Black Hole Perturbation in Einstein-Maxwell-Dilaton Gravity

Martín G. W. Richarte
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Stability Analysis

- What does it mean the stability analysis of a black hole solution? If the precise initial data related to the BH solution are perturbed a bit, the basic features of the corresponding solutions do not change much.
- However, the black hole stability conjecture is a challenging problem in the analysis of nonlinear partial differential equations and is one of the major open problems in mathematical relativity.
- There are various levels of increasing difficulty in proving the stability of generic perturbations, $\mathscr{P}[\phi] = 0$ on $\mathscr{M}$ (OP). One may consider the linearized version of the OP, $d\mathscr{P}[\psi] = 0$ on $\mathscr{M}$, with $\psi(t, r, \theta, \phi) = e^{-i\omega t} S(r, \theta, \phi)$. For the LOP, one may show that: 1- No modes exponentially grow in time **(mode stability)** and 2-All solutions of the LOP decay in time **(full linear stability)**. Solutions to the full nonlinear equation (OP) decay in time **(full nonlinear stability)**.

Time-line-I

The theory of metric perturbations of static black hole spacetimes is an old and well-studied subject:

- In 1957, Regge and Wheeler analyzed the linear, (axial) metric perturbations, of the Schwarzschild metric. In a suitable gauge, metric perturbations decouple into even-parity and odd-parity perturbations. They presented the master Regge-Wheeler equation for an invariant scalar ϕ component of the metric, i.e. $\Box_m \phi = V\phi$ with $V = (4/r^2)(1-2m/r)$.
- In 1971, Price showed that for large t and fixed r, the scalar waves decay as $t^{-(2\ell+2)}$; an effect known as "Price tails".
- In 1974, Moncrief presented a gauge-invariant formalism for studying BH perturbation. Later, in 1979, he examined the mode stability for the RN metric.

Time-line-II

- In 1973, Teukolsky employed a curvature perturbation approach based on the Newman-Penrose (NP) formalism to study the stability of the Kerr family. He noticed that the extreme curvature components, relative to a principal null frame, are gauge invariant and satisfy decoupled, separable, wave equations. The equations, bearing the name of Teukolsky, are roughly of the form $\square_{a,m}\psi = \mathscr{L}[\psi]$, where $\mathscr{L}[\psi]$ is a first order linear operator in ψ.

- In 1973, Press and Teukolsky studied numerically the mode stability of gravitational perturbation ($s = 2$) the Kerr black hole for $0 \leq a < M$.

- In 1975, Chandrasekhar initiated a transformation theory relating the two approaches. He exhibited a transformation which connects the Teukolsky equations to a Regge-Wheeler type equation. Though originally it was meant only to unify the Regge-Wheeler approach with that of Teukolsky, the Chandrasekhar transformation, and various extensions of it, turn out to play an important role in the field.

Time-line-III

1. In 1979, Zerilli obtained the dynamical master equation associated with linear polar perturbations.
2. Wald (1978) and Kay (1986) showed the boundness of the perturbations of the Schwarzschild background. If the initial data $(\phi_{(\ell,m)}(0,r), \partial_t\phi_{(\ell,$ has compact support then the perturbations are bounded: $|\phi_{(\ell,m)}(t,r)| < K_{(\ell,m)}$ for all t and $r > 2M$ [**Wald-Kay's Thm**].
3. In 1979-80, Gerlach and Sengupta formulated a covariant and gauge-invariant formalism to describe the metric and matter perturbations of a generic spherically symmetric spacetime.
4. In 1989, Whiting explored the mode stability of linearized gravitational perturbations of Kerr black holes and showed the lack of exponentially growing modes, for the Teukolsky equation on Kerr.

Time-line-IV

PERTURBATIONS OF A ROTATING BLACK HOLE. I. FUNDAMENTAL EQUATIONS FOR GRAVITATIONAL, ELECTROMAGNETIC, AND NEUTRINO-FIELD PERTURBATIONS*

SAUL A. TEUKOLSKY†
California Institute of Technology, Pasadena
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An analytic representation for the quasi-normal modes of Kerr black holes

By E. W. LEAVER
Department of Physics, University of Utah, Salt Lake City, Utah 84112, U.S.A.

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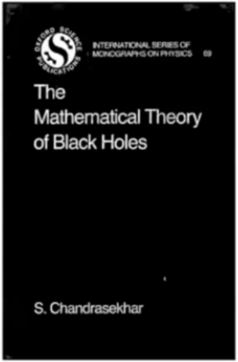

Mode stability of the Kerr black hole

Bernard F. Whiting
Institute of Field Physics, Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27599-3255

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There are no exponentially growing modes for separated solutions $\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta)$ of the Einstein equation

- in Schwarzschild [Regge-Wheeler (1957), Zerilli (1970)]
- in Reissner-Nordström [Moncrief (1979)]
- in Kerr [Teukolsky (1972), Whiting (1989)]

Time-line-V

1. In 2007, Dafermos and Rodnianski (without using spherical harmonics) showed the general stability of scalar waves [**Dafernos-Rodniaski 'sThm**].
2. In 2010, Aretakis showed that for solution to $\Box_g \Psi = 0$ on extremal RN one has analogous boundedness results as in the non-extremal case, and similar decay results away from the horizon [**Aretakis'Thm**].
3. In 2013, Dotti studied the non-modal linear stability of the Schwarzschild black hole. He proved that the perturbation could be described by two scalar fields: gauge invariant combinations of perturbed algebraic and differential invariants of the Weyl tensor. These fields measure the geometry distortion caused by a generic perturbation and are shown to be pointwise bounded on the outer region $r \geq 2M$ [**Dotti's Thm**].

