Hyperbolicity in Physics

Oscar Reula

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Hyperbolicity: causal propagation in PDEs

- Systems of Partial Differential Equations such that:
  - There exists a hypersurface in "space" such that:
    - Giving “arbitrary datum” on it:
      - There exists a unique solution in some "region".
      - These solutions depend continuously on the data (depends on the assumed topology [Sobolev spaces vs. Gevrey spaces]).

Socio/psycho-logy page

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- Why is this important?
- Today we have computers and can find general solutions.
  - Theory fully developed when simulations started.
  - Entered very late in GR
- Most textbooks in physics are older than the advances in PDE's.
  - Are based in a handful of exact solutions (very important also, but stay in the realm of Cauchy-Kowalevski, convergence of series??)
  - Landau / Eckart Theory of relativistic dissipative fluids.

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  - ~~Landau / Eckart Theory of relativistic dissipative fluids.~~

Outline

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- Lecture I:
  - Examples: Wave equation, and Electromagnetism, GR.
  - Old stuff < 1950. Cauchy-Kowalevski Theorem.
  - The question of region of influence or finite propagation.
- Lecture II:
  - The question of well posedness and predictability.
  - Symmetric and Strongly hyperbolic systems.
- Lecture III:
  - General systems (constraints)
  - Examples: Force-Free in Euler's Potentials.
  - Open problems.
  - Numerical applications (Jets around Kerr).

Examples:

$$u^a \nabla \phi = F(\phi)$$

$$\partial_t \phi = -\vec{v} \cdot \nabla \phi + F(\phi) \qquad u^a = (1, v^i);$$

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$$\partial_t \rho = -\vec{v} \cdot \nabla \rho - \rho \cancel{\nabla \cdot \vec{v}}$$

Advection equation

Examples:

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\begin{aligned}
\partial_t \vec{E} &= \nabla \wedge \vec{B} + J \\
\partial_t \vec{B} &= -\nabla \wedge \vec{E} \\
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\end{aligned}
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Electromagnetism

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Electromagnetism

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\begin{aligned}
\partial \rho &= -\rho \nabla \cdot \vec{v} - \vec{v} \cdot \nabla \rho \\
\rho \partial \vec{v} &= -\rho (\vec{v} \cdot \nabla) \vec{v} - \vec{v}\, (\vec{v} \cdot \nabla \rho) - \nabla p(\rho)
\end{aligned}
$$

Fluid

Examples:

$$\nabla_a \nabla^a \phi = F(\phi)$$

$$g^{\mu\nu} \nabla_\mu \nabla_\nu g^{\sigma\rho} = F^{\sigma\rho}(g^{\mu\nu}, \partial_\lambda g^{\mu\nu})$$

- Klein-Gordon
- General Relativity
- Electromagnetism
- Yang-Mills
- Fluids
- etc.

Cauchy problem

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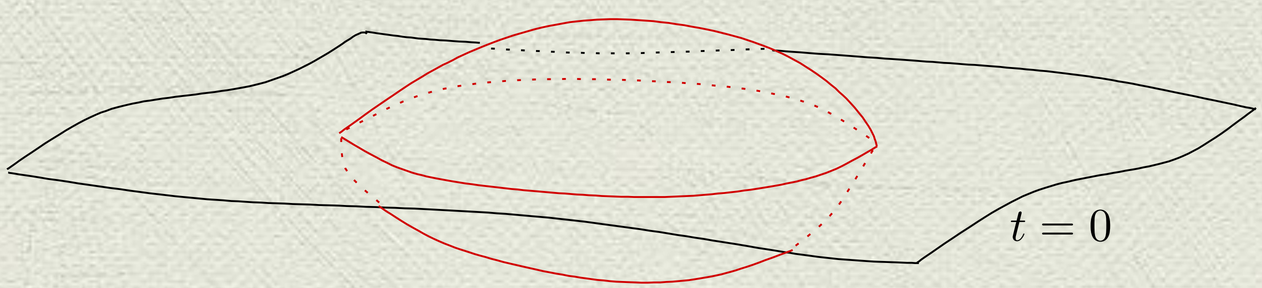

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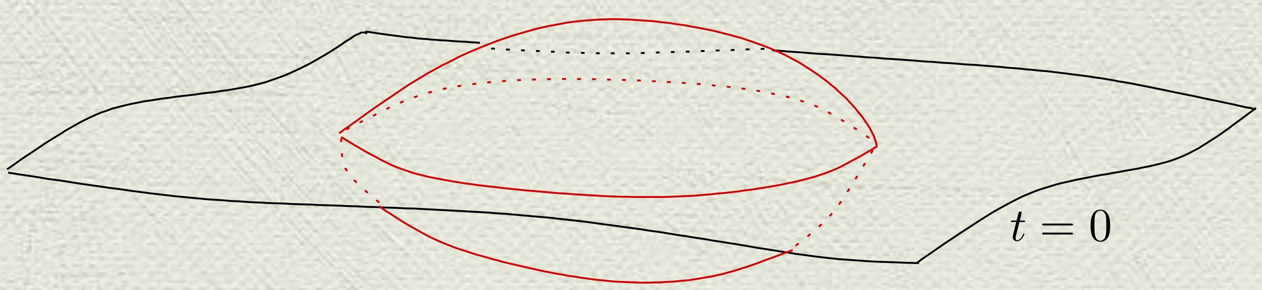

$$u^\alpha\big|_{t=0} = f^\alpha$$

$$\begin{aligned} u^\alpha(x,\delta t) &= u^\alpha(x,0) + \delta t\, \partial_t u^\alpha(x,0) \\ &= f^\alpha(x) + \delta t \left[A^{\alpha i}{}_\beta(fix,0)\nabla_i f^\beta + B^\alpha(f,x,0)\right] \end{aligned}$$

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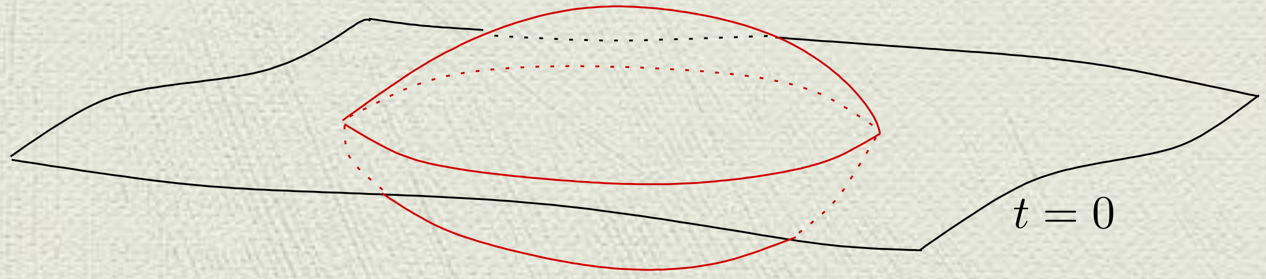

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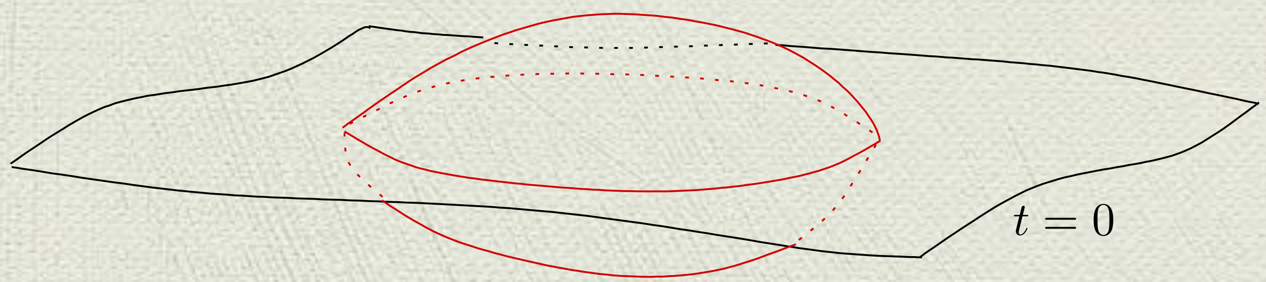

Analytic

Cauchy problem

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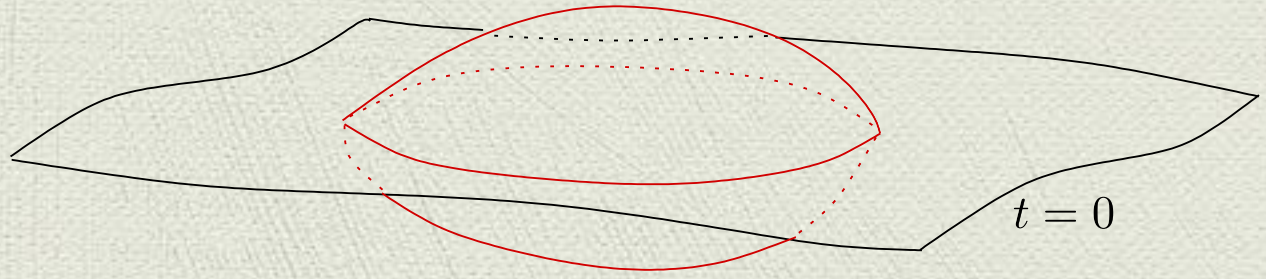


Analytic

Then there exists a unique, analytic solution in a nbh of t=0

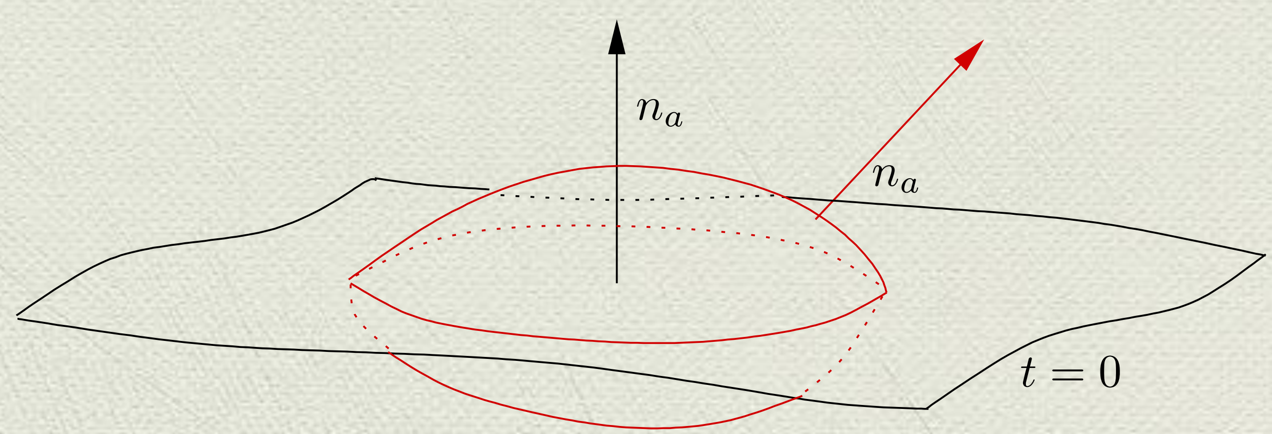
Cauchy problem

- Write f(x) in Taylor series, around x_0.
- Write u(t,x) in Taylor series around $(t=0,x_0)$.
- Get all coefficients for the u(t,x) series from the f(x) series using the equations.
- Show that convergence of Taylor series for f(x) implies convergence of Taylor series for u(t,x).

Cauchy Problem

Definition: Given a manifold M, and a set of fields on it, u^α, a characteristic surface of a quasi-linear equation $A^{\alpha a}{}_\beta(u)\nabla_a u^\beta = F^\alpha(u)$ is a hypersurface such that its normal, n_a at every point satisfies,

$$\det(A^{\alpha a}{}_\beta n_a) = 0.$$

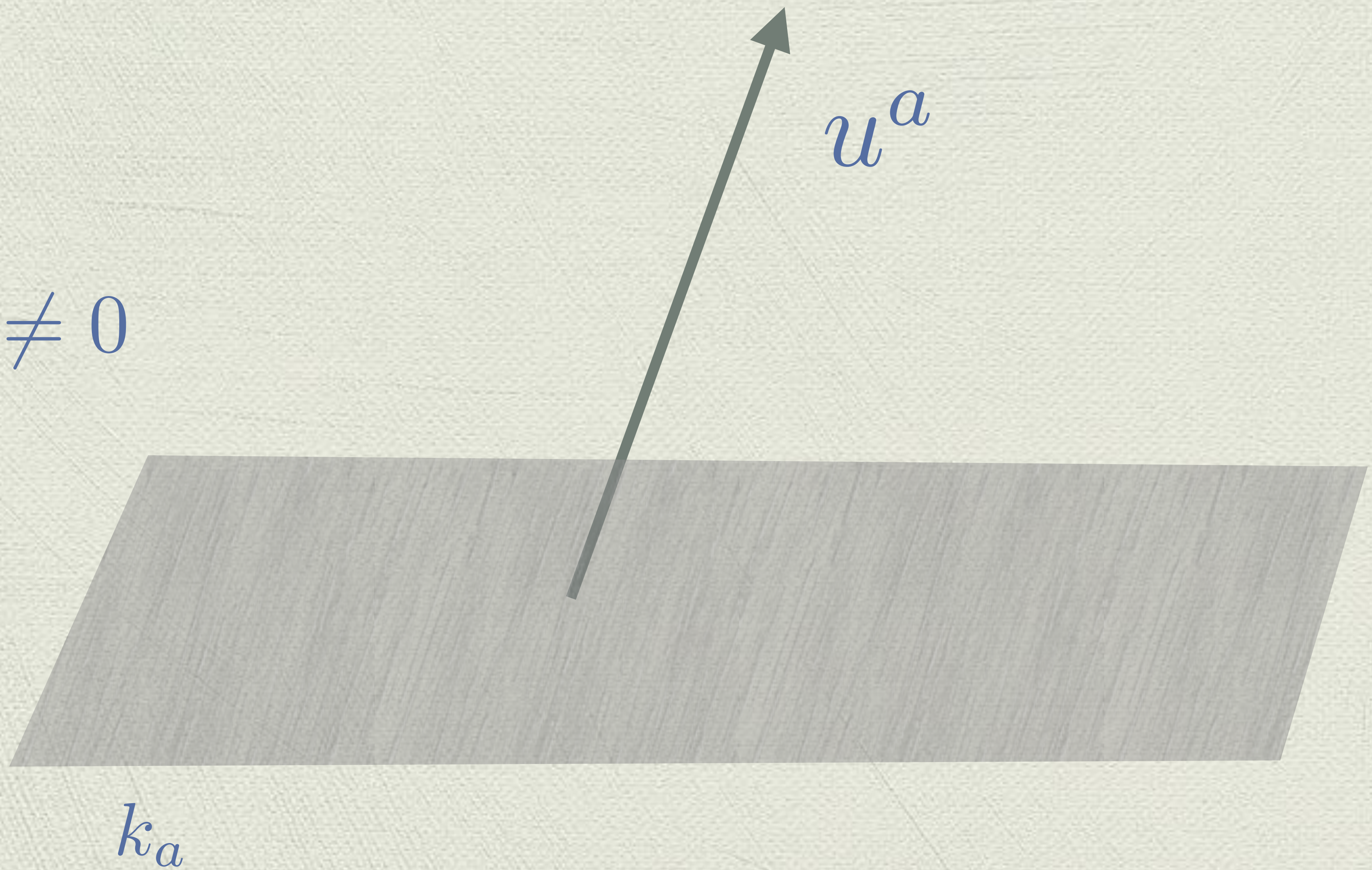


For this course forget general relativity, the metric it is not fundamental here!

