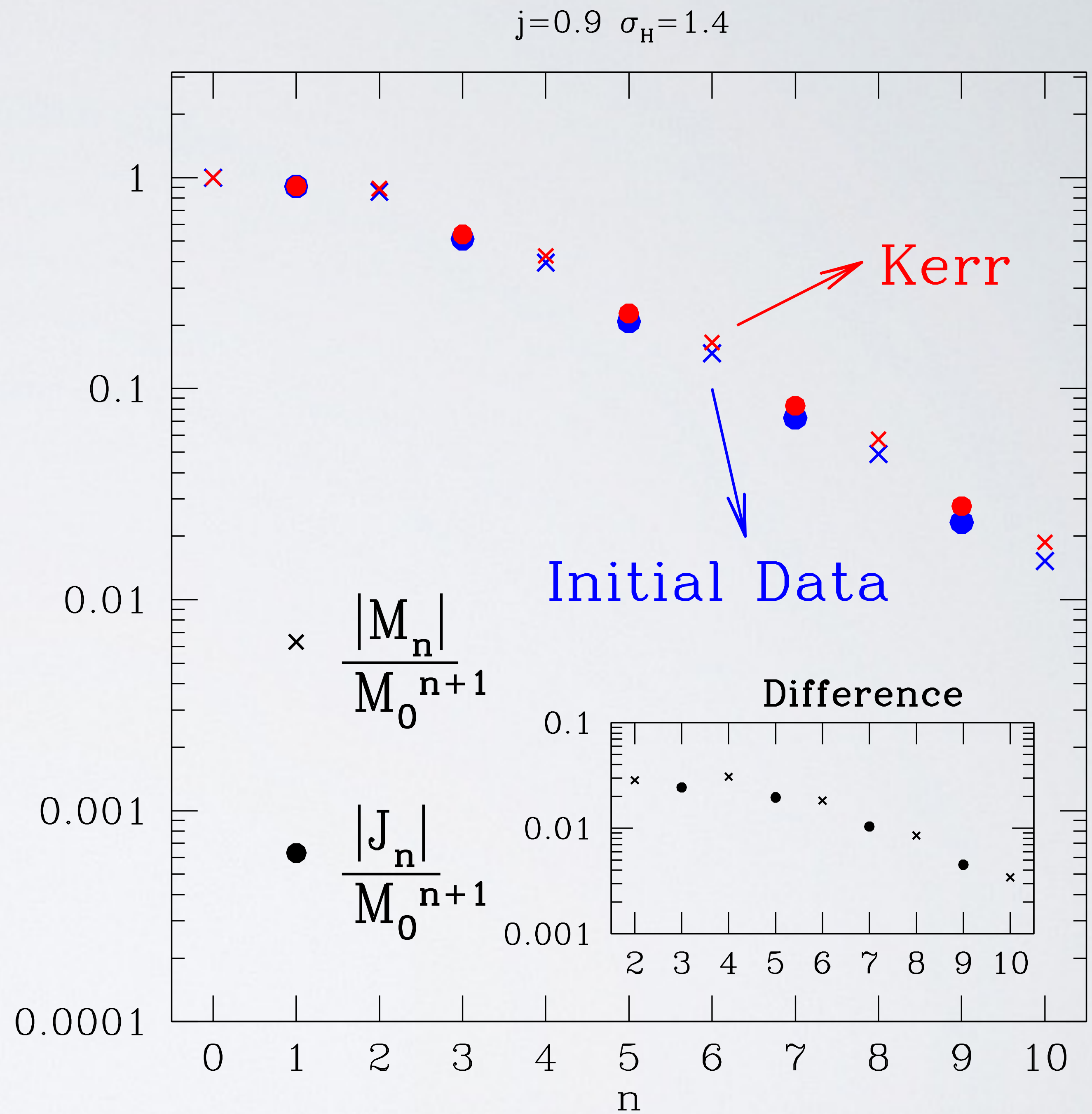




- Small perturbation regime (no new trapped surfaces)

Mass and angular multipoles





- Small perturbation regime (no new trapped surfaces)
- Solution is a perturbed rotating black hole

Black hole inequalities

- Rigidity:

  -For $\Delta\sigma_{\mathcal{H}} >> \Delta\sigma_{\mathcal{H}}^{II}$

$$M_B \to M$$
$$j_B \to j$$

"Strong perturbation" with the same parameters

$$\epsilon_{AB} \to 1$$

- Similar behaviour also in J.L. Jaramillo, N. Vasset and M. Ansorg ('08) : in terms of ADM mass

  - Is it still a perturbed black hole or is it Kerr?
  - Does $\epsilon_{AB} = 1 \Rightarrow$ Kerr ?

- Possible way to answer this question is to compute the multipoles for "strong perturbations" (future work)