Time-line-VI

1. In 2018, P. Hintz showed the non-linear stability of Schwarzschild de-Sitter BH.
2. In 2021, Dafermos *et al.* studied the nonlinear stability of the Schwarzschild family of black holes and the subextremal Kerr family [600 pages].
3. In 2022, Giorgi, Klainerman, and Szeftel reported wave equation estimates and the nonlinear stability of slowly rotating Kerr black holes [900 pages].

	LINEAR STABILITY	NON-LINEAR STABILITY
Schwarzschild	Dafermos-Holzegel-Rodnianski (2016), Hung-Keller-Wang (2017), Johnson (2018)	Klainerman-Szeftel (2017) for axially symmetric polarized perturbations, Dafermos-Holzegel-Rodnianski-Taylor (2021)
Kerr	Dafermos-Holzegel-Rodnianski (2017), Ma (2017), Andersson-Bäckdahl-Blue-Ma (2019), Häfner-Hintz-Vasy (2019)	Klainerman-Szeftel (2021), **G.**-Klainerman-Szeftel (upcoming) for small angular momentum
Reissner-Nordström	**G.** (2019, 2020) in the full subextremal range	?
Kerr-Newman	**G.** (2020, 2021, upcoming) for small angular momentum	?

Theory+ Background

EMD theory-I

- Consider an EFT with a Hilbert-Einsten term $\mathscr{R}$, a dilaton field ϕ plus a coupling between the $\mathscr{U}(1)$ gauge field A_μ and the dilaton field [Cremmer+,' 78&Ferrara+, '95]:

$$\mathscr{S} = \int_{\mathscr{M}} d^4x\sqrt{-g}\left(\kappa\mathscr{R} - \frac{\alpha_\phi}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{\alpha_F}{4}e^{-\phi}F_{\mu\nu}F^{\mu\nu}\right).$$

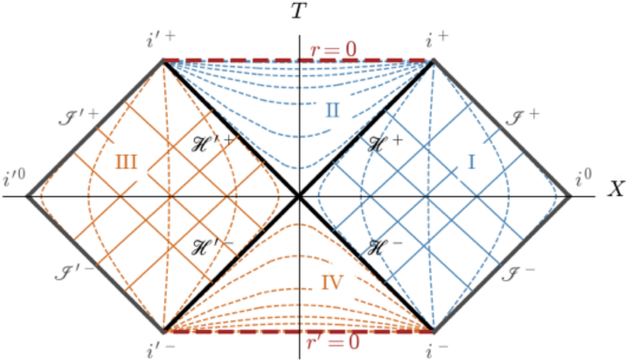
- In a local chart $\mathscr{W}^\mu = (t, r, \theta, \varphi)$, the theory admits a black hole solution with one horizon, $\mathscr{M} = \mathbb{R}_+ \times \mathbb{R}_{\geq r_h} \times \mathbb{S}^2$, [cf. Gibbons&Maeda, '88/Clement et al., 2002]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + h(r)d^2\Omega,\ F = \frac{1}{\sqrt{2}r_0}dr \wedge dt,$$
$$e^{2\phi} = \frac{r}{r_0}, f(r) = \frac{r-b}{r_0}, h(r) = rr_0, \tag{1}$$

where $b = 4M \geq 0$ and $r_0 = Q\sqrt{2} \geq 0$.

EMD theory-II

- In the Penrose-Carter diagram, the black hole region is $\mathcal{B}^+ = \mathscr{M} \backslash J^-(\mathscr{J}^+)$ (region I) and the (future) horizon, $\mathcal{H}^+ = \partial\mathcal{B}^+ = \partial I^-(\mathscr{J}^+)$, is located at $r_h = b$.



EMD theory-III

- Future null infinity $-\mathscr{I}^+$–is defined to be the endpoints of all future-directed null geodesics along which $r \to +\infty$. It has the topology of $\mathbb{R}\times\mathbb{S}^2$ with the function U talking values in $\mathbb{R}$. Thus a null hypersurface S_U intersects $\mathscr{I}^+$ at infinity in a two sphere, $S_{\infty,U}$.

- However, the black hole in the EMD theory has a different topology provided the radius vanishes at $\mathscr{I}^+$, namely $\mathbb{R}\times\{p\}$.

- Another difference is that the conformal factor has $\nabla C|_{\mathscr{I}^{\pm}_{EMD}} = 0$ whereas $\nabla C|_{\mathscr{I}^{\pm}_{Schwarzschild}} \neq 0$.

EMD theory-IV

- Take two real and two complex null vectors $-X_\alpha X^\alpha = 0$ with $X^\alpha = \{l, n, m, \bar{m}\}^\alpha$– such that the following cross-normalization is adopted,

$$\begin{aligned} l^\alpha n_\alpha &= -1, \quad m^\alpha \bar{m}_\alpha = 1 \\ l^\alpha m_\alpha &= 0, \quad n^\alpha \bar{m}_\alpha = 0, \end{aligned}$$

- The Weyl scalars are defined as several contractions among the Weyl tensor and the null basis,

$$\begin{aligned} \Psi_0 &= -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \\ \Psi_1 &= -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta, \\ \Psi_2 &= -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta, \\ \Psi_3 &= -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta, \\ \Psi_4 &= -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \end{aligned}$$

EMD theory-V

- Only the gravitational monopole of the source, $\Psi_2 = -\frac{2r+b}{12r^2 r_0^2}$, is different from zero!
- To determine which kind of Petrov-Pirani-Penrose spacetime the black hole solution represents, one computes the following algebraic quantities:

$$I = \Psi_0\Psi_4 - \Psi_1\Psi_3 + 3\Psi_2^2,\ K = \Psi_4^2\Psi_1 - 3\Psi_4\Psi_3 + 2\Psi_3^3$$

$$L = \Psi_4\Psi_2 - \Psi_3^2,\ D = I^3 - 27J^2,\ N = 12L^2 - \Psi_4^2 I$$

$$J = \det\begin{pmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{pmatrix}$$

- It turns out to be a Petrov D spacetime provided $D = K = N = 0$, $I \neq 0$, and $J \neq 0$.