Finite propagation

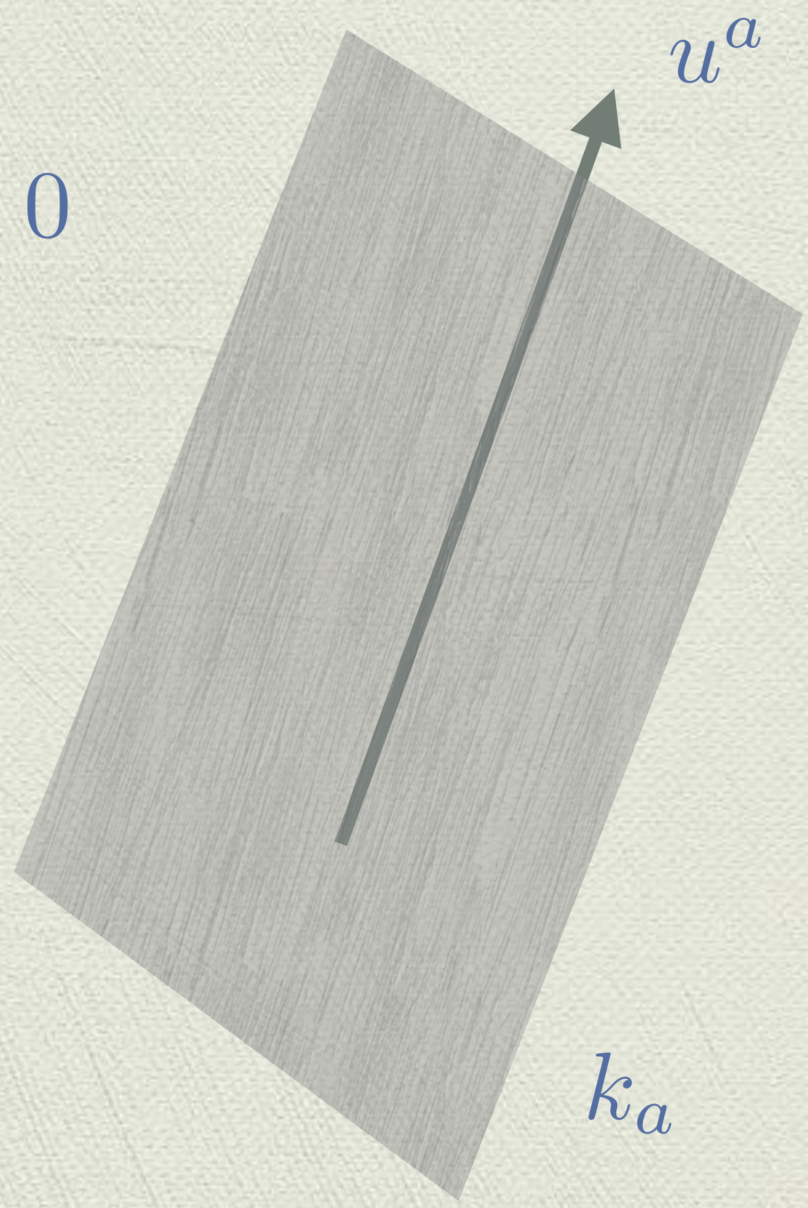
$$u^a \nabla \phi = F(\phi) \qquad \partial_t \phi = -\vec{v} \cdot \nabla \phi + F(\phi) \qquad u^a = (1, v^i);$$

$$u^a k_a \neq 0$$



Finite propagation

$u^a \, k_a = 0$



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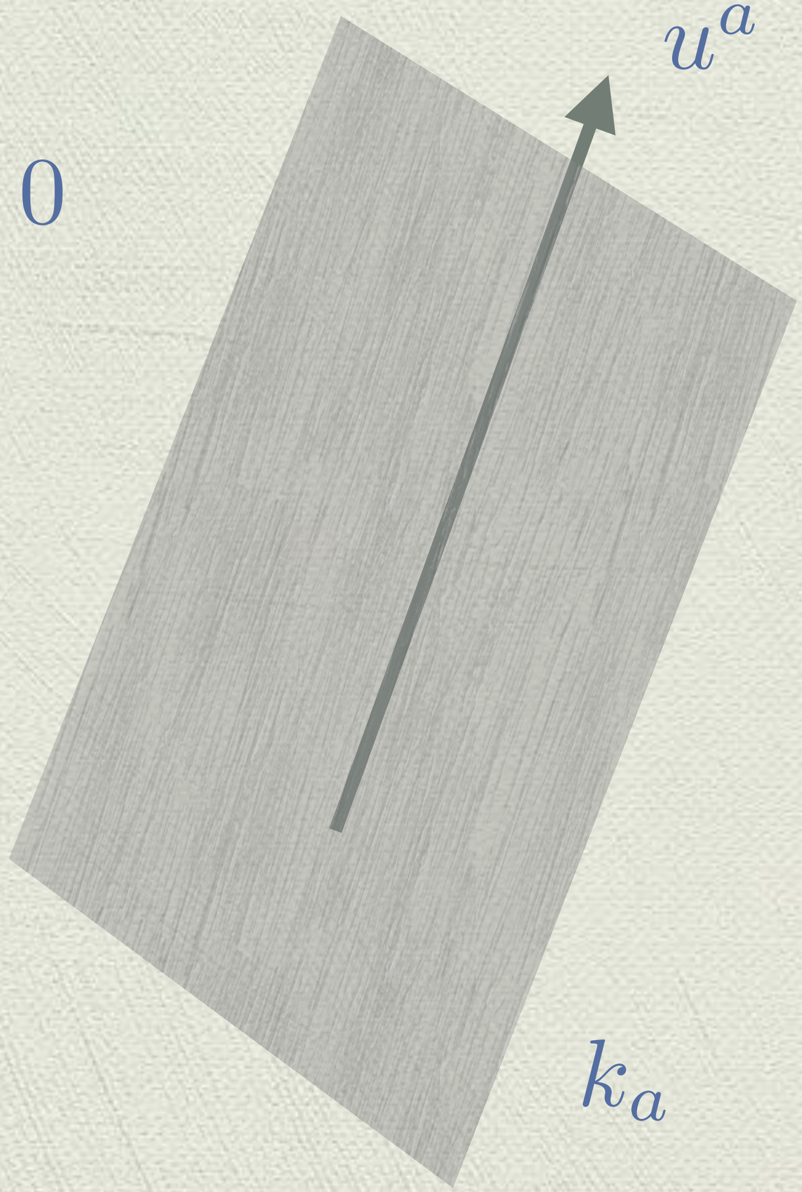


Planes at which data can not be given (over determined).

Plane wave solutions

High Frequency Limit

$$\phi(t,x) = \phi_0(t,x) + \varepsilon\, \phi_1(t,x)\, e^{\frac{f(t,x)}{\varepsilon}} \qquad \varepsilon \to 0$$

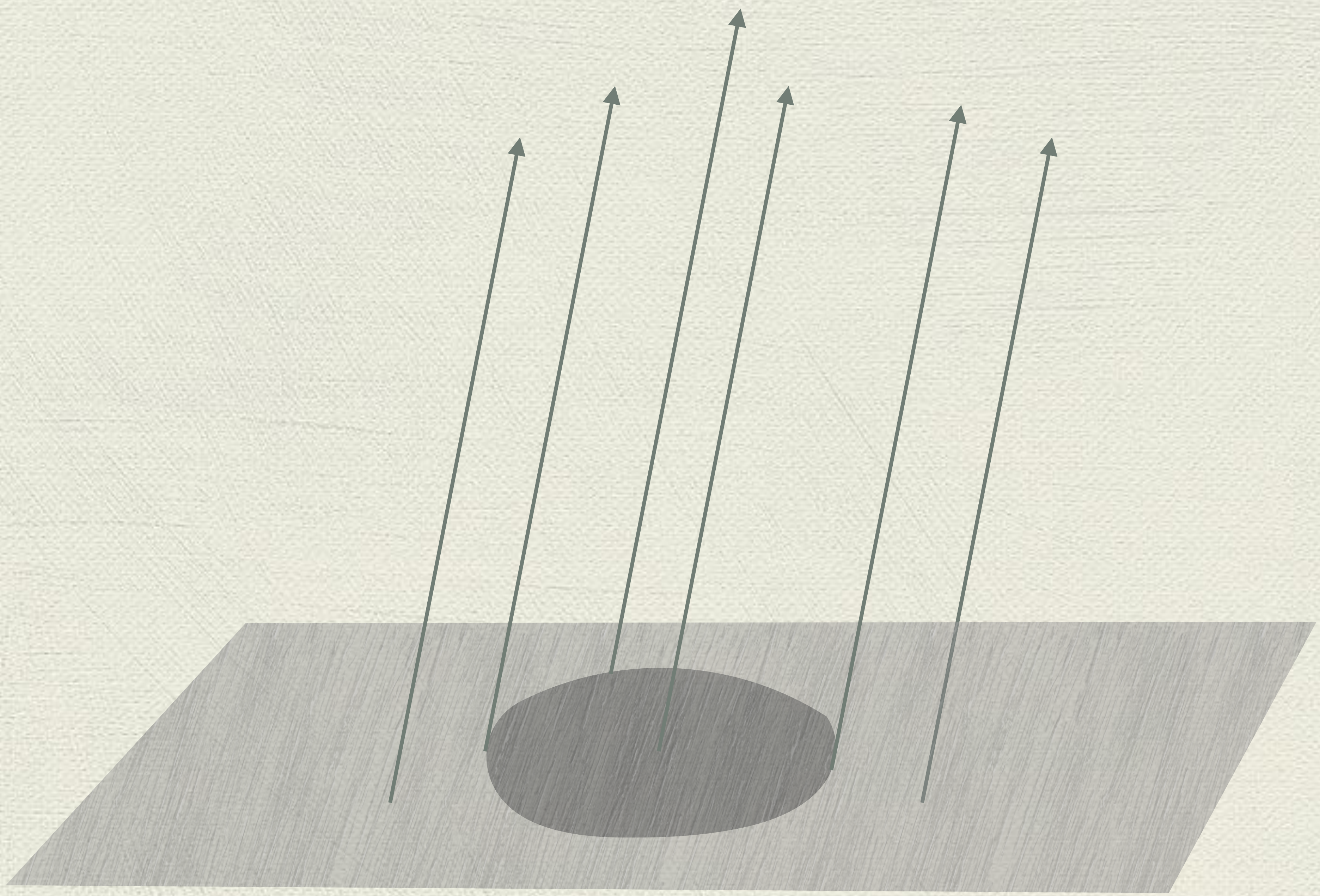
$$\partial_t \phi = -\vec{v} \cdot \nabla \phi + F(\phi) \qquad u^a = (1, v^i);$$

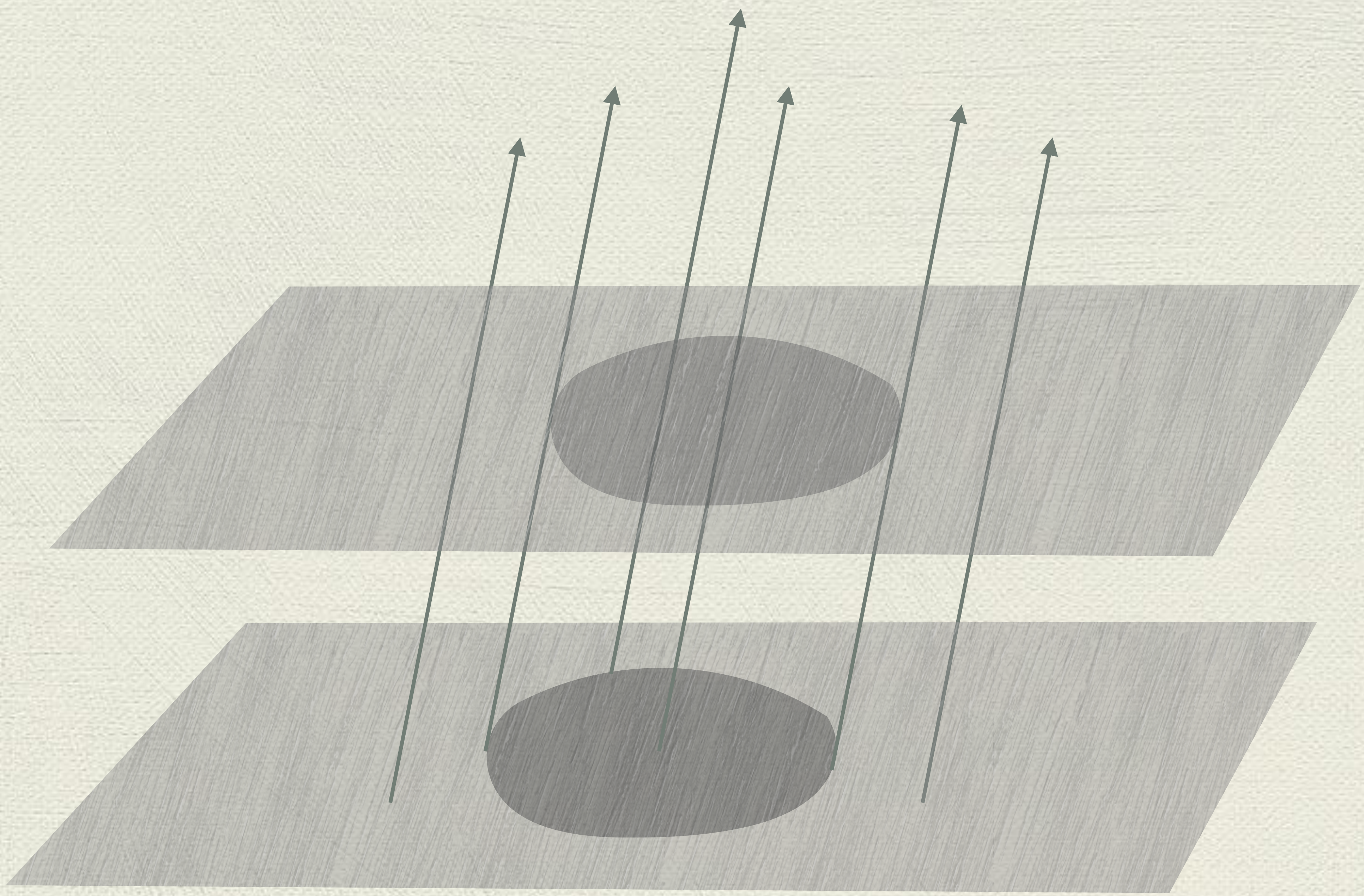
$$\phi_1 \partial_t f = v^i(t,x) \phi_1 \nabla_i f$$

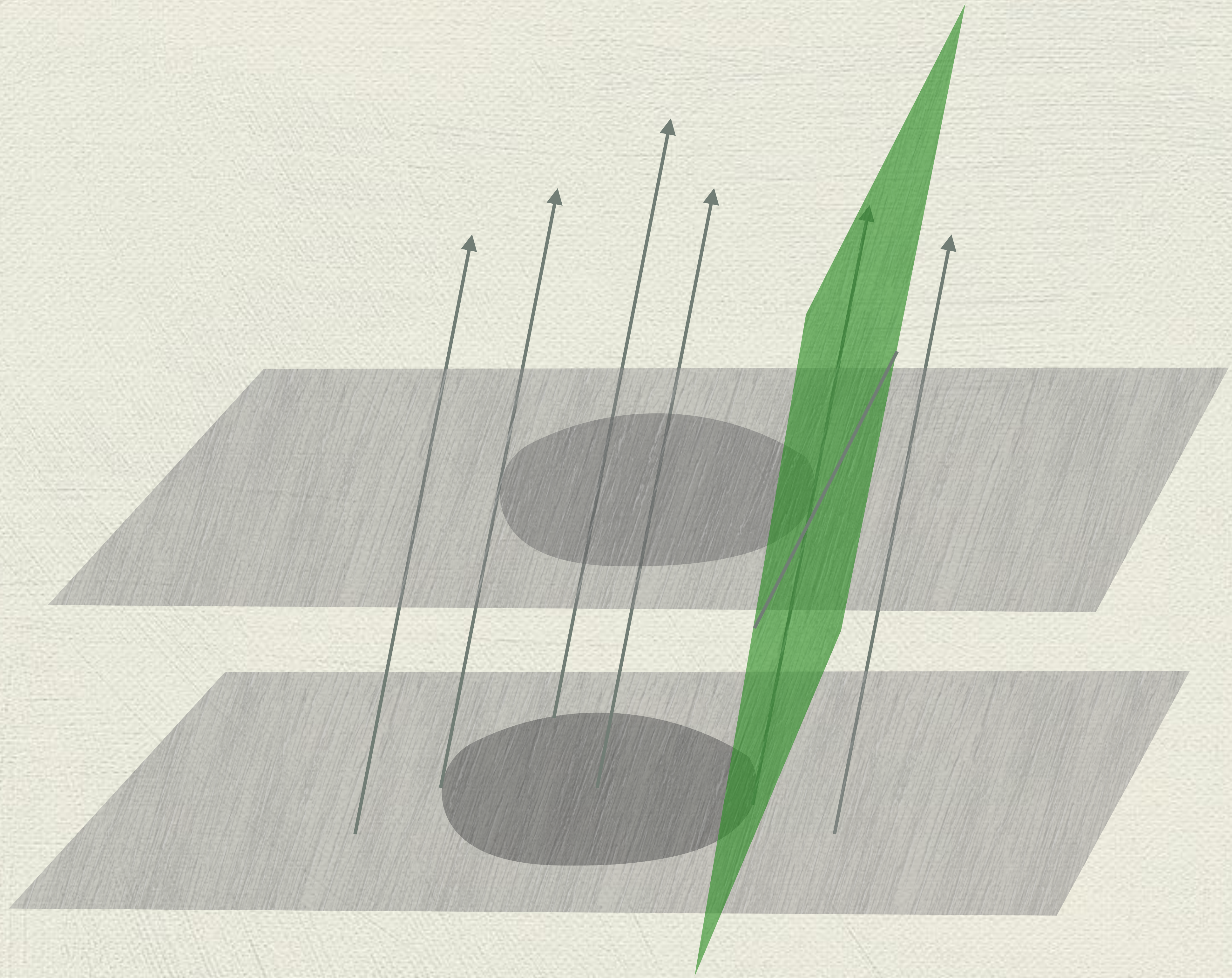
$$\phi_1 k_0 = v^i(t,x) k_i \phi_1 \qquad k_a := \nabla_a f$$

$$\Longleftrightarrow \qquad u^a k_a = 0$$

Local plane waves







High Frequency Limit

$$u^\alpha(t,x) = u_0^\alpha(t,x) + \varepsilon\, u_1^\alpha(t,x)\, e^{\frac{f(t,x)}{\varepsilon}} \qquad \varepsilon \to 0$$

$$\partial_t u^\alpha = A^{\alpha i}{}_\beta(u,x,t)\nabla_i u^\beta + B^\alpha(u,t,x)$$