EMD theory-VI

- This means that the Weyl tensor can be expressed as

$$C_{\alpha\beta\gamma\delta} = \Psi_2\Big(V_{\alpha\beta}U_{\gamma\delta} + U_{\alpha\beta}V_{\gamma\delta} + W_{\alpha\beta}W_{\gamma\delta}\Big),$$

where the bivectors are constructed as an antisymmetric combination of the null vectors,

$$\begin{aligned} U_{\alpha\beta} &= -2l_{[\alpha}\bar{m}_{\beta]}, \\ V_{\alpha\beta} &= 2n_{[\alpha}\bar{m}_{\beta]}, \\ W_{\alpha\beta} &= 2m_{[\alpha}\bar{m}_{\beta]} - 2n_{[\alpha}l_{\beta]}. \end{aligned}$$

Warped Spacetimes+Decomposition

Warped Spacetimes+Decomposition-I

- Consider warped spacetimes-$\mathscr{M} = \mathscr{O} \times \mathscr{K}$- such that the line element can be written as

$$ds^2 = g_{\alpha\beta} dz^\alpha \otimes dz^\beta = g_{ab} dx^a \otimes dx^b + r^2(x)\gamma_{AB} dy^A \otimes dy^B, \quad (2)$$

where each manifold has a covariant derivative, i.e, $(\mathscr{M}, g_{\alpha\beta}, \nabla_\alpha)$, $(\mathscr{O}, g_{ab}, \nabla_a)$, and $(\mathscr{K}, \gamma_{AB}, \mathscr{D}_A)$.

- If $\mathscr{K} = \mathbb{S}^2$ is closed (compact without boundary), any vector field on $\mathscr{K}$ decomposes as

$$V^A = U^A + \mathscr{D}^A S, \quad \mathscr{D}_A U^A = 0.$$

Warped Spacetimes+Decomposition-II

- If the submanifold $\mathscr{K}$ is Einstein-type ($\mathscr{R}_{AB} \propto \gamma_{AB}$), then any symmetric 2-rank tensor on $\mathscr{K}$ contains a scalar field S, a divergence-free vector field V_A, and a TT tensor field Q_{AB}:

$$T_{AB} = Q_{AB} + \mathscr{D}_{(A}V_{B)} + (\mathscr{D}_A\mathscr{D}_B + \frac{1}{n}\gamma_{AB}\Box)S + \frac{1}{n}\gamma_{AB}J,$$

where $n = \mathrm{dim}K$, the TT part is $Q^A_A = 0, \mathscr{D}^B Q_{AB} = 0$, and $J = T^A_A$.

- Terms in the above expression are orthogonal to each other, with the pairing product defined as $(U, V) = \int_K V^{*A} V_A$

Warped Spacetimes+Decomposition-III

- Any symmetric 2^{nd} rank tensor field on $\mathscr{M} = \mathscr{O} \times \mathscr{K}$, such as the metric perturbation itself,

$$h_{\alpha\beta} = \begin{pmatrix} h_{ab} & h_{aB} \\ h_{Ab} & h_{AB} \end{pmatrix},$$

can be decomposed from the $\mathscr{K}$-viewpoint as a set of $\mathscr{K}$-scalar, divergence free vector, and TT-tensor fields as follows:

  - h_{ab} 3-scalar fields in 4D,
  - $h_{aB} = h_{aB} + \mathscr{D}_B h_a$ 2-scalars and 2-vectors,
  - $h_{AB} = h^{TT}_{AB} + \mathscr{D}_{(A} h_{B)} + (\mathscr{D}_A \mathscr{D}_B + \frac{1}{n}\gamma_{AB}\Box)S + \frac{1}{n}\gamma_{AB} J$ (2-scalars, 1-vector, 1-tensor).

- Maxwell field is decomposed as $A_\alpha = (A_a, A_A) = (A_a, U_A + \mathscr{D}_A S)$: 1-scalar and 1-div-free vector.

Warped Spacetimes+Decomposition-IV

- Group theory arguments indicate that scalar, vector, and tensor modes of h_{ab} do not mix, so one gets a 3-separate untangled set of equations: scalar mode equations (even), vector modes (axial) and tensor modes (absent in 4D with spherical horizons). The isometry group of $\mathscr{M}$ is $\mathbb{R}_t \times SO(3) \times P$.
- One may use a **modal decomposition**: Introduce the operator J_1, J_2, J_3 canonical basis of KVFs on sphere: $\mathbf{J}^2 = \sum_{m=1}^{3} (\mathscr{L}_{J_m})^2$.
- One can further decompose the perturbation in harmonics modes over $\mathbb{S}^2$ and according to its parity:
    - $\mathbf{J}^2 h_{\alpha,\beta}^{(\ell,m)} = \ell(1+\ell) h_{\alpha,\beta}^{(\ell,m)}, \quad \mathbf{J}_3 h_{\alpha,\beta}^{(\ell,m)} = imh_{\alpha,\beta}^{(\ell,m)}$ with $|m| \leq \ell$.
    - $\mathbf{P} h_{\alpha,\beta}^{(\ell,m)} = \pm(-1)^\ell h_{\alpha,\beta}^{(\ell,m)}$ (Axial/Polar).

Axial Perturbation

Second order perturbation in the EMD theory

- Perturb all fundamental fields at first order in κ, i.e, $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$, $A_\mu = \bar{A}_\mu + \kappa\delta A_\mu$, and $\phi = \bar{\phi} + \kappa\delta\phi$. Retaining the Lagrangian density up to $\mathscr{O}(\kappa^2)$,

$$\mathscr{L}^{(2)}_{EH} = \sqrt{-\bar{g}}\Big[-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h \nabla^\alpha h - \frac{1}{2}\nabla_\alpha h \nabla_\beta h^{\alpha\beta}$$
$$+ \frac{1}{2}\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu} + \frac{1}{8}\bar{\mathscr{R}}(h^2 - h^\mu_\nu h^\nu_\mu) + \frac{1}{2}(2h^\mu_\lambda h^{\lambda\rho} - h h^{\mu\rho})\bar{\mathscr{R}}_{\mu\rho}\Big],$$
$$\mathscr{L}^{(2)}_{F} = -\frac{\alpha_F}{4}\sqrt{-\bar{g}}\Big[Z(\bar{\phi})\mathscr{F}^{(2)} + \frac{h}{2}Z(\bar{\phi})\mathscr{F}^{(1)} + \frac{\overline{F}}{8}Z(\bar{\phi})(h^2 - 2h^{\mu\nu}h_{\mu\nu})$$
$$Z_{\bar{\phi}}(\bar{\phi})\mathscr{F}^{(1)}\delta\phi + \frac{h}{2}\bar{F}Z_{\bar{\phi}}(\bar{\phi})\delta\phi + Z_{\bar{\phi}\bar{\phi}}(\bar{\phi})\bar{F}\delta\phi^2\Big],$$
$$\mathscr{L}^{(2)}_{K} = -\frac{\alpha_\phi}{2}\sqrt{-\bar{g}}\Big[\mathscr{K}^{(2)} + \frac{h}{2}\mathscr{K}^{(1)} + \frac{\overline{K}}{8}(h^2 - 2h^{\mu\nu}h_{\mu\nu})\Big],$$

where $Z(\phi) = e^{-2\phi}$.

Axial perturbation-I

- Axial perturbations are characterized by $h = 0$. The bases of the covector and tensor for the metric perturbation over $\mathbb{S}^2$ involve $\{E_{AB}\partial^B Y^{(\ell,m)}\}$ and $\{E_{(AC}\mathscr{D}^C\mathscr{D}_{B)}Y^{(\ell,m)}\}$, where $\gamma_{AB} = \mathrm{diag}(1, sin^2\theta)$ and $E_{AB} = \sqrt{\gamma}\epsilon_{AB}$. Initially, the non-zero components of the perturbation are h_{ta}, h_{ra}, and h_{AB}.
- Due to the invariance of the theory under diffeomorphism, one may use gauge transformations ($x^\mu \to x^\mu + \xi^\mu$) to remove unphysical variables through the relations $h_{\mu\nu} \to h_{\mu\nu} + \mathscr{L}_\xi g_{\mu\nu}$ and $A_\mu \to A_\mu + \mathscr{L}_\xi A_\mu + \nabla_\mu \Lambda$. This leads to $h_{AB} = 0$ (**Regge-Wheeler gauge**).
- The gauge-invariant variables to write down the axial perturbation are
  1. $h_{tA} = \sum_{(\ell,m)} \mathbf{h}_0(x) E_{AB}\partial^B Y^{(\ell,m)}$ (metric contribution)
  2. $h_{rA} = \sum_{(\ell,m)} \mathbf{h}_1(x) E_{AB}\partial^B Y^{(\ell,m)}$ (metric contribution)
  3. $\delta A_B = \sum_{(\ell,m)} \mathbf{U}(x) E_{BA}\partial^A Y^{(\ell,m)}$ (electromagnetic part)

Axial perturbation-II

- The action at the second order in the Regge-Wheeler gauge ($h_2 = 0$) can be written as

$$S^{(2)}_{axial} = \sum_{(\ell,m)} \int_{\mathscr{O}} dtdr \mathscr{L}^{(2)}_{(\ell,m)}$$

- The Lagrangian is given by [cf. Gannouji &Rodríguez Baez, 2022]

$$\begin{aligned}\mathscr{L}^{(2)}_{(\ell,m)} &= a_1^{(\ell,m)} \mathbf{h}_0^2 + a_2^{(\ell,m)} \mathbf{h}_1^2 + 2a_3^{(\ell,m)} a_4^{(\ell,m)} \ \mathbf{U}(\mathbf{h'}_0 - \dot{\mathbf{h}}_1) \\ &+ a_3^{(\ell,m)} [\dot{\mathbf{h}}_1^2 + \mathbf{h'}_0^2 + 4\frac{r'}{r}\mathbf{h}_0 \mathbf{h'}_1 - 2\dot{\mathbf{h}}_0 \mathbf{h'}_1] \\ &+ a_5^{(\ell,m)} \ \mathbf{U}\mathbf{h}_0 + a_6^{(\ell,m)} \ \mathbf{U}^2 + a_7^{(\ell,m)} \dot{\mathbf{U}}^2 + a_8^{(\ell,m)} \mathbf{U'}^2,\end{aligned}$$

where the coefficients $a_i^{(\ell,m)}(r, \ell)$ for all $1 \le i \le 8$.