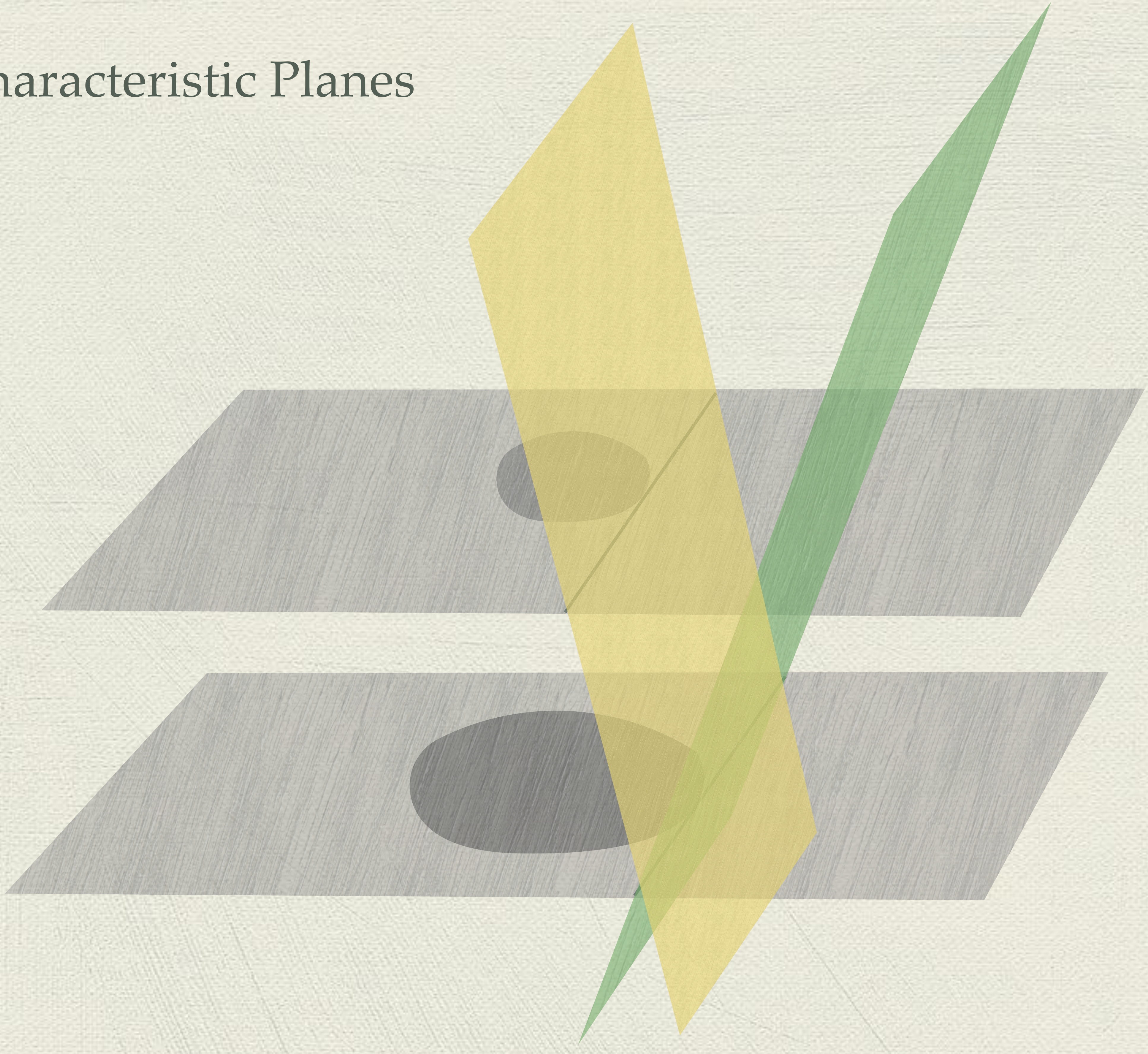
$$u_1^\alpha \partial_t f = A^{\alpha i}{}_\beta(u_0,x,t) u_1^\beta \nabla_i f$$

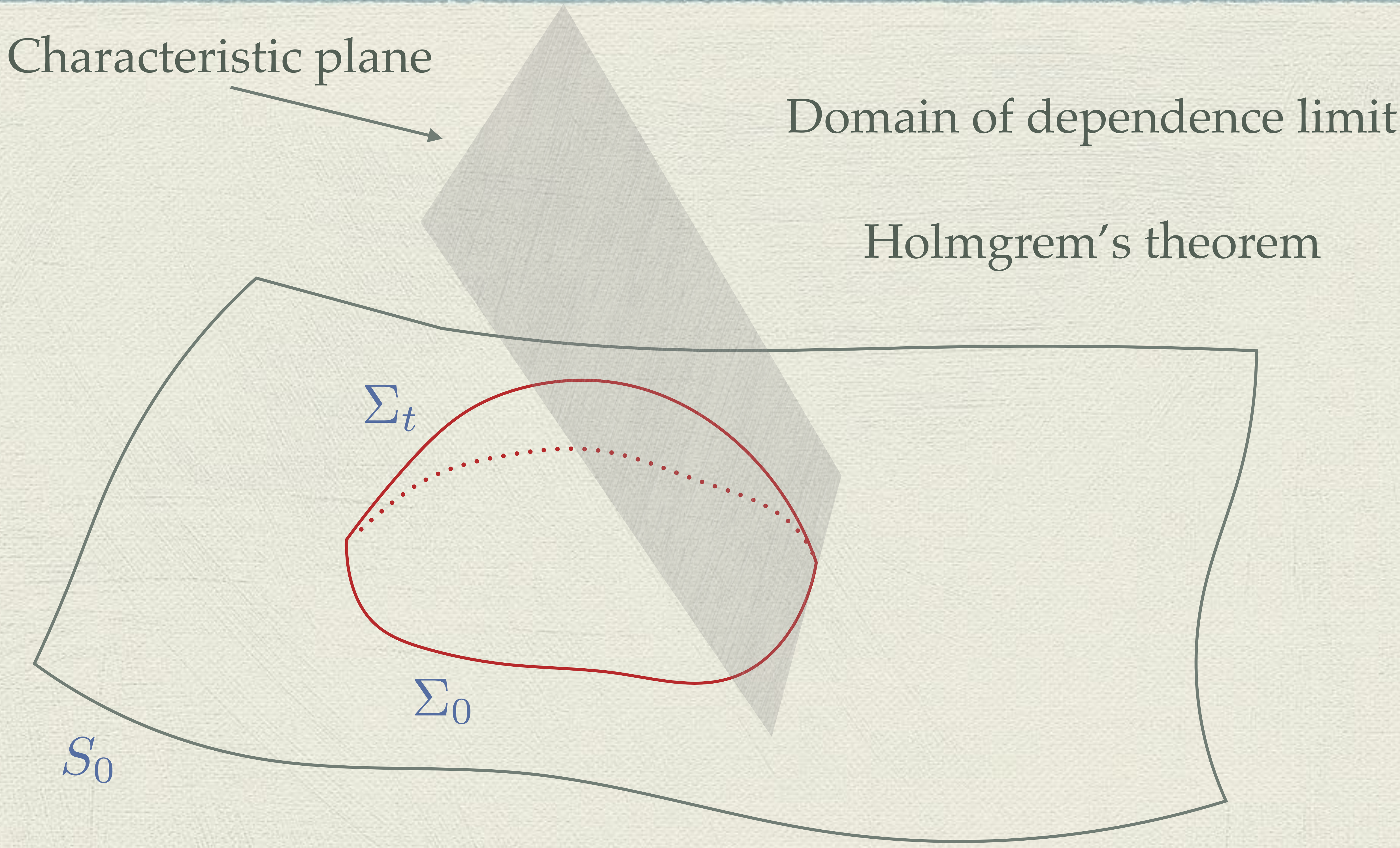
$$u_1^\alpha k_0 = A^{\alpha i}{}_\beta(u_0,x,t) k_i u_1^\beta$$

Set of plane waves

$$\det(\delta^\alpha{}_\beta\, k_0 - A^{\alpha i}{}_\beta\, k_i) = 0 \qquad k_a \in \mathcal{C}$$

Characteristic Planes



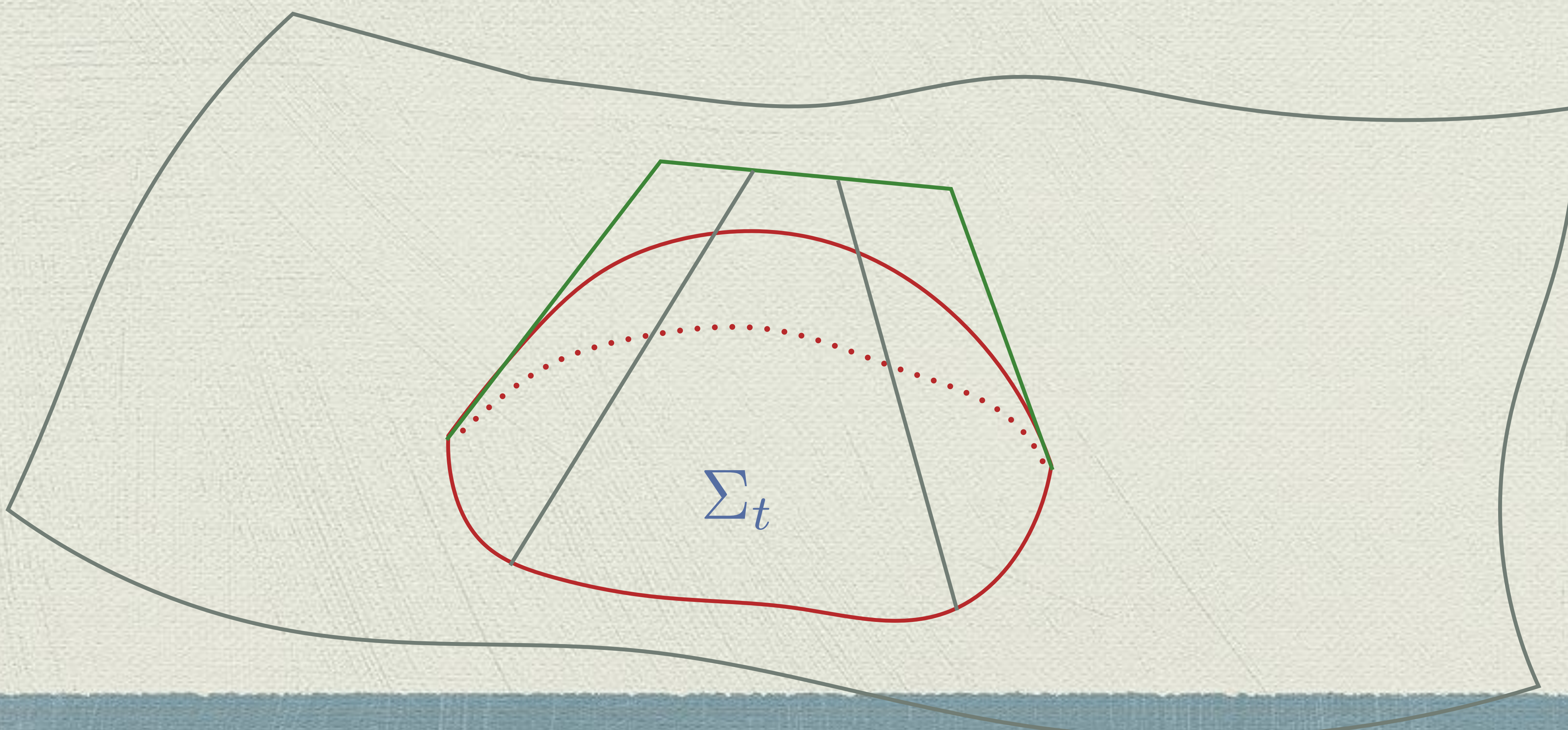


$$\Psi(\Sigma_0, t) = \Sigma_t \in \mathcal{M} \quad \Psi(\Sigma_0, 0) = \Sigma_0, \qquad \nabla t \notin \mathcal{C}$$

Local Cauchy development

$$D(\Sigma_0, \Psi) := \cup_t \Sigma_t \in \mathcal{M}$$

$$D(\Sigma_0, \Psi) \leq D(\Sigma_0, \Psi') \;\; if \;\; D(\Sigma_0, \Psi) \subseteq D(\Sigma_0, \Psi')$$



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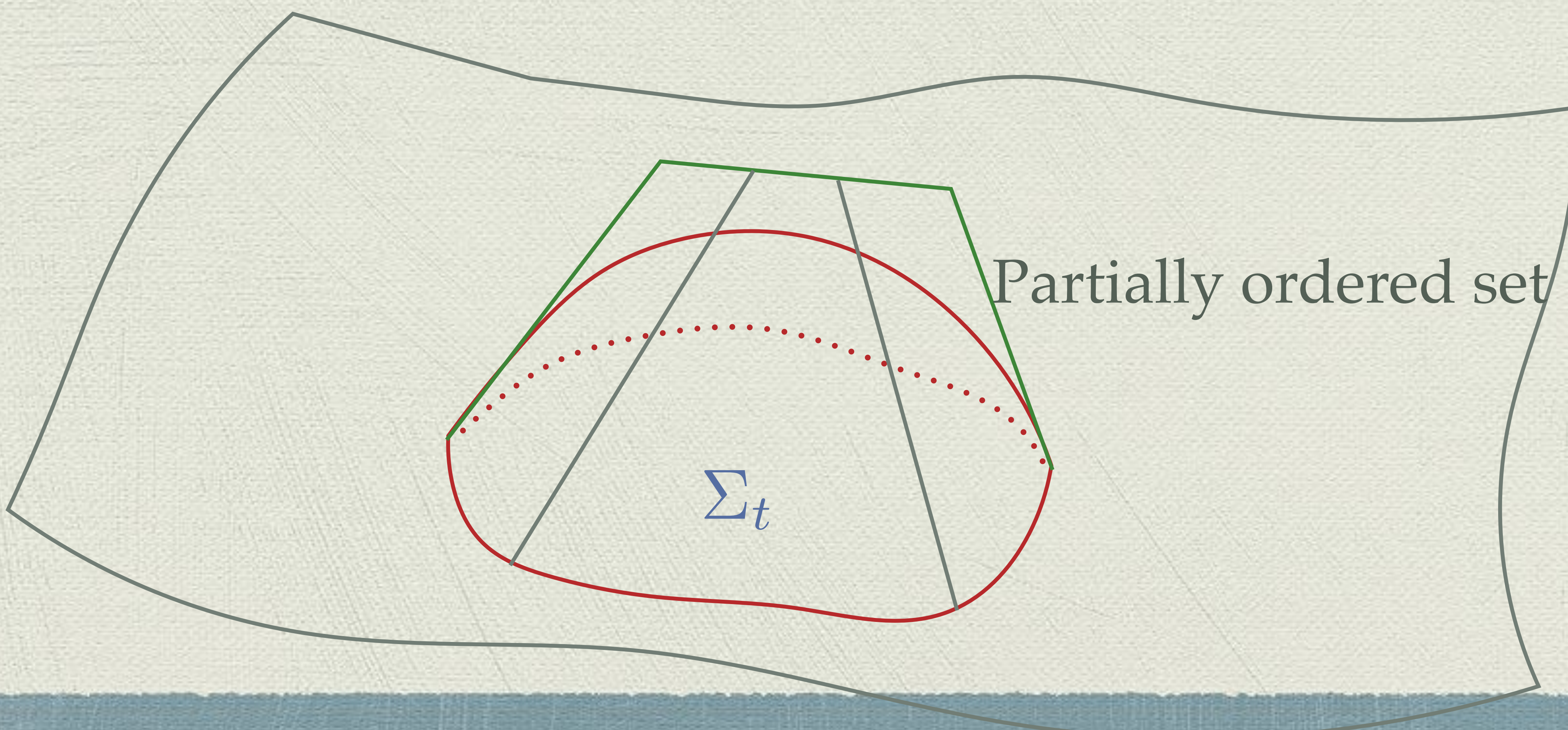
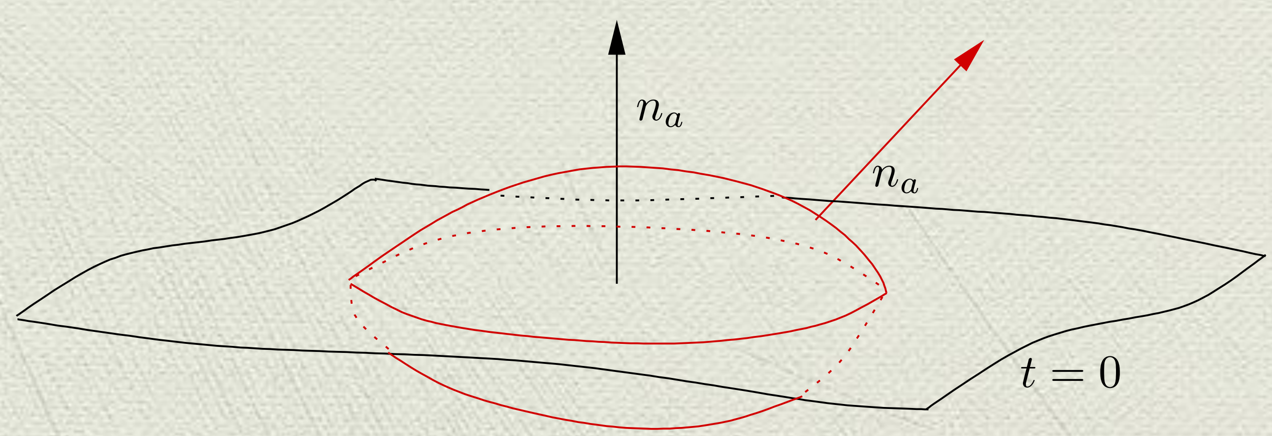


Cauchy Problem

- Cauchy–Kowalevski theorem: If ${A^{\alpha i}}_{\beta}$ and B^{α} are analytic, then for each analytic initial data f^{α} there exists a nbh. around Σ_0 and an analytic, unique, solution satisfying the initial conditions, $u|_{\Sigma_0} = f$.
- Holmgrem's theorem: Even when f is not analytic, if a solution exists, it is unique in a region bounded by characteristic surfaces.
- Everything seemed to be OK

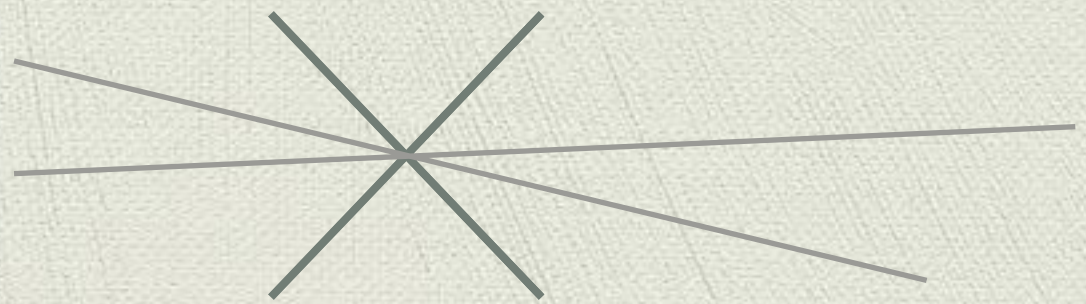


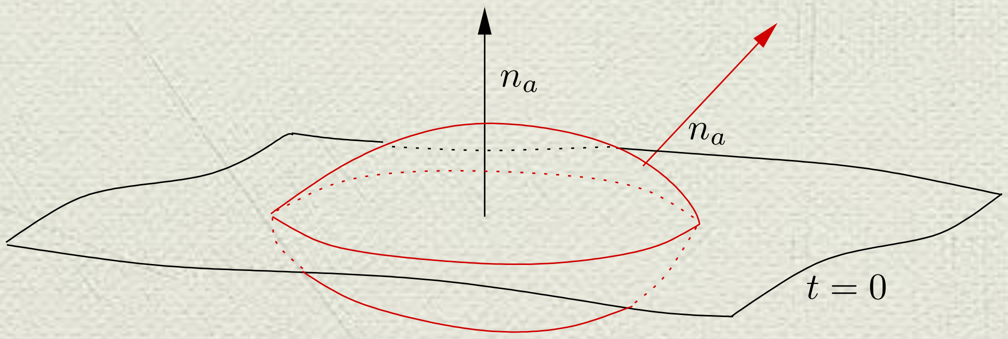
Example: Wave equation

$$\phi_{,11} - \phi_{,22} = f$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{,1} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{,2} = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad u := \phi_{,1} \quad v := \phi_{,2}$$

- In this case $0 = det(A^\mu n_\mu) = n_1^2 - n_2^2 \implies n_\mu = (1, \pm 1)$ so there are characteristic surfaces.
- Hyperbolic equations, characteristics are maximum propagation speeds.
- Concept of domain of dependence.





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Eigenvectors of

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$$\partial_t c^\pm = \pm\partial_x c^\pm$$

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$$u(t,x) = c^+(t,x)v_+ + c^-(t,x)v_- \qquad c^\pm(t,x) = C^\pm(k)e^{\omega t \pm ikx}$$

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$$\begin{array}{ll} \lambda_+ = 1 & v_+ = (1,1) \\ \lambda_- = -1 & v_- = (1,-1) \end{array}$$

$$c^+(0,x) = e^{ix} + \frac{1}{10000} e^{i1000x}$$

$$c^+(t,x) = e^{it} e^{ix} + \frac{e^{i1000t}}{10000} e^{i1000x}$$

Question:

$$\det(Wave\ Equation) = g^{ab} k_a k_b$$

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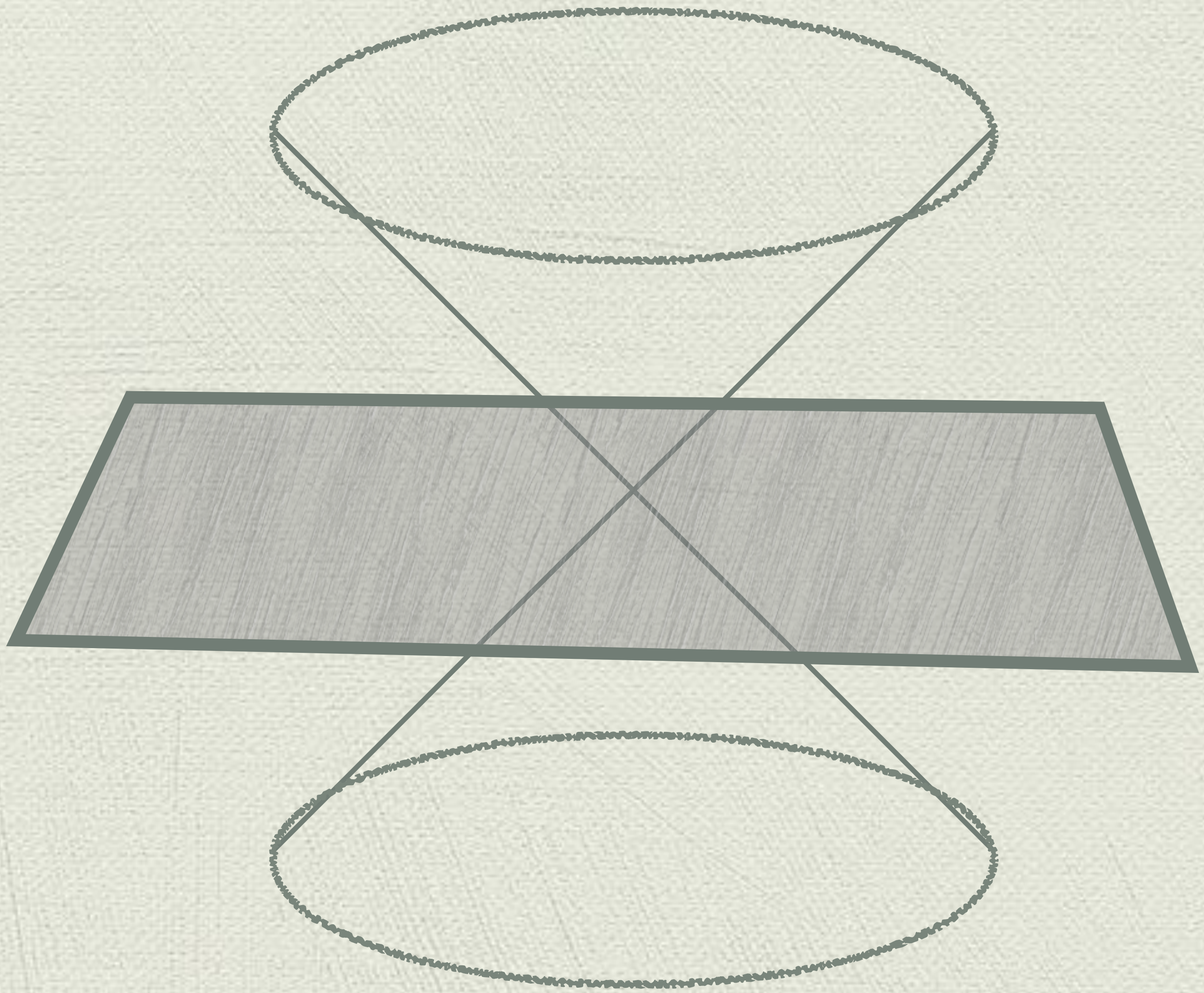

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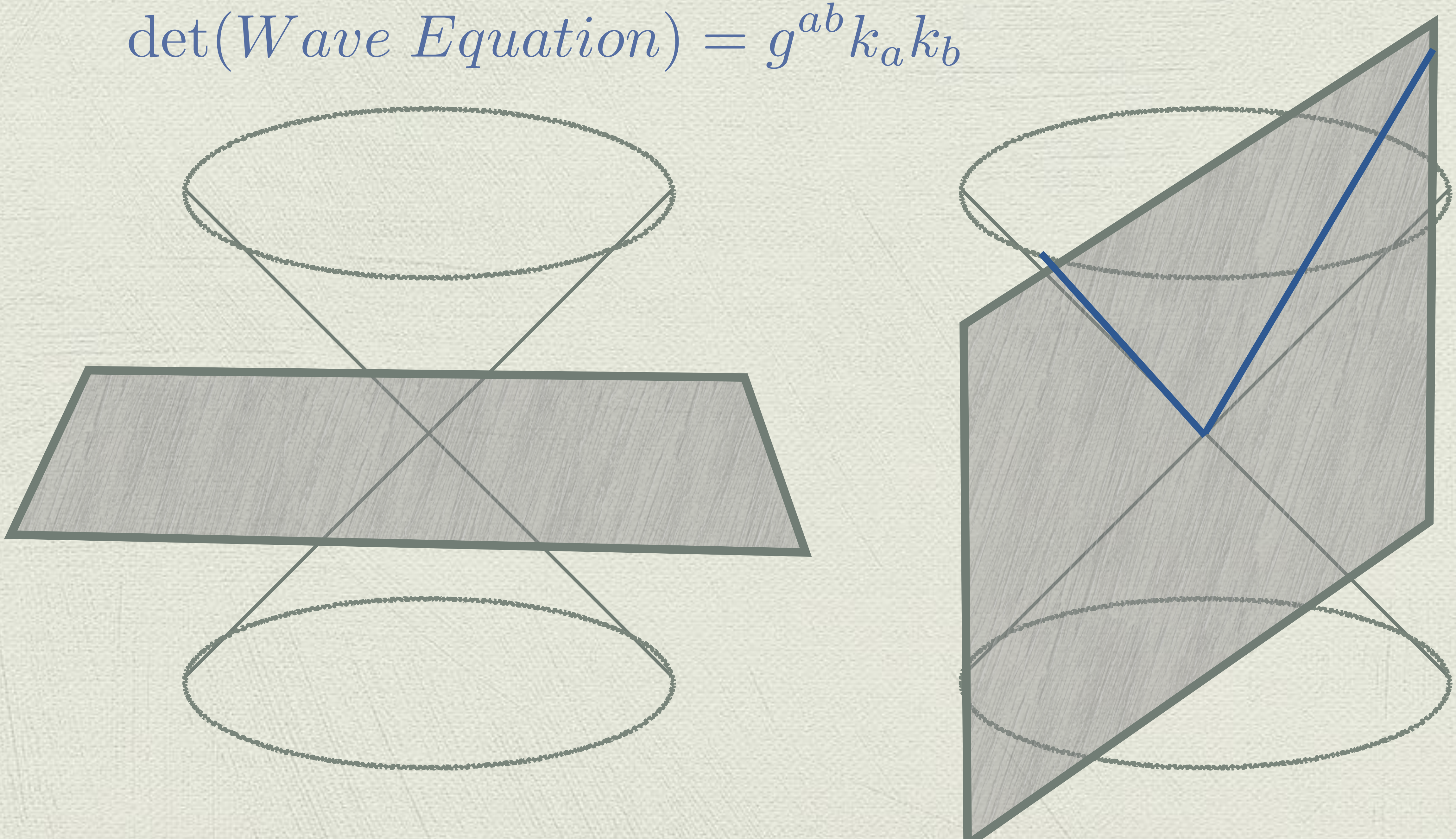

Example: Laplacian

$$\phi_{,11} + \phi_{,22} = f$$

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- In this case $0 = det(A^\mu n_\mu) = n_1^2 + n_2^2 \implies n_\mu = 0$ so there is no characteristic surfaces.
- Elliptic equations, if you know the solution on a nbh. of a point you know it everywhere.
- Something funny...

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- Cauchy-Kowalevski gives only existence for analytic initial data.
- There are equations which only have solutions if the data is analytic.
- In many cases if a solution exists it is not a continuos function with respect to the initial data. (Not well posed.)
- No good for physics! There is no predictability.

Summary Lecture I

- Old stuff is very geometrical and elegant.
- But only for analytic data.
- Does not responds questions about existence of non-analytic solutions (and their data). Recall that generic quasi-linear equations develop shocks.
- Does not address the question of predictability (classical).
- It is not enough to incorporate all equations in physics (Constraints — Gauge).

Grav 17

Huerta Grande, Córdoba, Argentina
3-7 de abril de 2017

- Abhay Ashtekar
- Florian Beyer
- Görg Frauendiener
- Helmut Friedrich
- Gabriela Gonzalez
- Luis Lehner
- Alejandro Perez
- Jorge Pullin
- Martín Reiris
- Olivier Sarbach
- Lilian Stemstad
- Manuel Tiglio
- Gabriela Vila
- Robert Wald

Lecture II: Hyperbolicity

- Continuous Dependence of Solutions to the Cauchy Problem.
- Constant Coefficient Systems.
- First Order Systems and Different types of hyperbolicity: Symmetric Hyperbolic, Strongly Hyperbolic, Weakly Hyperbolic, Leray Hyperbolicity.
- Energy Estimates.
- Causality: Filling the cones.
- Covariant definition of Strong Hyperbolicity.

Constant coefficient system (on a torus)

$$u_t(t,x) = P(\partial)u(t,x) \qquad P(\partial_j) = \sum_{|\alpha|\leq m} A^\alpha D_\alpha$$

non-characteristic surface

$$u(t,x) = \sum_k e^{ix\cdot k}\hat{u}(t,k) \qquad \hat{u}(t,k) := \int e^{-ix\cdot k}u(t,x)$$

$$\hat{u}_t = P(ik)\hat{u} \quad \Longrightarrow \quad \hat{u}(t,k) = e^{P(ik)t}u(0,k)$$

$$f(x) = \sum_{|k|<r} e^{ix\cdot k}\hat{f}(k) \quad \Longrightarrow \quad u(t,x) = \sum_{|k|<r} e^{ix\cdot k}e^{P(ik)t}u(0,k)\hat{f}(k)$$

- Good for finite series (Cauchy-Kowalevski)

Constant coefficient system

- When do solutions at a given time have a continuous dependence with respect to their initial data? *Hadamard*
- This happens if and only if $|e^{P(ik)}| < C$ independent of k. (When considering the same norm for initial data and solution at the given time).
- **IF:** $||u(t,\cdot)||^2 = ||\hat{u}(t,\cdot)||^2 \leq C||\hat{f}(\cdot)||^2 = C||f(\cdot)||^2$
- **ONLY IF:** Otherwise there exists a sequence $\{(k_l, \hat{f}_l)\}$ with $||\hat{f}_l|| = 1$ and $|e^{P(ik_l)}\hat{f}_l| = l$

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Reproduce this calculation!

Constant coefficient system

Constant coefficient system

- For Laplacian equation it is not bounded, grows exponentially with $|k|$.
- For Wave equation it is bounded by one, $P(ik)$ is diagonalizable and has imaginary eigenvalues.
- For Heat equation it is not bounded, either $|e^{P(ik)}|$ or $|e^{-P(ik)}|$ grows exponentially with $|k|^2$.

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Without stability we can not compute!

Constant coefficient system

Constant coefficient system

- Consider now first order systems $P(\partial) = A^a \partial_a$
- When is $|e^{P(ik)}| \leq C$?

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- Answer: If and only if, for each k_a, $\quad A^a k_a$ has real eigenvalues and a complete set of eigenvectors (diagonalizable).
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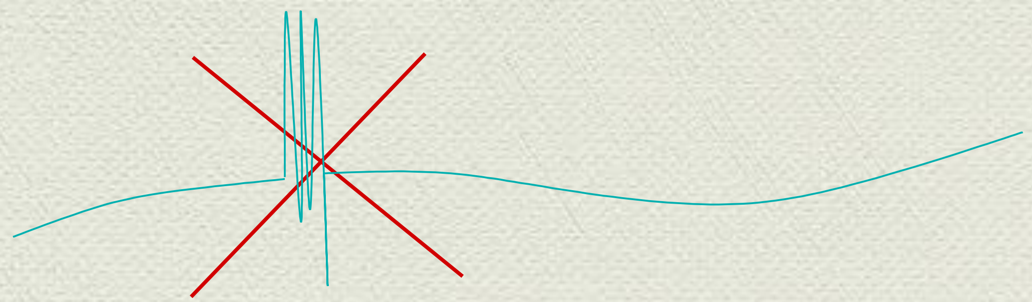
- Answer: If and only if, for each k_a, $\quad A^a k_a$ has real eigenvalues and a complete set of eigenvectors (diagonalizable).
- Definition of **Strong hyperbolicity** for constant coefficient systems
- Alternatively: It is SH if there exist $H = H(k)$, positive definite bilinear form such that $H(k)A^a k_a$ is symmetric.

Constant coefficient system

- Consider now first order systems $P(\partial) = A^a \partial_a$
- When is $|e^{P(ik)}| \leq C$?
- Answer: If and only if, for each k_a, $A^a k_a$ has real eigenvalues and a complete set of eigenvectors (diagonalizable).
- Definition of **Strong hyperbolicity** for constant coefficient systems
- Alternatively: It is SH if there exist $H = H(k)$, positive definite bilinear form such that $H(k) A^a k_a$ is symmetric.
- **Symmetric hyperbolicity:** If H does not depend on k_a. Most physical systems can be cast in Symmetric Hyp. form.

Strong hyperbolicity

- High frequency limit.

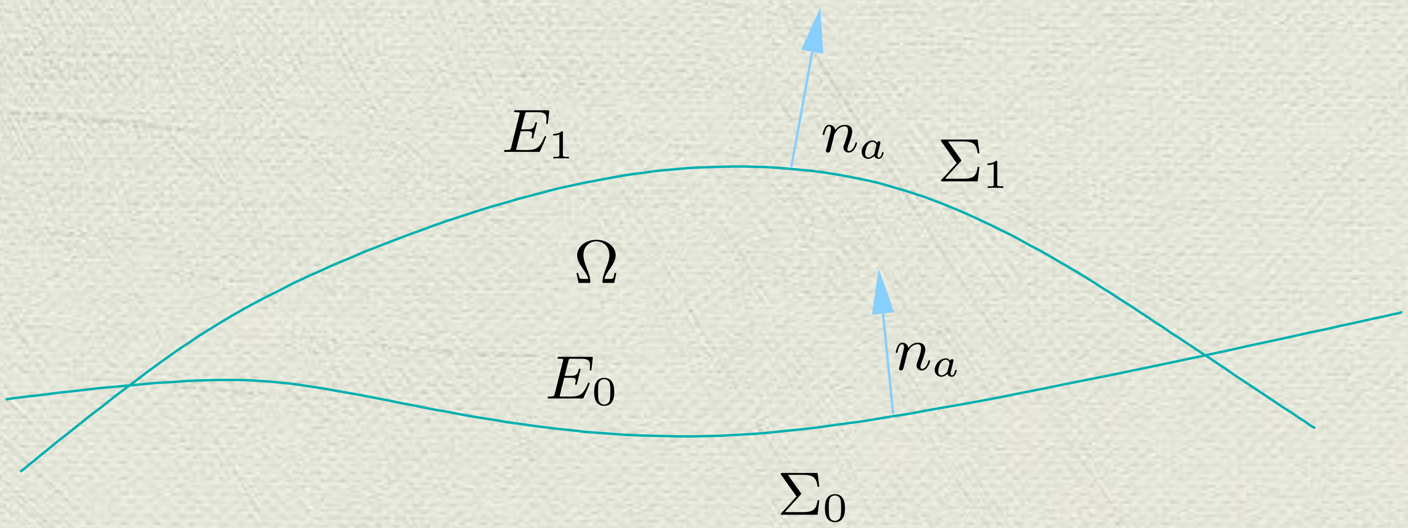


- Strong-Hyperbolicity: ⟷ Waves can be decomposed into a complete set of plane waves, each one of them with purely imaginary frequency.