Counting the DOFs

Intermezzo-I

- A natural question is: How many degrees of freedom does the system have?
- Even though there are three functions, $\mathbf{U}$, $\mathbf{h}_0$, and $\mathbf{h}_1$, not all the EMDFEs are algebraically independent.
- To extract the true DOFs , it is convenient to use the Dirac's method for a system with (hidden) constraints.
- Using $\mathscr{L}^{(2)}$ one constructs the total Hamiltonian through a Legrendre transformation:

$$\mathscr{H}_T = \mathscr{H}_c + \int_{\Sigma_t} dr \lambda_0 \Pi_0,$$

where the computation of the canonical momenta reveals a first constraint ($\Pi_0 \simeq 0$) on the space phase of the field configurations defined by the pairs $\mathscr{W} = \{(\mathbf{U}, \Pi_u), (\ \mathbf{h}_0, \Pi_0), (\mathbf{h}_1, \Pi_1)\}$.

Intermezzo-II

- The consistency of the first constraint leads

$$\dot{\Pi}_0 = \left\{\Pi(r), \mathscr{H}_T(r')\right\}_{PB} = 2a_1\ \mathbf{h}_0 + a_5\ \mathbf{U} = \psi_0.$$

- The consistency of the condition $\psi_0 = 0$ on $\mathscr{W}$ leads to a second constraint

$$\dot{\psi}_0 = \left\{\psi(r), \mathscr{H}_T(r')\right\}_{PB} = 2a_1\lambda_0 + a_5\ \dot{\mathbf{U}} = \eta_0.$$

- Taking $\eta_0 = 0$ on $\mathscr{W}$ one obtains the Lagrange multiplier $\lambda_0 = -a_5\ \dot{\mathbf{U}}/2a_1$.
- The algebra of the constraints $\{\Pi(r), \psi(r')\}_{PB} = -2a_1$ points that the system only has two second-class constraints.
- **DOFs** $= \frac{1}{2}[dim\mathscr{W} - 2\mathscr{P} - \mathscr{S}] = 2.$ One proves that the EFT for the axial perturbations only has **2 DOF**.

- Removing the hidden constraints and defining $\mathbf{X}^T = (\mathbf{z}, \mathbf{U})$, one constructs a new Lagrangian that is a quadratic non-degenerated form:

$$\mathcal{L}^{(2)}_{axial} = \mathbf{K}_{ij}\dot{\mathbf{X}}^i\dot{\mathbf{X}}^j - \mathbf{L}_{ij}\mathbf{X'}^i\mathbf{X'}^j - \mathbf{M}_{ij}\mathbf{X}^i\mathbf{X}^j.$$

- The matrices for the EFT based on the perturbed Lagrangian are

$$\mathbf{K}_{ij} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad \mathbf{L}_{ij} = -\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad \mathbf{M}_{ij} = -\begin{pmatrix} \gamma_1 & \frac{\sigma}{2} \\ \frac{\sigma}{2} & \gamma_2 \end{pmatrix}$$

- For the gravitational perturbation, one avoids a **ghost instability** provided $\alpha_1 > 0$ for $\ell \geq 2$ and $r > r_h$.

- The mass term $\mathbf{M}_{ij}$ requires further diagonalization, especially if one is interested in exploring tachyon instabilities!

Mechanical Stability

Mechanical Stability-I

- Define a new set of variables $\mathbf{q} = rM_P\mathbf{z}$ and $\mathbf{A}_v = \mathbf{U}\sqrt{\alpha_F(\lambda-2)}$. Introducing a new vector $\Psi_j = (\mathbf{q}, \mathbf{A}_v)$, the Lagrangian takes the following form,

$$\mathscr{L}^{(2)} = \frac{\lambda^2}{4(\lambda-2)} \sum_{i=q,A_V} [\dot{\Psi}_i^2 - (\partial_x \Psi_i)^2 - \mathbf{V}_{ij}\Psi_i\Psi_j],$$

  where $\lambda = \ell(\ell+1)$, $\ell \geq 2$, and the tortoise coordinate is defined through the relation $\partial_x = f(r)\partial_r$. This is a set of coupled ODEs that looks like wave equations with a source term encoded in the non-diagonal matrix $\mathbf{V}_{ij}$. The E-L equation are $-\partial_t^2\Psi_i + \partial_x^2\Psi_i - \mathbf{V}_{ij}\Psi_j = 0$.

- In another base, $\Psi_j^{\pm} = \mathbf{P}\Psi_j$, the source term becomes diagonal:

$$\mathbf{V}_{ij}^{\pm} = \begin{pmatrix} V_+ & \\ 0 & V_- \end{pmatrix}, \quad V_{\pm} = \frac{f}{rr_0}[(\ell+\frac{1}{2})^2 - \frac{3b}{4r} \mp \sqrt{2\ell(\ell+1)}] > 0.$$

Mechanical stability-II

- Stability means that no perturbation grows unbounded in time. For that, one Fourier transforms as $\Psi_j = e^{-i\omega t}\Phi_j$ such that one gets a Schrödinger-like equation $\mathcal{H}_{ij}\Phi_j = \omega^2\Phi_i$ where $\mathcal{H}_{ij} = -\delta_{ij}\partial_x^2 + \mathbf{V}_{ij}$.
- The frequency $E = \omega^2$ appears as the eigenvalues of the operator $\mathcal{H}_{ij}$.
- Unstable modes are equivalent to purely imaginary ω, $E = \omega^2 < 0$.
- The stability of the spacetime is related to the positivity of the spectrum $\sigma(\mathcal{H}_{ij}) > 0$.
- **Theorem**: If the operator $\mathcal{H}_{ij}\Phi_j = \omega^2\Phi_i$ is a **self-adjoint** and **positive definite** on a Hilbert space $L^2(\mathbb{R}, dx)$ then $\sigma(\mathcal{H}_{ij}) \subseteq \mathbb{R}_+$ **[cf. Reed and Simon]**.