Strong hyperbolicity

Strong hyperbolicity

- Allows energy estimates:



Strong hyperbolicity

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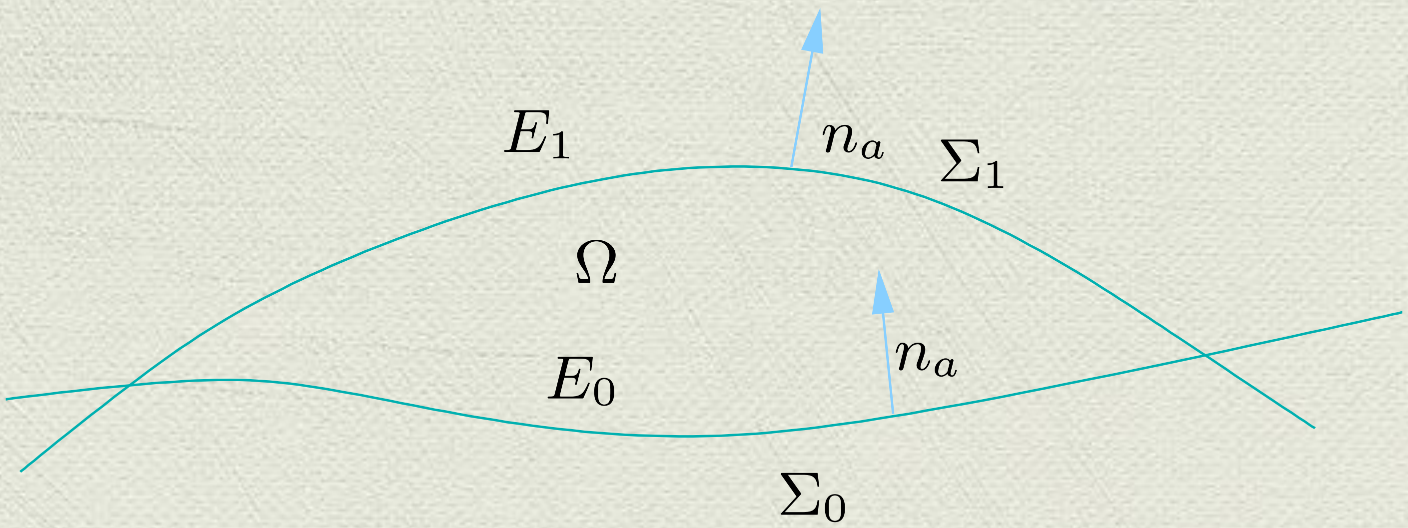


- Symmetric Hyperbolic: Usual energies

- Strongly Hyperbolic: Pseudodifferential Calculus

Symmetric Hyperbolicity (Covariant Version)

$$A^{\alpha a}{}_{\beta} \nabla_a u^{\beta} = J^{\alpha}$$

Is **Symmetric Hyperbolic** if there exists $H(u)_{\gamma\alpha}$ and k_a such that:

- $H_{\gamma\alpha} A^{\alpha a}{}_{\beta}$ is symmetric, and
- $H_{\gamma\alpha} A^{\alpha a}{}_{\beta} k_a$ is positive.

Example:

$$h_1[u_1^a \nabla_a \phi_1 = F_1(\phi_1)]$$

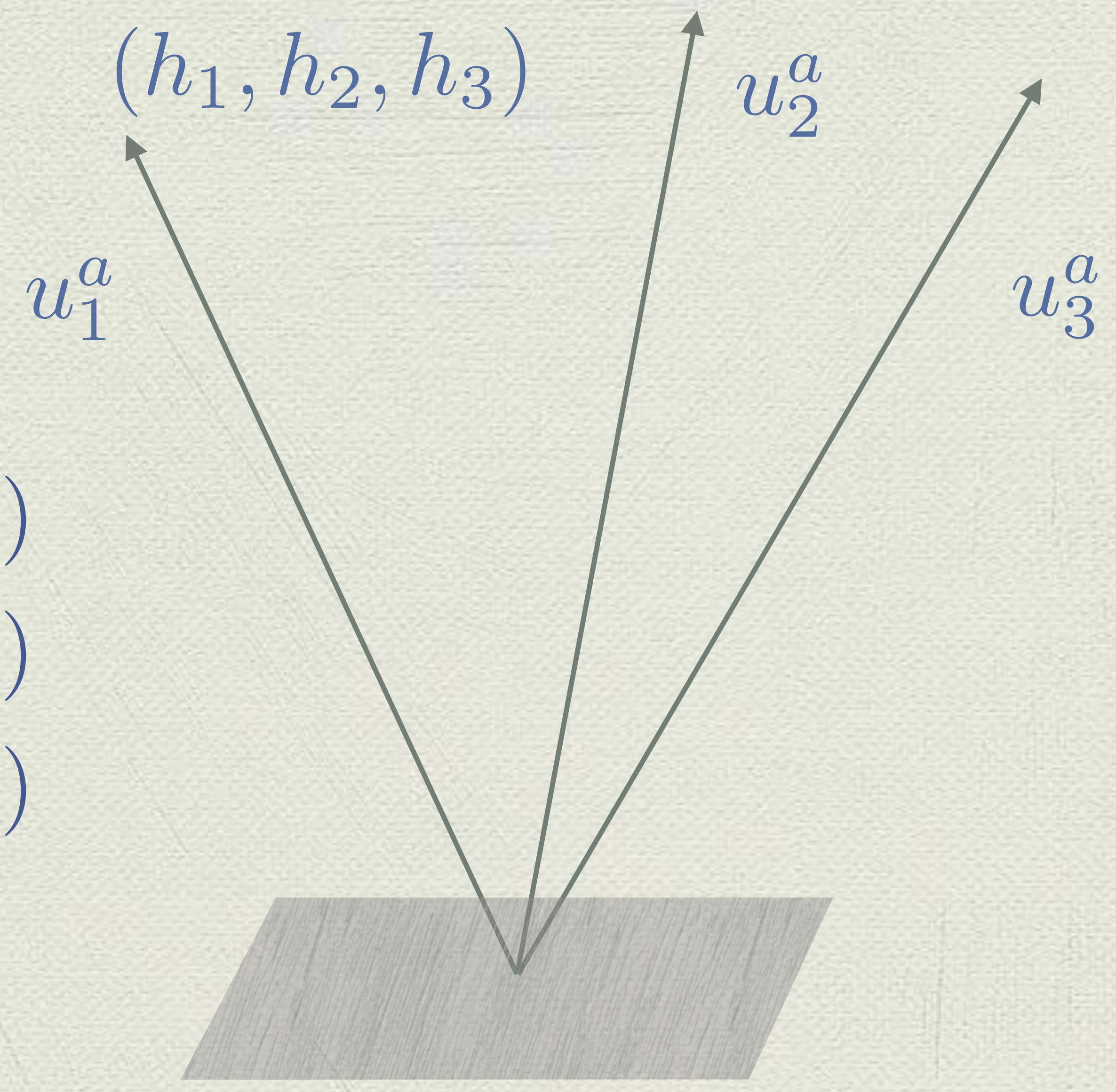
$$h_1 u_1^a k_a > 0$$

Example:

$$
\begin{aligned}
u_1^a \nabla_a \phi_1 &= F_1(\phi_1, \phi_2, \phi_3) \\
u_2^a \nabla_a \phi_2 &= F_2(\phi_1, \phi_2, \phi_3) \\
u_3^a \nabla_a \phi_3 &= F_3(\phi_1, \phi_2, \phi_3)
\end{aligned}
$$

Example:

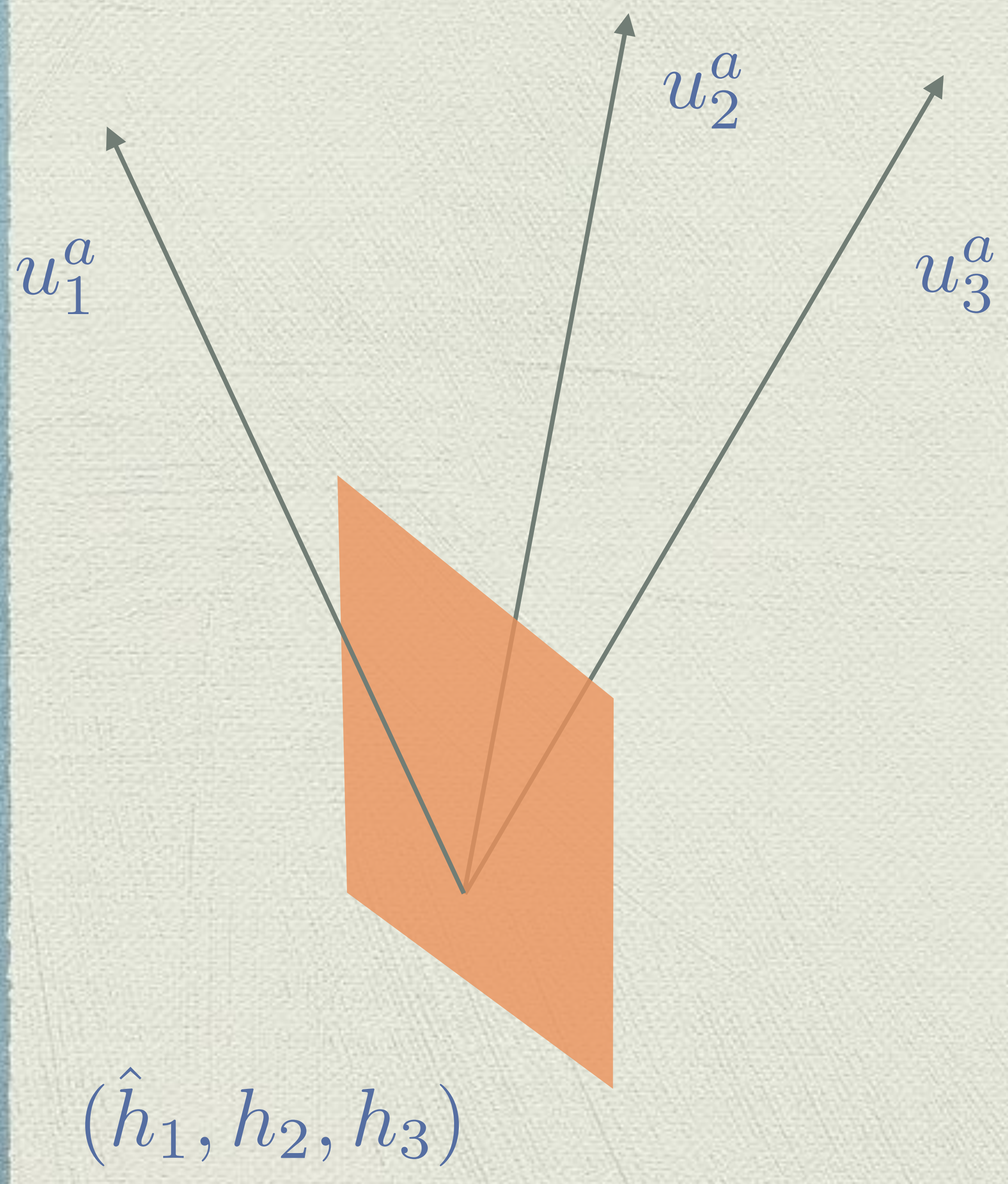
$$
\begin{aligned}
u_1^a \nabla_a \phi_1 &= F_1(\phi_1, \phi_2, \phi_3) \\
u_2^a \nabla_a \phi_2 &= F_2(\phi_1, \phi_2, \phi_3) \\
u_3^a \nabla_a \phi_3 &= F_3(\phi_1, \phi_2, \phi_3)
\end{aligned}
$$



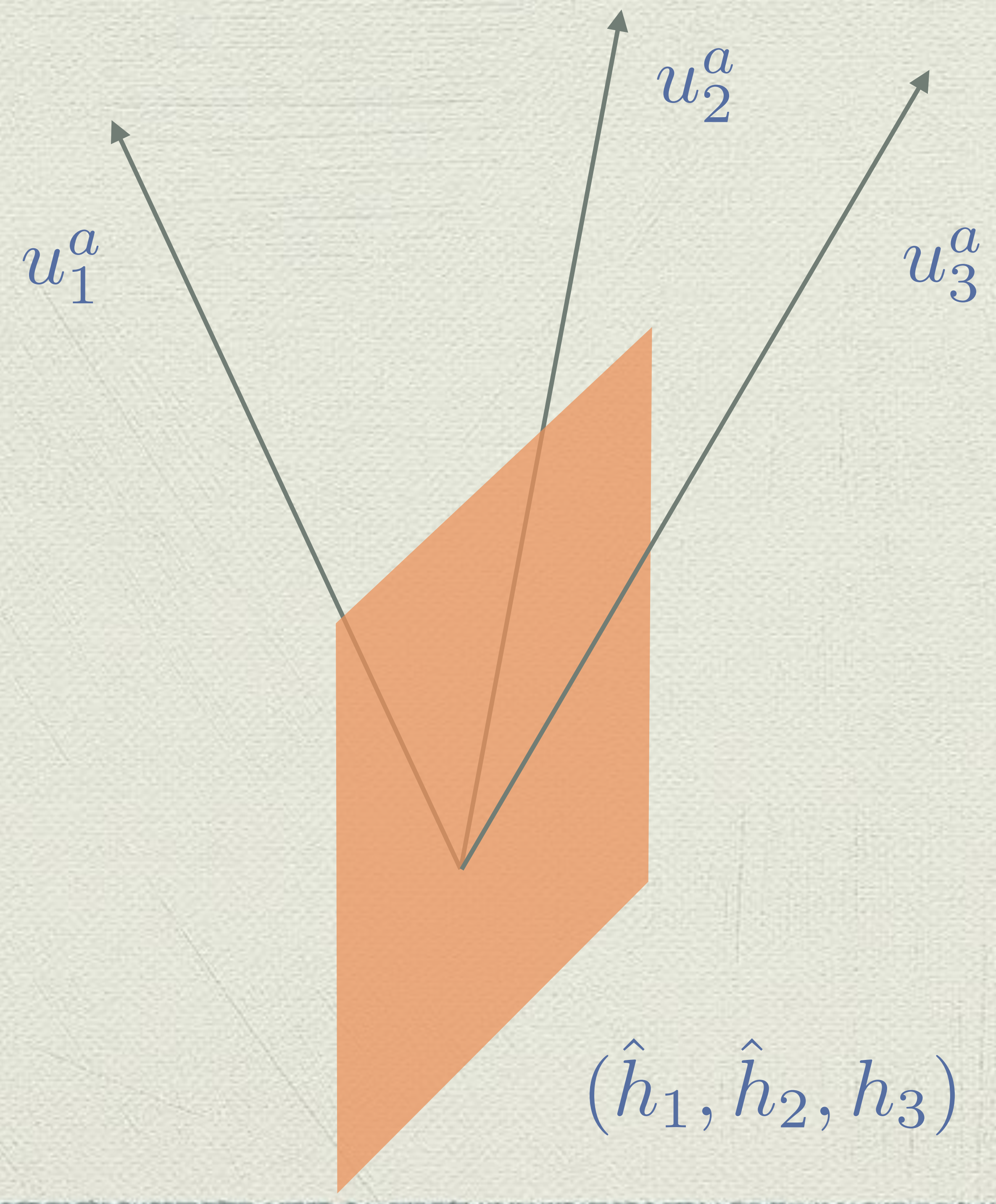
$$(h_1 u_1^a k_a > 0, h_2 u_2^a k_a > 0, h_3 u_3^a k_a > 0)$$

Example:

Example:



Example:



Strong Hyperbolicity (covariant)

Is **Strongly Hyperbolic** if there exists $H(u)_{\gamma\alpha}(k)$ and $\hat{k}_a$ such that:

- $H_{\gamma\alpha}(k)A^{\alpha a}{}_{\beta}k_a$ is symmetric, and
- $H_{\gamma\alpha}(k)A^{\alpha a}{}_{\beta}\hat{k}_a$ is symmetric and positive.

General Quasilinear Systems

General quasilinear equation:

$$A^a{}_{A\alpha}(u)\nabla_a u^\alpha = F_A(u)$$

$$u^\alpha = \{set\ \ of\ \ tensors\ \ in\ \ some\ \ manifold\ \ M\}$$

Fiber bundle over M of dimension m

$$F_A(u) = \{set\ \ of\ \ tensors\ \ also\ \ in\ \ M\}$$

Fiber bundle over M of dimension q

They can be also maps from manifolds into manifolds. (The differential among these bundles is always well defined)

Quasilinear systems of equations

Quasilinear systems of equations

Electromagnetism

$$\nabla_{[a}F_{bc]} = 0 \quad 4$$

$$\nabla^a F_{ab} = J_b \quad 4$$

$$F_{ab} \quad 6$$

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Wave equation

$$\nabla_a \phi = l_a \quad 4$$

$$\nabla_{[a} l_{b]} = 0 \quad 6$$

$$\nabla^a l_a = J \quad 1$$

$$(\phi, l_a) \quad 5$$

Quasilinear systems of equations

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There are constraints.

$$\varepsilon^{tabc}\nabla_a F_{bc} = 0 \qquad \nabla^a F_{at} = 0 \qquad 2$$

Quasilinear systems of equations

Electromagnetism			Wave equation	
$\nabla_{[a}F_{bc]} = 0$	4		$\nabla_a \phi = l_a$	4
$\nabla^a F_{ab} = J_b$	4		$\nabla_{[a}l_{b]} = 0$	6
			$\nabla^a l_a = J$	1
F_{ab}	6		(ϕ, l_a)	5

There are constraints.

$\nabla_i \phi = l_i \quad \epsilon^{tijk}\nabla_j l_k = 0$ 6

Symmetric hyperbolic systems (a la Geroch)

- Geroch paper on SH systems

$$A^a{}_{A\alpha}(u)\nabla_a u^\alpha = F_A(u)$$

Is symmetric hyperbolic or admits a hyperbolization if there exists $H_\beta{}^A$ such that:

$$H_\beta{}^A A^a_{A\alpha} = H_{(\beta}{}^A A^a_{|A|\alpha)}$$

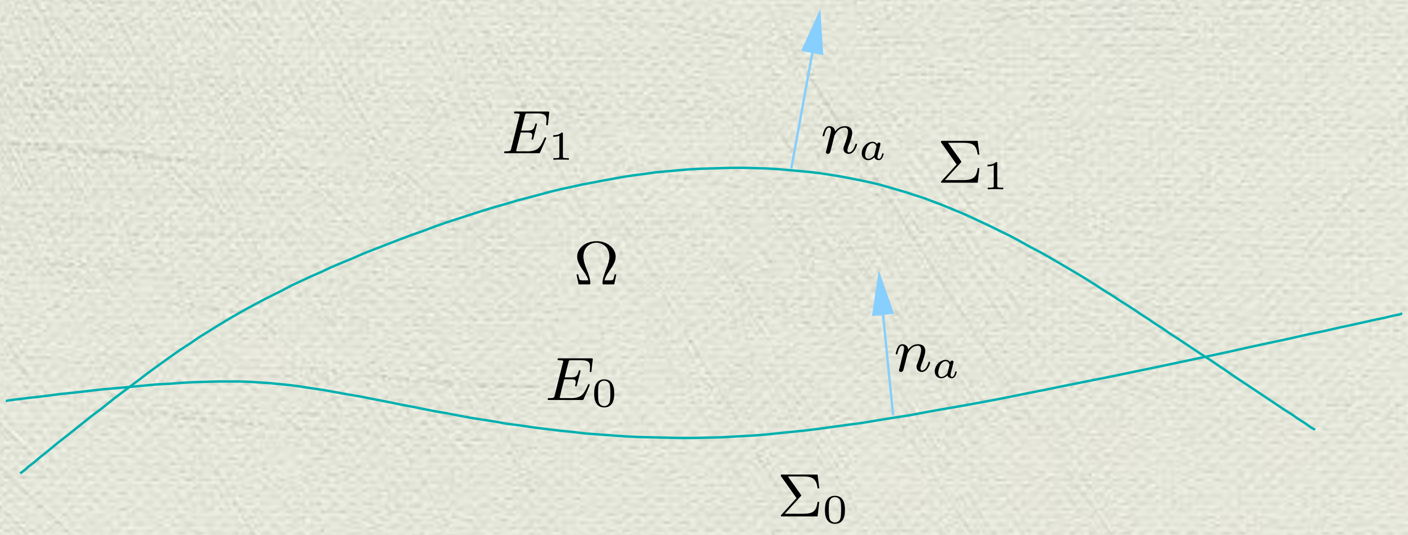
is symmetric and there exists n_a such that

$$H_\beta{}^A A^a_{A\alpha} n_a$$

is positive definite.

Strong hyperbolicity

- Allows energy estimates:



Symmetric hyperbolic systems

$$H_\beta{}^A A^a_{A\alpha} n_a$$

$$J^a := H_\beta{}^A A^a{}_{A\alpha} u^\beta u^\alpha \qquad \nabla_a J^a := H_\beta{}^A F_A u^\beta$$

A mathematical energy is defined by,

$$\mathcal{E}(t) = \int_{\Sigma_t} H_\beta{}^A A^a_{A\alpha} u^\beta u^\alpha n_a \, dV$$

$$\tilde{\mathcal{E}}(t) \leq \mathcal{F}(\tilde{\mathcal{E}}(0))$$

Symmetric hyperbolic systems

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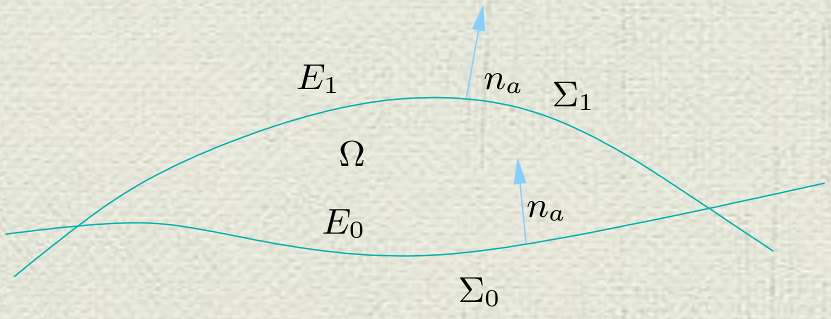


Symmetric hyperbolic systems

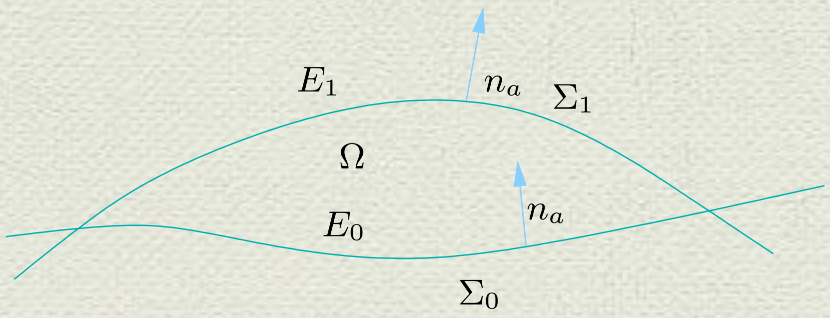
$$\int_\Omega \nabla_a J^a dV = \int_{\Sigma_t} J^a n_a dS - \int_{\Sigma_0} J^a n_a dS$$

$$\begin{aligned}
\mathcal{E}(t) &= \mathcal{E}(0) + \int_\Omega H_\alpha{}^A J_A u^\alpha dV \\
&\leq \mathcal{E}(0) + \int_0^t |F|^2 dV + \int_0^t \int_{\Sigma_{\tilde{t}}} \mathcal{E}(\tilde{t})\, dS\, d\tilde{t}
\end{aligned}$$

Linear case, quasilinear case more complicated (Sobolev)

Symmetric hyperbolic systems

$$\int_\Omega \nabla_a J^a dV = \int_{\Sigma_t} J^a n_a dS - \int_{\Sigma_0} J^a n_a dS$$

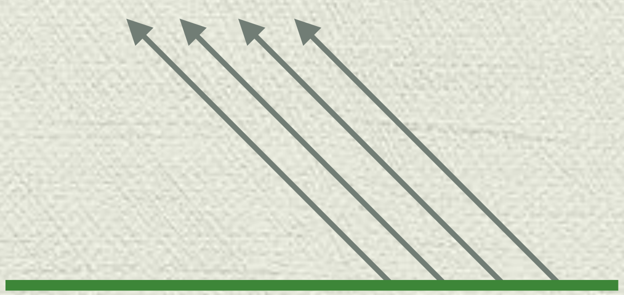


$$\begin{aligned}\mathcal{E}(t) &= \mathcal{E}(0) + \int_\Omega H_\alpha{}^A J_A u^\alpha dV \\ &\leq \mathcal{E}(0) + \int_0^t |F|^2 dV + \int_0^t \int_{\Sigma_{\tilde{t}}} \mathcal{E}(\tilde{t})\, dS\, d\tilde{t}\end{aligned}$$

Linear case, quasilinear case more complicated (Sobolev)

Energy norm: examples

$$l^a \nabla_a u = 0, \quad l^a = (1, -1) \qquad\qquad u_t = u_x$$



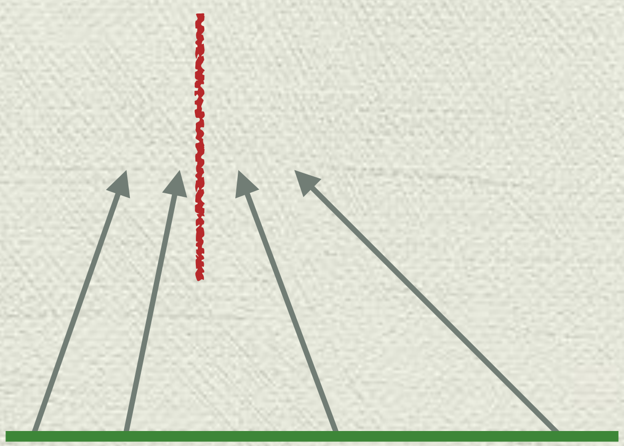
$$\mathcal{E}(t) = \int_{\Sigma_t} u(t,x)^2 dx$$

$$\frac{d}{dt}\mathcal{E}(t) = 2\int_{\Sigma_t} u u_x \, dx = \int_{\Sigma_t} (u^2)_x \, dx = u^2(t,b) - u^2(t,a) = 0$$

$$\mathcal{E}_1(t) = \int_{\Sigma_t} \{u^2 + u_x^2\} dx$$

Energy norm: examples

$$l^a \nabla_a u = 0, \quad l^a = (1, -u) \qquad u_t = u u_x$$



$$\frac{d\mathcal{E}(t)}{dt} = \int_{\Sigma_t} 2 u u_t \, dx = \int_{\Sigma_t} 2 u^2 u_x \, dx = \int_{\Sigma_t} \frac{2}{3} (u^3)_x \, dx = 0$$

Energy norm: examples

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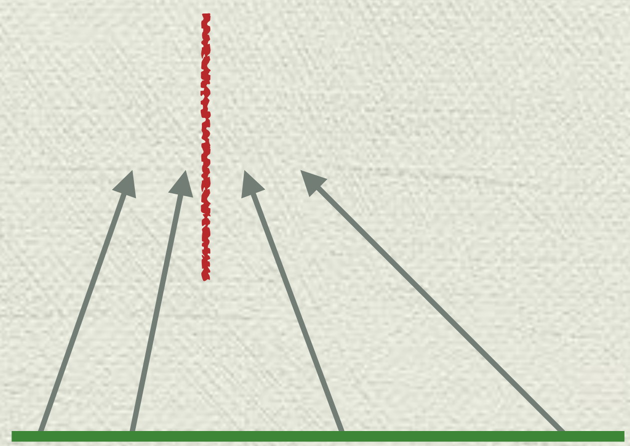

$$\mathcal{E}(t) := \int_{\Sigma_t} u^2 dx$$

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$$u_t = uu_x \longrightarrow u_{xt} = (u_x)^2 + uu_{xx}$$

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$$\frac{d}{dt}\mathcal{E}_1(t) = \int_{\Sigma_t} 2u_x u_{xt}\, dx = \int_{\Sigma_t} 2u_x[(u_x)^2 + uu_{xx}]\, dx$$

$$\frac{d}{dt}\mathcal{E}_1(t) = \int_{\Sigma_t} [2(u_x)^3 + u(u_x)^2_x]\, dx = \int_{\Sigma_t} (u_x)^3\, dx$$

Energy norm: examples

$$u_{xxt} = 3(u_x)^2 u_{xx} + u u_{xxx}$$

Energy norm: examples

$$u_{xxt} = 3(u_x)^2 u_{xx} + u u_{xxx}$$

$$\frac{d}{dt}\mathcal{E}_2(t) = \int_{\Sigma_t} 2u_{xx}u_{xxt}\, dx = \int_{\Sigma_t} 2u_{xx}[3(u_x)^2 u_{xx} + u u_{xxx}]\, dx$$

$$\frac{d}{dt}\mathcal{E}_2(t) = \int_{\Sigma_t} 4(u_x)^2 (u_{xx})^2\, dx \le \max\,(u_x)^2 \int_{\Sigma_t} (u_{xx})^2\, dx \le \max\,(u_x)^2 \mathcal{E}_2(t)$$

Energy norm: examples

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$$\max\,(u_x)^2 \leq C\mathcal{E}_2(t)$$

Sobolev inq.

Energy norm: examples

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$$\frac{d}{dt}\mathcal{E}_2(t) = \int_{\Sigma_t} 4(u_x)^2(u_{xx})^2\,dx \leq \max\,(u_x)^2 \int_{\Sigma_t} (u_{xx})^2\,dx \leq \max\,(u_x)^2 \mathcal{E}_2(t)$$

$$\max\,(u_x)^2 \leq C\mathcal{E}_2(t) \qquad \frac{d}{dt}\mathcal{E}_2(t) \leq C\mathcal{E}_2(t)^2$$

Sobolev inq.

Symmetric hyperbolic systems

$$H_\beta{}^A A^a{}_{A\alpha} n_a > 0 \qquad\qquad H_\beta{}^A A^a{}_{A\alpha} \hat{n}_a > 0$$

$$H_\beta{}^A A^a{}_{A\alpha} (x n_a + y \hat{n}_a) > 0 \ \forall \ \ x, y > 0$$

The set of all co-vectors for which the above expression is positive is a cone. The cone of Cauchy planes. Its boundary is called the characteristic cone.

Causality is derived from this cone structure.

Symmetric hyperbolic systems

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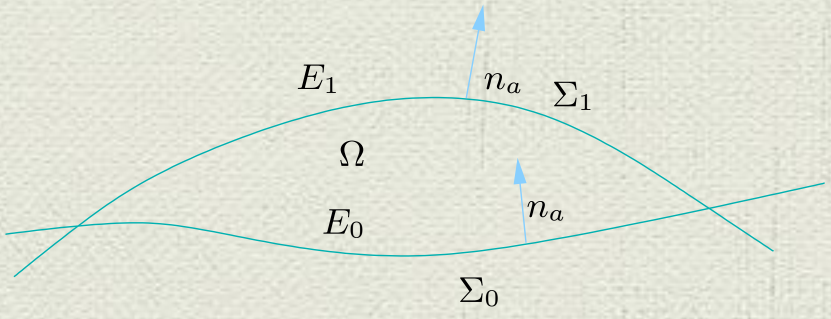


Causality is derived from this cone structure.

Symmetric hyperbolic systems

$$\mathcal{C} := \{k_a \in T_p^* \mid H_\alpha{}^A A^{\beta a}{}_{A\beta} k_a > 0\}$$ Causal Cone

The boundary of it is called the "characteristic cone".

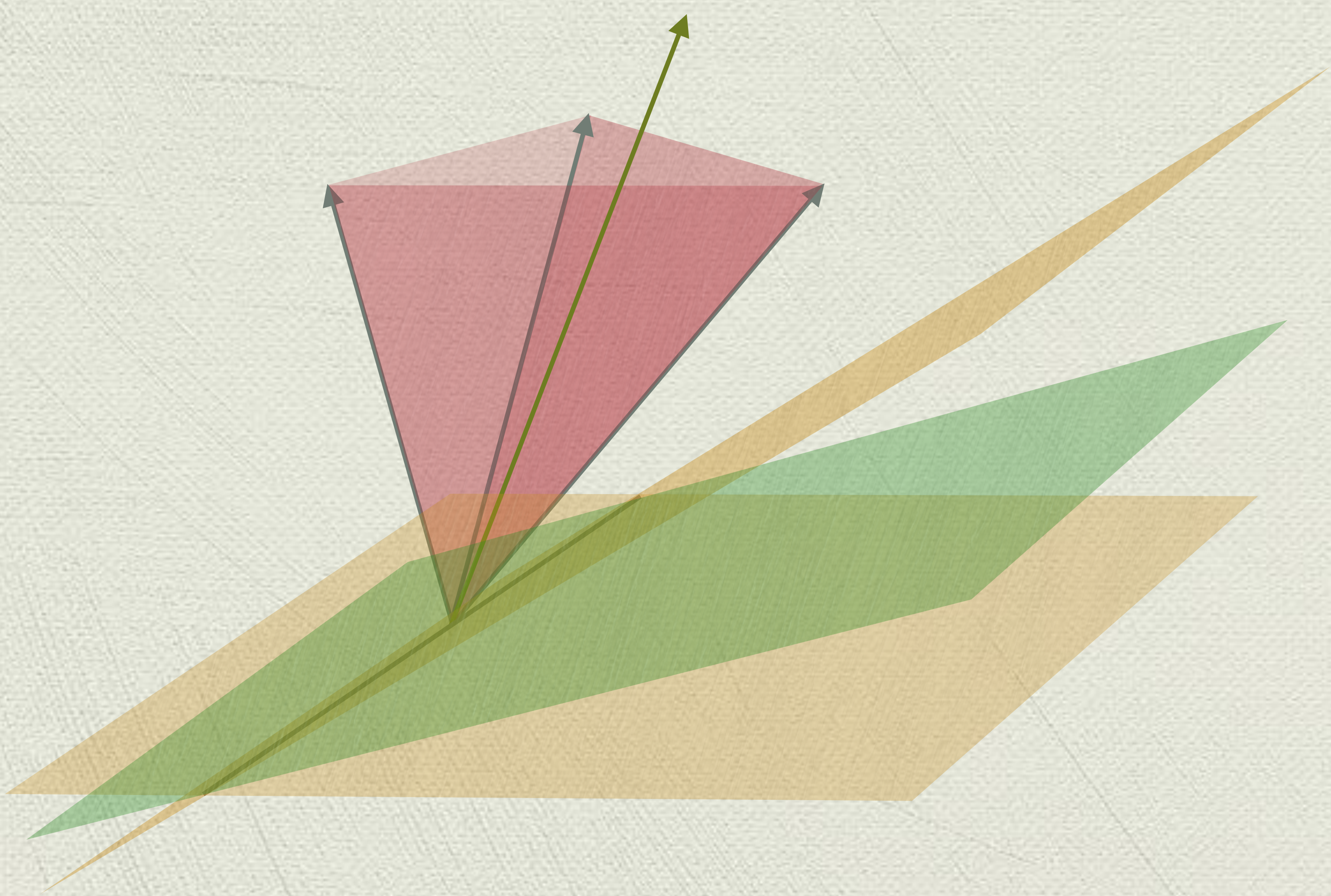
Co-cone of propagation vectors

$$\mathcal{C}^* := \{p^a \in T_p \mid p^a k_a \geq 0 \; \forall \; k_a \in \mathcal{C}\}$$

Symmetric hyperbolic systems

$$\mathcal{C} := \{k_a \in T^*_p \mid H_\alpha{}^A A^{\beta a}{}_{A\beta} k_a > 0\}$$

$$\mathcal{C}^* := \{p^a \in T_p \mid p^a k_a \geq 0 \,\forall\, k_a \in \mathcal{C}\}$$



Lecture II: Summary

- The first order evolution equations which have continuous dependence with respect to initial data (in normed spaces) and are stable under lower order perturbation are the strongly hyperbolic systems.
- Most physical system are symmetric hyperbolic, including some relativistic dissipative fluids
- Symmetrizers give a extremely good tool for causality (filled (real) cones and co-cones)
- But those cones might not be identical to the true causality properties of the complete physical system

Lecture III: Examples and Techniques, Open Problems, some simulations.