Mechanical stability-III

- To show that the above theorem holds, several steps are needed. Define the inner product over $L^2(\mathbb{R}, dx)$:

$$(\Phi, \Phi) = \int dx \sum_i \Phi_i^* \Phi^i.$$

- The positive condition is $(\Phi, \mathcal{H}\Phi) \geq 0 \; \forall \; \Phi \in \; \mathcal{D}(\; \mathcal{H})$. This (sufficient) condition implies that given a well-behaved initial data, of compact support, Φ remains bounded for all time. One obtains

$$(\Phi, \mathcal{H}\Phi) = \int dx \Big(|\Phi_1' + S_1\Phi_1|^2 + |\Phi_2' + S_2\Phi_2|^2 + |\Phi_2' + S_2\Phi_2|^2 \Big)$$
$$+ \int dx \Big(|\frac{(\lambda - 2)}{r^2} f\Phi_1 + \sqrt{G}\Phi_2|^2 + \frac{\lambda f}{r^2} |\Phi_2|^2 \Big) > 0$$

Mechanical stability-IV

- One employs Weyl' s theorem to prove that the operator is essentially self-adjoint (Thm X.6 RS). Arbitrary solutions of the ODE $\mathcal{H}\phi = E\phi$ on $\mathbb{R}$ has two L.I solutions for any complex E.

  1. If for some $E \in \mathbb{C}$ every solution in this two dim. space is square integrable (locally) near $x = -\infty$, i.e., for some constant $d < 0$

     $$\int_d^{-\infty} dx |\phi|^2 = \lim_{\delta \to -\infty} \int_d^{\delta} dx |\phi|^2 < \infty,$$

     then this also will be the case for any other $E \in \mathbb{C}$. One says the Hamiltonian is limit circle (LC) at $x = -\infty$, otherwise is limit point (LP).

  2. The condition of being LC/LP at $x = +\infty$ is defined in the similar way.

  3. If the endpoints are either regular or regular singular points of the ODE, a Frobenius series solution around these points is enough to determine the LC/LP character at the endpoints.

Mechanical stability-V

- Weyl's theorem states that the associated Hamiltonian is essentially self-adjoint if the endpoints are classified as LPs. This happens to be case for the Hamiltonian associated with the even perturbations.
- If the operator $\mathcal{H}_{ij}$ is positive definite and self-adjoint, recalling that $E = \omega^2$, one concludes that its spectrum belongs $\mathbb{R}_+$.
- Black holes in the EMD theory are stable under axial perturbation!
- Is this procedure enough to claim that the perturbations are bounded?

Mechanical stability-VI

- Let's double-check the previous statement. One way is by looking at the perturbations of geometrical invariants near the endpoints and confirming that the perturbations are bounded and subdominant. Define the averaging over the 2-sphere as follows,

$$\langle\overline{\mathscr{R}^{\mu\nu\sigma\rho}}\mathscr{R}_{\mu\nu\sigma\rho}\rangle_{\mathbb{S}^2} := \int_{\mathbb{S}^2} d^2\Omega\overline{\mathscr{R}^{\mu\nu\sigma\rho}}\mathscr{R}_{\mu\nu\sigma\rho}$$
$$= \int_{\mathbb{S}^2} d^2\Omega\overline{\mathscr{R}^{\mu\nu\sigma\rho}_{(0)}}\mathscr{R}_{\mu\nu\sigma\rho\ (0)} + \int_{\mathbb{S}^2} d^2\Omega\overline{\mathscr{R}^{\mu\nu\sigma\rho}_{(1)}}\mathscr{R}_{\mu\nu\sigma\rho\ (1)}$$

- At spatial infinity i^0, one arrives at

$$\langle\overline{\mathscr{R}^{\mu\nu\sigma\rho}}\mathscr{R}_{\mu\nu\sigma\rho}\rangle_{\mathbb{S}^2} = \frac{488\pi}{16(rr_0)^4} + \frac{2\kappa(2\ell+1)\omega^2}{(rr_0)^4[\ell(\ell+1)-2]^2}.$$

- $\langle\overline{\mathscr{R}^{\mu\nu\sigma\rho\ (1)}}\mathscr{R}_{\mu\nu\sigma\rho\ (1)}\rangle_{\mathbb{S}^2} \ll \langle\overline{\mathscr{R}^{\mu\nu\sigma\rho\ (0)}}\mathscr{R}_{\mu\nu\sigma\rho\ (0)}\rangle_{\mathbb{S}^2}$ provided $\kappa \ll 1$.

The Zoo of Instabilities

The Zoo of Instabilities-I

- This result seems to be physically consistent with previous analysis for the axial perturbations of the Schwarzschild and Reissner-Nordström black holes.
- But the question is: What happens with other kinds of instabilities?
- Physically speaking, one must investigate at least four kinds of instabilities in the presence of the scalar field.
- **Top 4 instabilities**
  - Ghost
  - Gradient/Laplacian
  - Tachyon
  - Ostrogradsky

The Zoo of Instabilities-II

- To explore the existence of instabilities, it is much easier to work with the quadratic non-degenerated Lagrangian:

$$\mathscr{L}^{(2)} = \mathbf{K}_{ij}\dot{\mathbf{X}}^i\dot{\mathbf{X}}^j - \mathbf{L}_{ij}\mathbf{X'}^i\mathbf{X'}^j - \mathbf{M}_{ij}\mathbf{X}^i\mathbf{X}^j,$$

where the vector $\mathbf{X}^T = (\mathbf{z}, \mathbf{U})$.