- Wave equation: The hard way
- Wave equation: The easy way
- An example of a weakly hyperbolic system: Force-Free in Euler Potentials.
- Some simulations of Force-Free: Jets
- Open problems

Example: Wave equation

Example: Wave equation

$$\nabla^a \nabla_a \phi = J$$

$$\begin{aligned} \nabla_a \phi &= l_a \\ \nabla_{[a} l_{b]} &= 0 \\ \nabla^a l_a &= J \end{aligned}$$

Symmetrizer: map from the space of equations to the space of unknowns.

$$A^{An}{}_{\alpha} \nabla_n \varphi^{\alpha} = J^A$$

Example: Wave equation

Example: Wave equation

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Symmetrizer: map from the space of equations to the space of unknowns.

$$A^{An}{}_{\alpha} \nabla_n \varphi^{\alpha} = J^A$$

$$A^n{}_a = \delta^n{}_a, \quad A_{ab}{}^{nc} = \delta_{[a}{}^n \delta_{b]}{}^c, \quad A^{na} = g^{na}$$

Example: Wave equation

When does there exists a k_a such that the above bilinear form is positive?

It exists if and only if u^a is time-like and positive definite.

In this case the cone of positive k_a is the set of all covectors coming from vectors in the positive light cone (full).

Example: Wave equation

$$H^a = v^a, \quad H^{cab} = 2g^{c[a}u^{b]}, \quad H^a = u^a$$

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When does there exists a k_a such that the above bilinear form is positive?

Prove this!

It exists if and only if u^a is time-like and positive definite.

In this case the cone of positive k_a is the set of all covectors coming from vectors in the positive light cone (full).

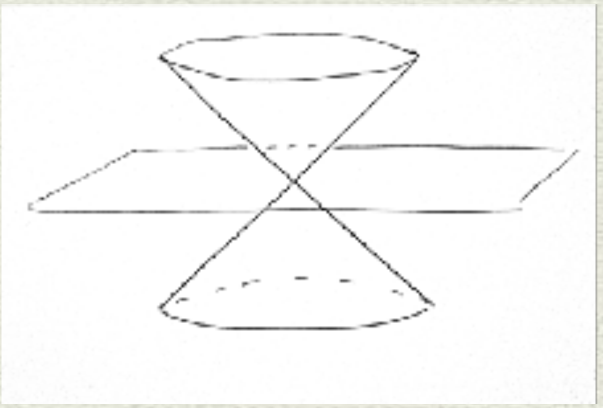
Pseudo-differential way

Wave Equation

$$\frac{\partial^2 \hat{\phi}}{\partial t^2} = -k^2 \hat{\phi} \qquad\qquad v := \partial_t \hat{\phi}, \qquad u := |k| \hat{\phi}$$

$$\frac{\partial}{\partial t} \begin{pmatrix} v \\ u \end{pmatrix} = |k| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

$$i\omega = \pm i|k|, \qquad \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$



System strongly hyperbolic!

Strong Hyperbolicity (covariant)

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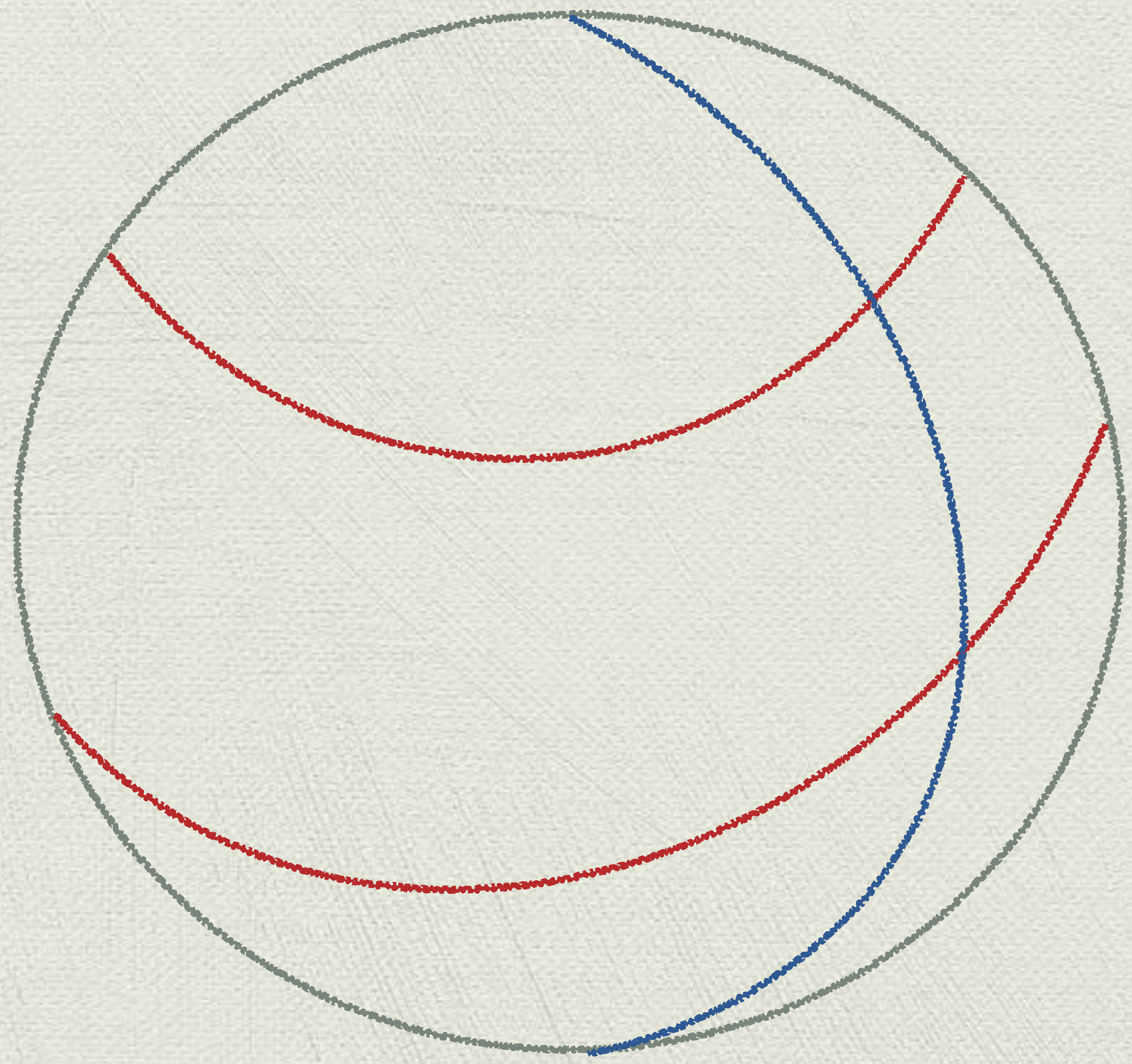
- $H_{\gamma\alpha}(k)A^{\alpha a}{}_{\beta}k_a$ is symmetric, and
- $H_{\gamma\alpha}(k)A^{\alpha a}{}_{\beta}\hat{k}_a$ is symmetric and positive.

Strong Hyperbolicity (covariant)

$$L(t)_a = tn_a + k_a \quad \text{such that } n_a \neq \alpha k_a$$



$$H_{\gamma\alpha}(k) A^{\alpha a}{}_{\beta} L(t)_a$$

On each intersection the dimension of the kernel is maximal

L(t) is not simply connected

Force-Free equations

Force-Free equations

They represent very diluted plasmas where only its high conductivity is important

$$T^{ab}_{T} = T^{ab}_{EM} + T^{ab}_{Pl}$$

$$\nabla_a T^{ab}_{T} = \nabla_a T^{ab}_{EM} + \nabla_a T^{ab}_{Pl} \approx \nabla_a T^{ab}_{EM} = F^{bc} \nabla_a F^{a}{}_{c}$$

$$\begin{aligned} F_{cb} \nabla_a F^{ab} &= 0 \\ \nabla_{[a} F_{bc]} &= 0 \end{aligned} \qquad \begin{aligned} G &:= F^{ab} F^{cd} \varepsilon_{abcd} = 0 \\ F &:= F^{ab} F_{ab} < 0 \end{aligned}$$

Force-Free equations

Force-Free equations

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$$\Downarrow$$

$$^*F_{ab} := -\frac{1}{2}\varepsilon_{abcd}F^{cd} = 2u_{[a}b_{b]}$$

Force-Free equations

Force-Free equations

$$
\begin{aligned}
F_{cb}\nabla_a F^{ab} &= 0 \\
\nabla_{[a}F_{bc]} &= 0
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\qquad
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\end{aligned}
$$

$$\downarrow$$

$$\mathcal{L}_u b^a = 0 \quad \longleftarrow \quad {}^*F_{ab} := -\tfrac{1}{2}\varepsilon_{abcd}F^{cd} = 2u_{[a}b_{b]}$$

Force-Free equations

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$$\begin{aligned} F_{cb}\nabla_a F^{ab} &= 0 \\ \nabla_{[a}F_{bc]} &= 0 \end{aligned}$$

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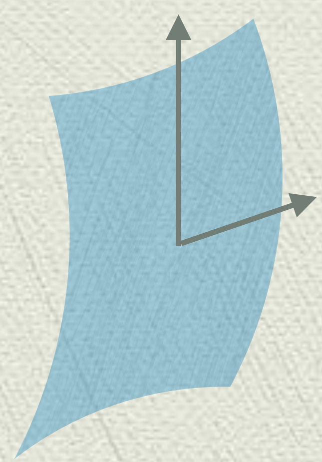
$${}^*F_{ab} := -\tfrac{1}{2}\varepsilon_{abcd}F^{cd} = 2u_{[a}b_{b]}$$

$$\mathcal{L}_u b^a = 0$$



Magnetic sheets

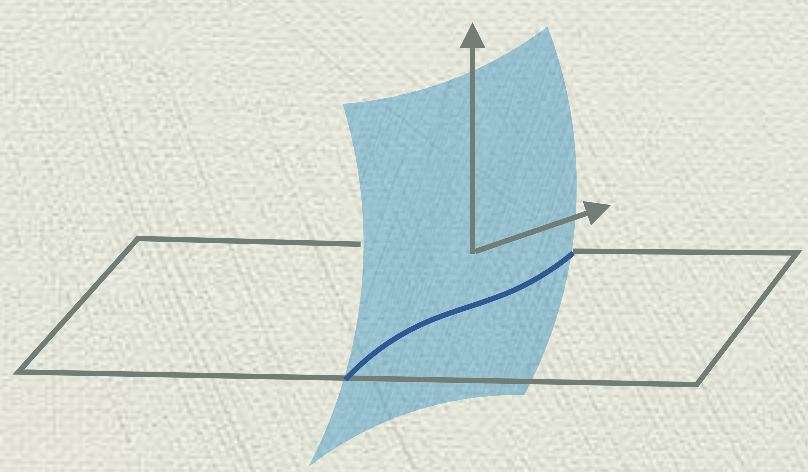
Force-Free equations

Force-Free equations

$$F_{cb}\nabla_a F^{ab} = 0$$
$$\nabla_{[a}F_{bc]} = 0$$

$$G := F^{ab}F^{cd}\varepsilon_{abcd} = 0$$
$$F := F^{ab}F_{ab} < 0$$

$$\mathcal{L}_u b^a = 0 \longleftarrow {}^*F_{ab} := -\frac{1}{2}\varepsilon_{abcd}F^{cd} = 2u_{[a}b_{b]}$$



Magnetic sheets

Force-Free equations

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$$\begin{aligned} F_{cb} \nabla_a F^{ab} &= 0 \\ \nabla_{[a} F_{bc]} &= 0 \end{aligned}$$

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Force-Free equations

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$$det(F^a{}_b) = G^2$$

Force-Free equations

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$$F := F^{ab}F_{ab} < 0$$

$$\nabla_a G = 0$$

$$det(F^a{}_b) = G^2$$

Symmetric Hyperbolic

Force-Free

$$\nabla_b F^{*ab} + \nabla^a \phi = \kappa n^a \phi$$ ⟵ Extended System

$$A^m_{A\alpha} \leftrightarrow \left\{ \left(F^{a[b} g^{c]m} \; , \; \frac{1}{2}\epsilon^{ambc} \; , \; F^{*bc} g^{am} \right) \; , \; (0 \; , \; g^{am} \; , \; 0) \right\}$$

$$\begin{aligned} \delta\hat{\Phi}^\alpha h_\alpha{}^A A^m_{A\beta} w_m \delta\Phi^\beta \quad &= \quad t^a P_a{}^b w_m \left[\hat{X}^{cm} X_{cb} + X^{cm} \hat{X}_{cb} - \frac{1}{2}(\hat{\mathbf{X}} \cdot \mathbf{X}) \delta^m_b \right] \\ &\quad -\frac{1}{2} t^a F^*_{ab} w_m \left[X^{bm} (\mathbf{F}^* \cdot \hat{\mathbf{X}}) + \hat{X}^{bm} (\mathbf{F}^* \cdot \mathbf{X}) \right] \\ &\quad -\frac{1}{2} k t^a w_a (\mathbf{F}^* \cdot \hat{\mathbf{X}})(\mathbf{F}^* \cdot \mathbf{X}) \\ &\quad + P_{ab} t^b w_m \left[\hat{X}^{*ma} \delta\phi + X^{*ma} \delta\hat{\phi} \right] - (t^a P_a{}^m w_m) \delta\phi\delta\hat{\phi} \end{aligned}$$

$$P_{ab} = [b_a b_b - b^2 u_a u_b]$$

F. Carrasco and OR

Pseudo-differential way

Force-Free in Euler Potentials

$$F_{cb}\nabla_a F^{ab} = 0$$
$$\nabla_{[a}F_{bc]} = 0$$

$$G := F^{ab}F^{cd}\varepsilon_{abcd} = 0$$

$$F_{ab} = \varepsilon^{ij}\nabla_a\phi_i\nabla_b\phi_j \ , \qquad A_a = \frac{1}{2}\varepsilon^{ij}\phi_i\nabla_a\phi_j$$

$$\varepsilon^{ij}\nabla_a\phi_k\nabla_c(\nabla^a\phi_i\nabla^c\phi_j) = 0, \qquad k = 1, 2,$$

$\phi_i = \phi_i^0$ Along the magnetic sheets

Pseudo-differential way

Force-Free in Euler Potentials

$$\partial_t U = \mathbb{A}\, U, \qquad U = \begin{pmatrix} \partial_t \hat{\phi}_i \\ |k|\hat{\phi}_i \end{pmatrix}, \quad i = 1, 2.$$

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & \frac{-k_1^2 - k_3^2}{|\vec{k}|} & -\frac{(\vec{\ell}_1 \cdot \vec{k})(\vec{\ell}_2 \cdot \vec{k})}{|\vec{k}| G_{22}} \\ 0 & 0 & -\frac{(\vec{\ell}_1 \cdot \vec{k})(\vec{\ell}_2 \cdot \vec{k})}{|\vec{k}| G_{11}} & \frac{-k_2^2 - k_3^2}{|\vec{k}|} \\ |\vec{k}| & 0 & 0 & 0 \\ 0 & |\vec{k}| & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} \ell_{ai} &:= \nabla_a \bar{\phi}_i, \\ G_{ij} &:= \ell_{ai} g^{ab} \ell_{bj} \end{aligned}$$