- The matrix $\mathbf{K}_{ij}$ is positive definite according to the Sylverster' s criterion $(K_{11} > 0 \cap \det(K_{ij}) > 0)$, which implies the **non-existence of ghost instabilities**. Neither ghost instability arises in the gravitational sector nor ghosts electromagnetic counterpart either.

The Zoo of Instabilities-III

- In the k-small limit, by solving the $det(\omega^2 K_{ij} - k^2 L_{ij}) = 0$ for the radial linear dispersion $\omega^2 = f^2 c_r^2 k^2$, one finds $c_r^2 = 1$.

- In the ℓ-small limit, by solving the $det(\omega^2 K_{ij} - \ell^2 L_{ij}) = 0$ for the angular linear dispersion relation $\omega^2 = (f/r^2) c_\Omega^2 \ell^2$, one finds that $c_\Omega^2 < 0$ for $r \in r_h(1, 1+\delta)$. Near the horizon there is an (**angular gradient instability**).

- The Ostrogradsky instability is absent, provided the reduced Lagrangian for the perturbation is a quadratic non-degenerated form with derivatives no higher than two. (**no Ostrogradsky instability**).

The Zoo of Instabilities-IV

- A tachyonic instability appears when the field has an effective negative mass squared.
- One must diagonalize the mass matrix and inspect the sign of its eigenvalues to explore the existence of a tachyon-like instability.
- One obtains that $M_{ij} = \mathrm{diag}(\kappa_+, \kappa_-)$ and the eigenvalues are given by

$$2\kappa_\pm = -(\gamma_1 + \gamma_2) \pm \sqrt{(\gamma_1 - \gamma_2)^2 + \sigma^2}$$

.

- The tachyonic instability could arise provided $\kappa_- < 0$. However, this is only relevant if $|\kappa_-| \leq H_0$. (**Existence of tachyonic instability**)!

Polar Perturbations

Polar perturbation-I

- The polar perturbations in the gravitational, electromagnetic, and scalar channels can be written as [cf. Gannouji &Rodríguez Baez, 2022],

$$
\begin{aligned}
h_{tt} &= \mathbf{h}_0(r,t)Y^{(\ell,m)}, \;\; h_{tr} = \mathbf{h}_1(r,t)Y^{(\ell,m)}, \;\; h_{rr} = \mathbf{h}_2(r,t)Y^{(\ell,m)} \\
h_{tA} &= \beta(r,t)\mathscr{D}_A Y^{(\ell,m)}, \;\; h_{rA} = \alpha_1(r,t)\mathscr{D}_A Y^{(\ell,m)}, \\
h_{AB} &= r^2[\mathbf{K}(r,t)\gamma_{AB}Y^{(\ell,m)} + \mathbf{G}(r,t)Y^{(\ell,m)}_{AB}], \\
\delta\phi &= \mathbf{p}(r,t)Y^{(\ell,m)}, \\
\delta A_t &= \mathbf{U}_0(r,t)Y^{(\ell,m)}, \\
\delta A_r &= \mathbf{U}_1(r,t)Y^{(\ell,m)}, \\
\delta A_B &= \mathbf{U}_2(r,t)\mathscr{D}_B Y^{(\ell,m)}.
\end{aligned}
$$

where $Y^{(\ell,m)}_{AB} := [\mathscr{D}_A\mathscr{D}_B + \frac{1}{2}\ell(\ell+1)\gamma_{AB}]Y^{(\ell,m)}$.

Polar perturbation-II

- From now, one will work with the uniform curvature gauge, namely $\mathbf{U}_2 = \mathbf{K} = \mathbf{G} = \beta = 0$ [cf. Gannouji &Rodríguez Baez, 2022]. The Lagrangian is given by

$$\mathscr{L}^{(2)}_{polar} = a_1\mathbf{h}_0^2 + a_2\mathbf{h}_1^2 + a_3\mathbf{h}_2^2 + a_4\alpha^2 + a_5\mathbf{h}_0\mathbf{h}_2 + a_6\mathbf{h}_0\alpha + a_7\mathbf{h}_0\mathbf{h'}_2 + a_8\mathbf{h}_0\alpha' + a_9\mathbf{h}_1\dot{\mathbf{h}}_2 + a_{10}\mathbf{h}_1\dot{\alpha} + a_{11}\dot{\alpha}^2 + b_1\mathbf{U}_0^2 + b_2\mathbf{U}_1^2 + b_3\mathbf{U'}_0^2 + b_4\dot{\mathbf{U}}_1^2 + b_5\mathbf{U'}_0\dot{\mathbf{U}}_1 + c_1\mathbf{p}^2 + c_2\dot{\mathbf{p}}^2 + c_3\mathbf{p'}_0^2 + d_1\dot{\mathbf{U}}_1\mathbf{h}_2 + d_2\mathbf{U'}_1\mathbf{h}_2 + d_3\alpha\mathbf{U}_0 + d_4\mathbf{U'}_0\mathbf{h}_0 + d_5\mathbf{h}_0\dot{\mathbf{U}}_1 + e_1\mathbf{h}_1\dot{\mathbf{p}} + e_2\mathbf{h}_2\mathbf{p'} + e_3\mathbf{p}_1\alpha + e_4\mathbf{h}_0\mathbf{p'} + e_5(\mathbf{U'}_0 - \dot{\mathbf{U}}_1)\mathbf{p} + e_6\mathbf{h}_0\mathbf{p} + e_7\mathbf{h}_2\mathbf{p} + e_8\mathbf{p'}\mathbf{p}.$$

- Only some of the variables are independent, so one needs to apply Dirac's method to discover the true DOFs of the polar Lagrangian.

Polar perturbation-III

- After removing 8-second class constraints from the theory, one can prove that the numbers of **DOFs**=3. The Lagrangian at second order for the polar sector can be recast as in terms of **h**, **A**, and **s**:

$$\begin{aligned}\mathscr{L}^{(2)}_{polar} =&\alpha_1\dot{\mathbf{h}}^2+\beta_1\mathbf{h'}^2+\gamma_1\mathbf{h}^2+\alpha_2\dot{\mathbf{A}}^2+\beta_2\mathbf{A'}^2+\gamma_2\mathbf{A}^2+\\ &\alpha_3\dot{\mathbf{s}}^2+\beta_3\mathbf{s'}^2+\gamma_3\mathbf{A}^2+\sigma_1\mathbf{hA}+\sigma_2\mathbf{h'A}+\sigma_2\mathbf{hA'}+\\ &\sigma_3\mathbf{h'A'}+\sigma_4\dot{\mathbf{h}}\dot{\mathbf{A}}+\eta_1\mathbf{hs}+\eta_2\mathbf{h's}-\eta_2\mathbf{hs'}+\eta_3\mathbf{h's'}+\\ &\eta_4\dot{\mathbf{h}}\dot{\mathbf{s}}+\nu_1\mathbf{As}+\nu_2\mathbf{sA'}-\nu_2\mathbf{s'A}+\nu_3\mathbf{s'A'}+\nu_4\dot{\mathbf{A}}\dot{\mathbf{s}}.\end{aligned}$$

  where the coefficients depend on (r,ℓ) whereas **A**, **h**, and **s** are fields on $\mathbb{R}_{r\geq r_h}\times\mathbb{R}_+$.

- In a compact form, one can write down,

$$\mathscr{L}^{(2)}_{polar}=\mathbf{K}_{ij}\dot{\mathbf{X}}^i\dot{\mathbf{X}}^j-\mathbf{L}_{ij}\mathbf{X'}^i\mathbf{X'}^j+\mathbf{D}_{ij}\mathbf{X'}^i\mathbf{X}^j-\mathbf{M}_{ij}\mathbf{X}^i\mathbf{X}^j,$$

  where $\mathbf{K}_{(ij)},\mathbf{L}_{(ij)},\mathbf{M}_{(ij)}\in\mathbb{R}^{3\times3}$ but $\mathbf{D}_{[ij]}\in\mathbb{R}^{3\times3}$.

Polar perturbation-IV

- There is no ghost instability provided that $\mathbf{K}_{11} > 0$, $\mathbf{K}_{11}\mathbf{K}_{22} - \mathbf{K}_{12}^2 > 0$, and $det(\mathbf{K}_{ij}) > 0$.
- The matrix $\mathbf{L}_{(ij)}$ is positive definite, and consequently, no Laplacian instability can rise in any limit (large k or large ℓ).
- The three eigenvalues of the mass matrix, $\mathbf{M}_{(ij)}$, are all positives. The tachyonic instability is absent. Besides, Ostrogradsky instability is not present in the polar sector.
- What about the mechanical stability?

Polar perturbation-V

- Considering the potential matrix $V_{ij} = \mathrm{diag}(V_1, V_2, V_3)$ with $V_i > 0$ for $r \in (r_h, \infty)$ and making a local analysis near the boundaries, one reveals that the endpoints are classified as LPs. Therefore, Weyl's theorem implies that the associated Hamiltonian is essentially self-adjoint. Besides, the operator is positive definite provided $(\phi, H\phi) > 0 \ \forall \ \phi \in Dom(H)$. By ThmX.6 , the system is stable under polar perturbations.

- Regarding the curvature invariant at spatial infinity, one finds that

$$\langle \overline{\mathscr{R}^{\mu\nu\sigma\rho}} \mathscr{R}_{\mu\nu\sigma\rho} \rangle_{\mathbb{S}^2} = \frac{488\pi}{16(rr_0)^4} + \frac{4\kappa(\ell(\ell+1)+1)^2\omega^4}{(rr_0)^8}.$$

- $\langle \overline{\mathscr{R}^{\mu\nu\sigma\rho\ (1)}} \mathscr{R}_{\mu\nu\sigma\rho\ (1)} \rangle_{\mathbb{S}^2} = \mathscr{O}(rr_0)^{-8} \langle \overline{\mathscr{R}^{\mu\nu\sigma\rho\ (0)}} \mathscr{R}_{\mu\nu\sigma\rho\ (0)} \rangle_{\mathbb{S}^2}$. This result ensures that polar perturbations remain small without demanding $\kappa \ll 1$.

Future Plans

- Confirm the analytical findings with the numerical computation of the QNMs to corroborate the linear (mode) stability for axial and polar perturbations (work in progress).

- Consider another EFT with operators of dimensions $\mathcal{O}_6$ and $\mathcal{O}_8$ and perform a stability analysis in that case.

- Study scalar-tensor theories and their stability using a different approach called the covariant and gauge-invariant formalism.

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