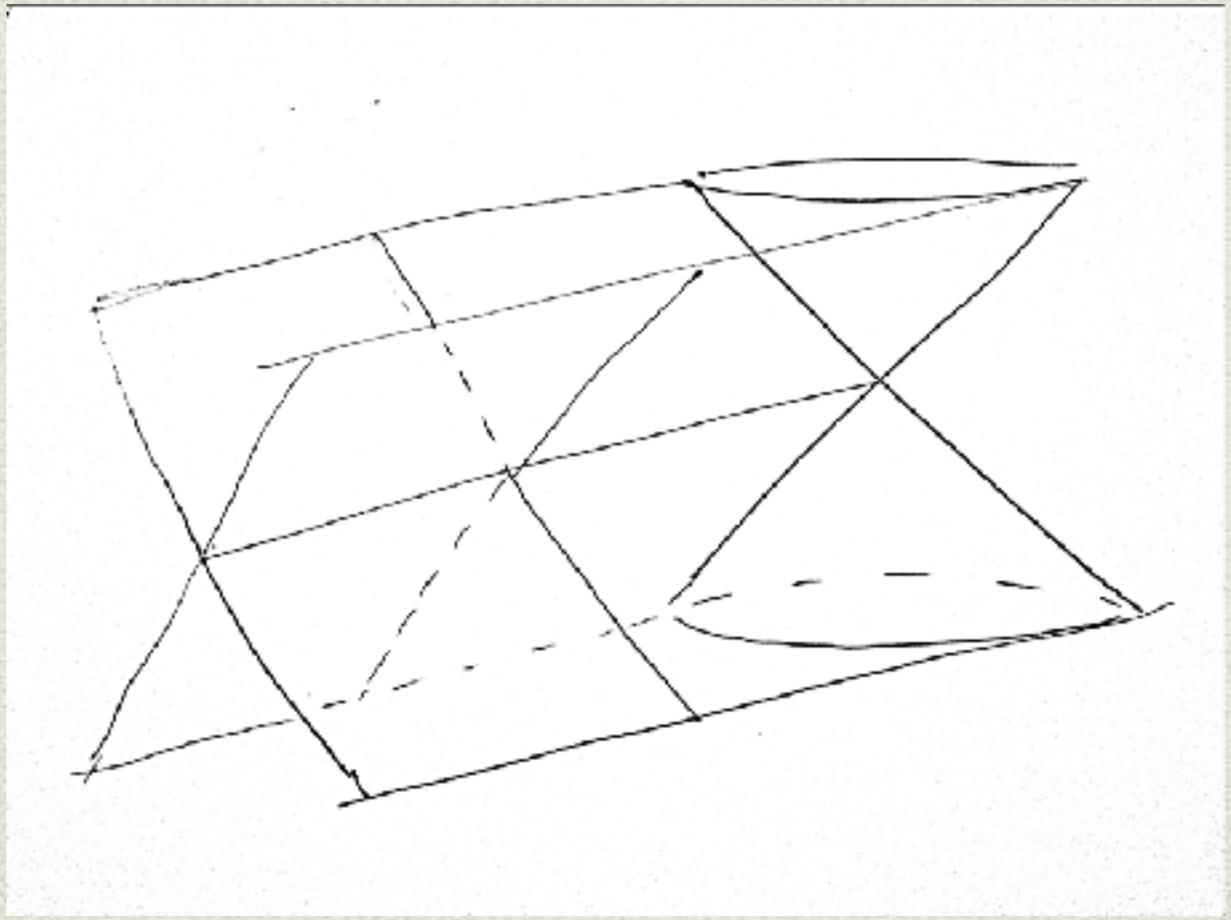
System is Ill Posed! $\quad k_0 = k_3 = 0$

Pseudo-differential way

Force-Free in Euler Potentials

$$(g^{ab}k_a k_b)([g^{cd} - G^{-1ij}\ell_i^c \ell_j^d]\, k_c k_d) = 0$$

$$k_0 = k_3 = 0$$

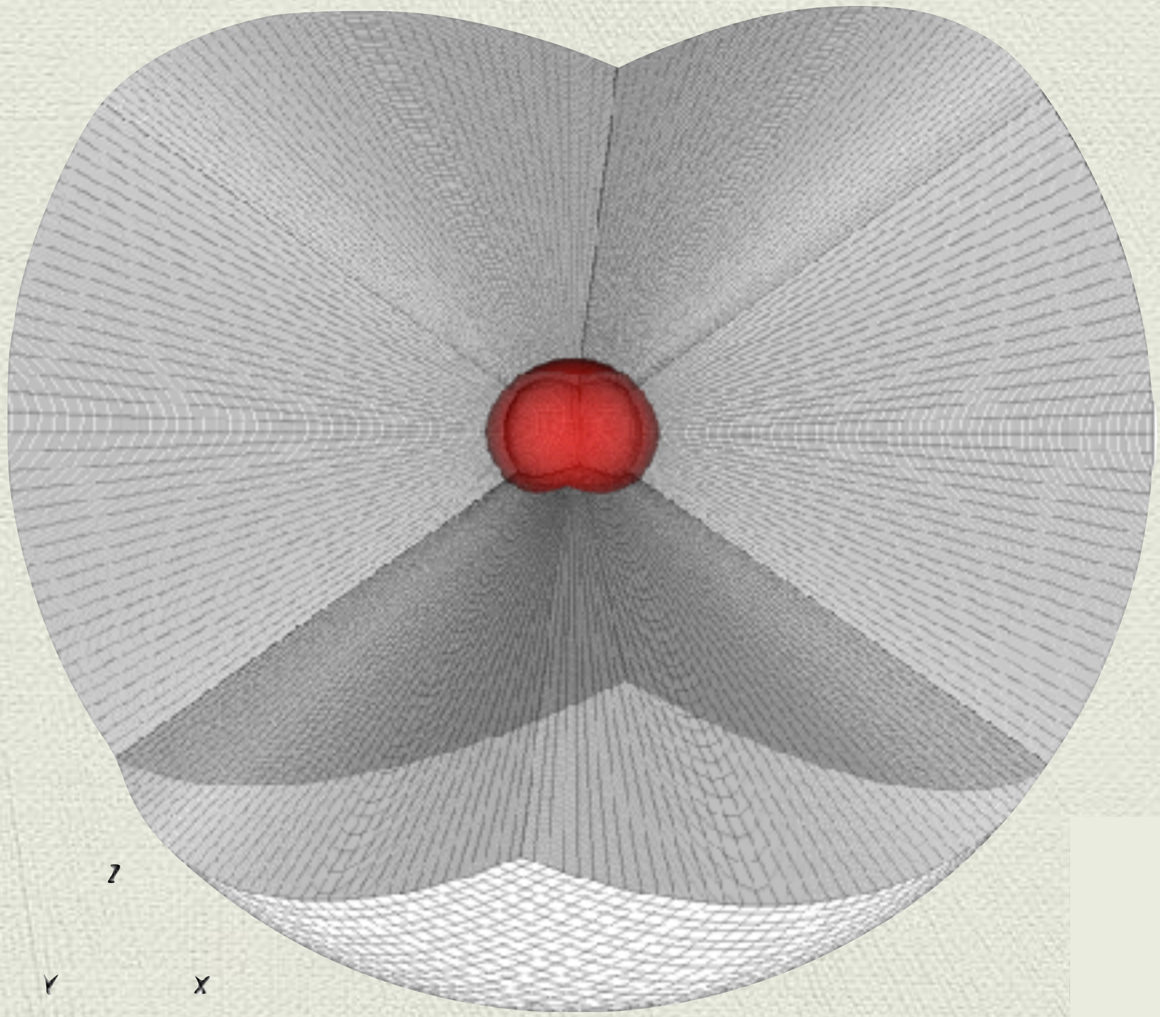


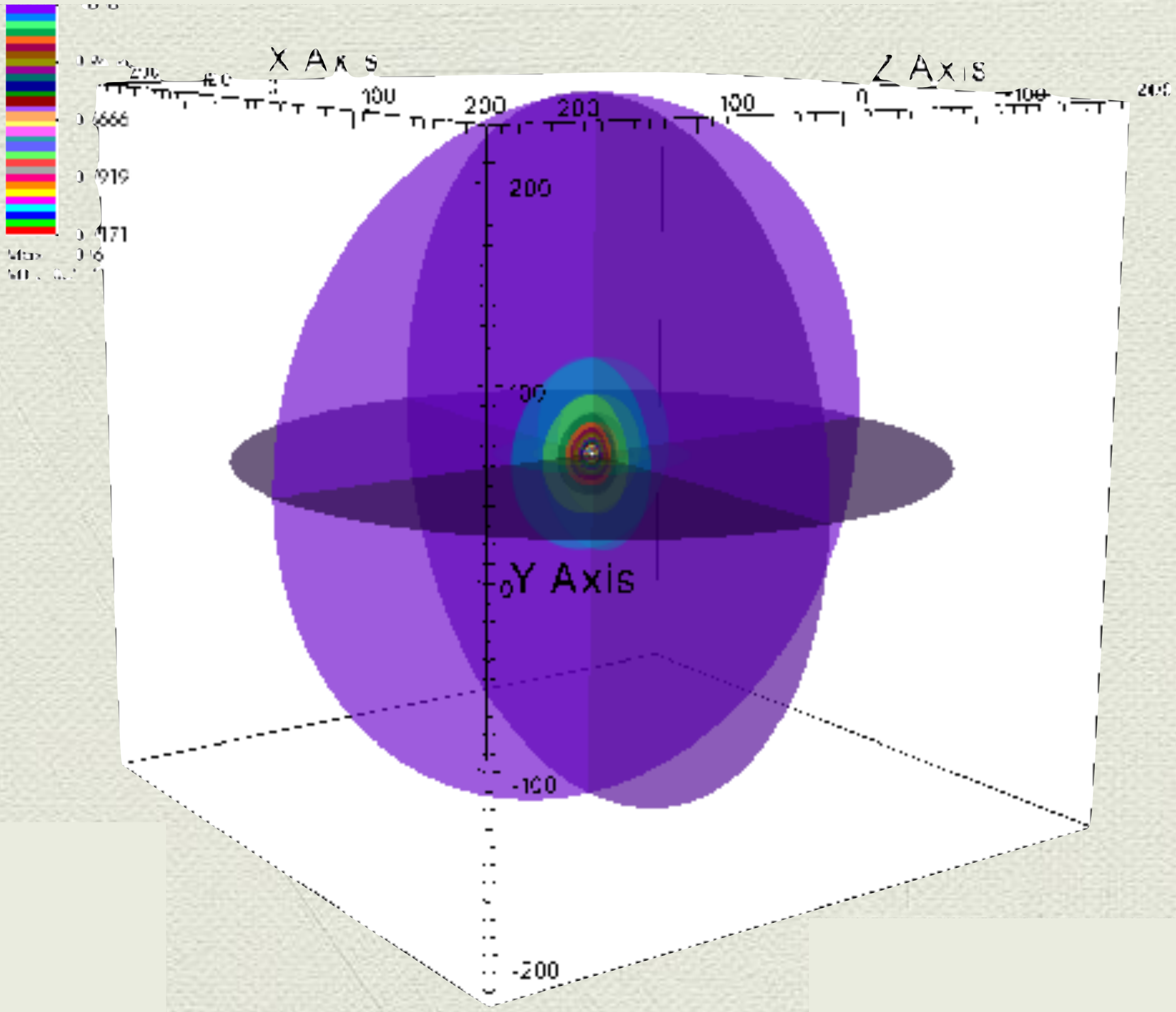
System is Ill Posed!

OR, and Marcelo E. Rubio

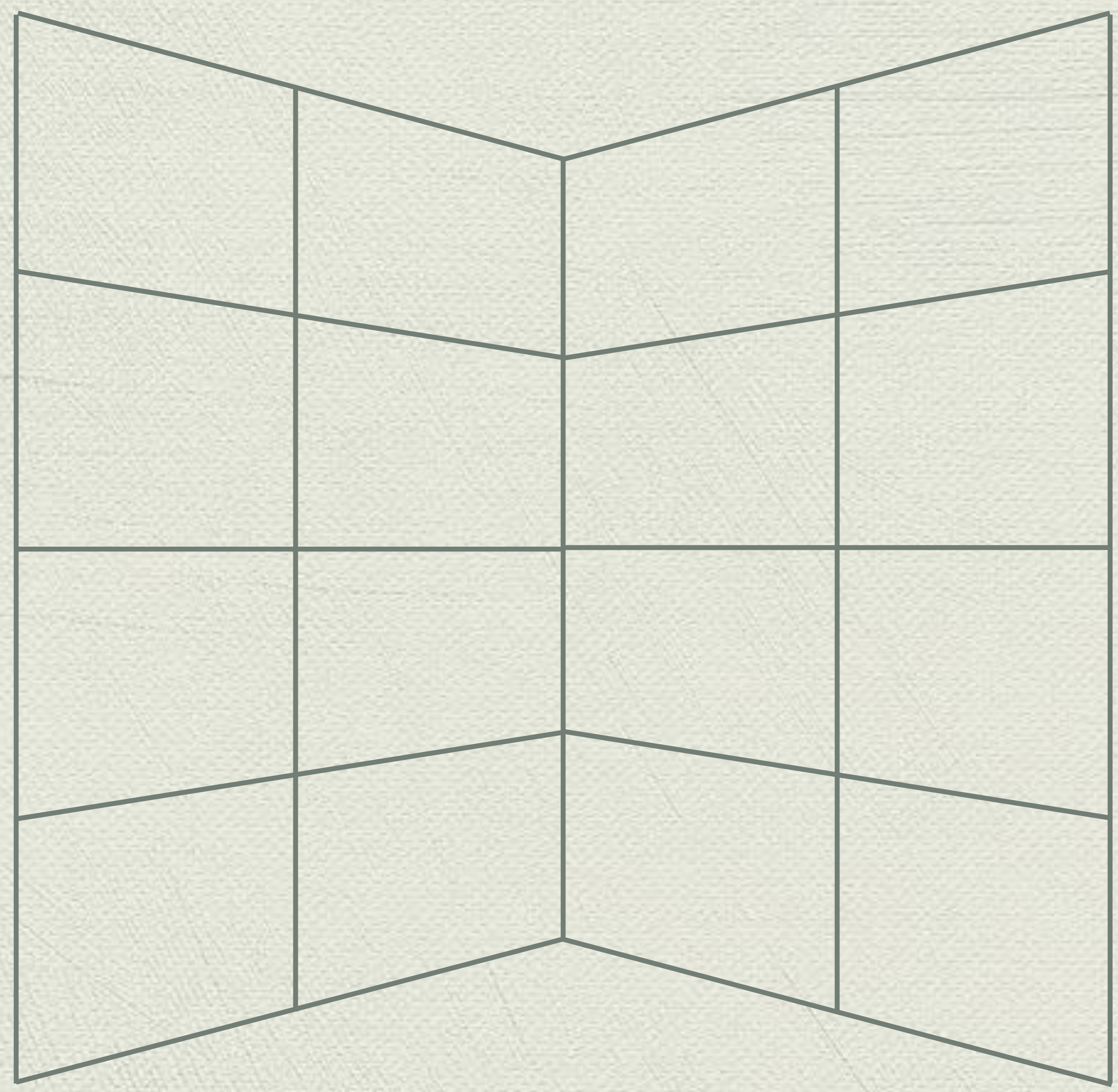
Force Free: theory and numerics

F. Carrasco Tesis

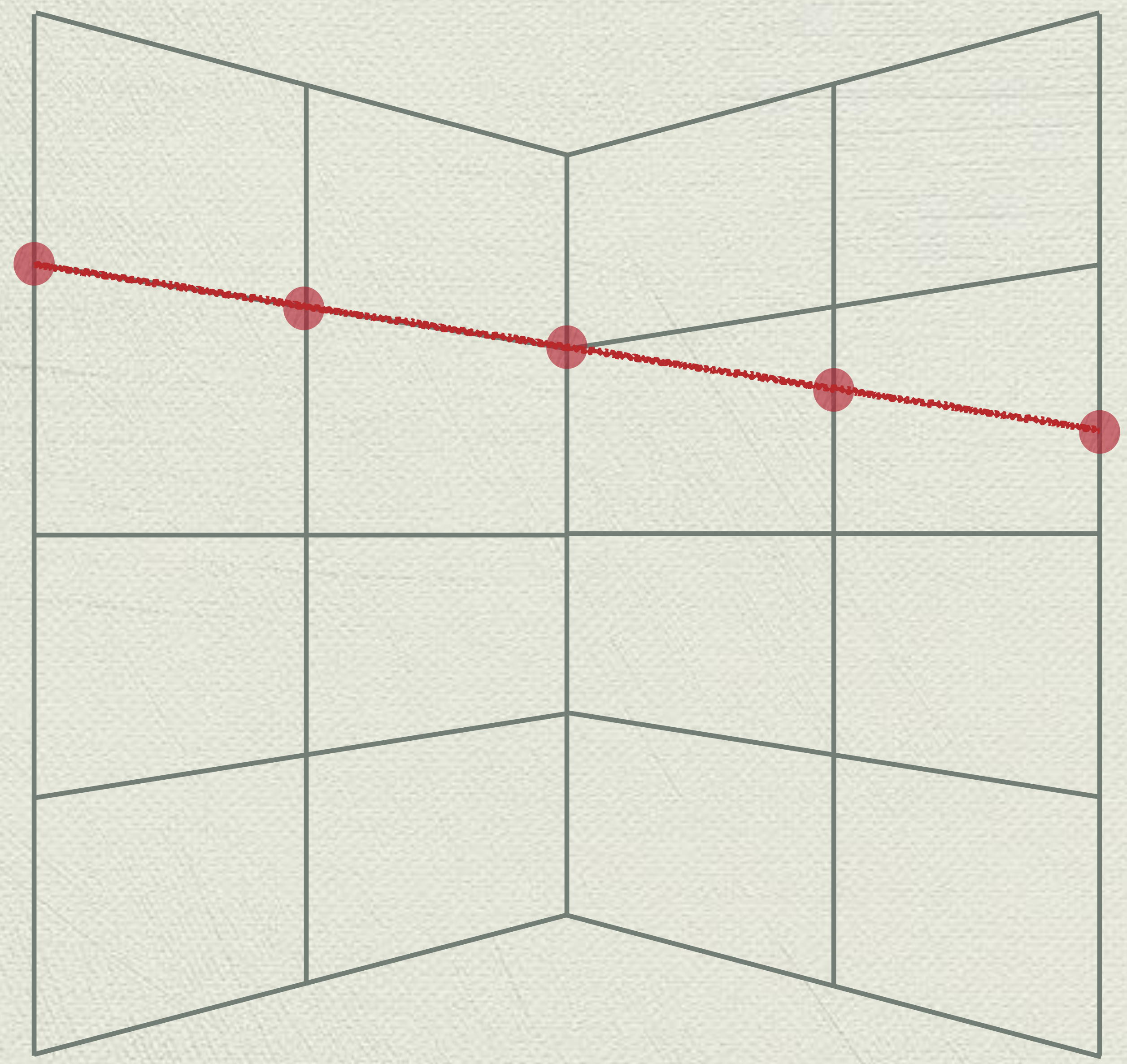




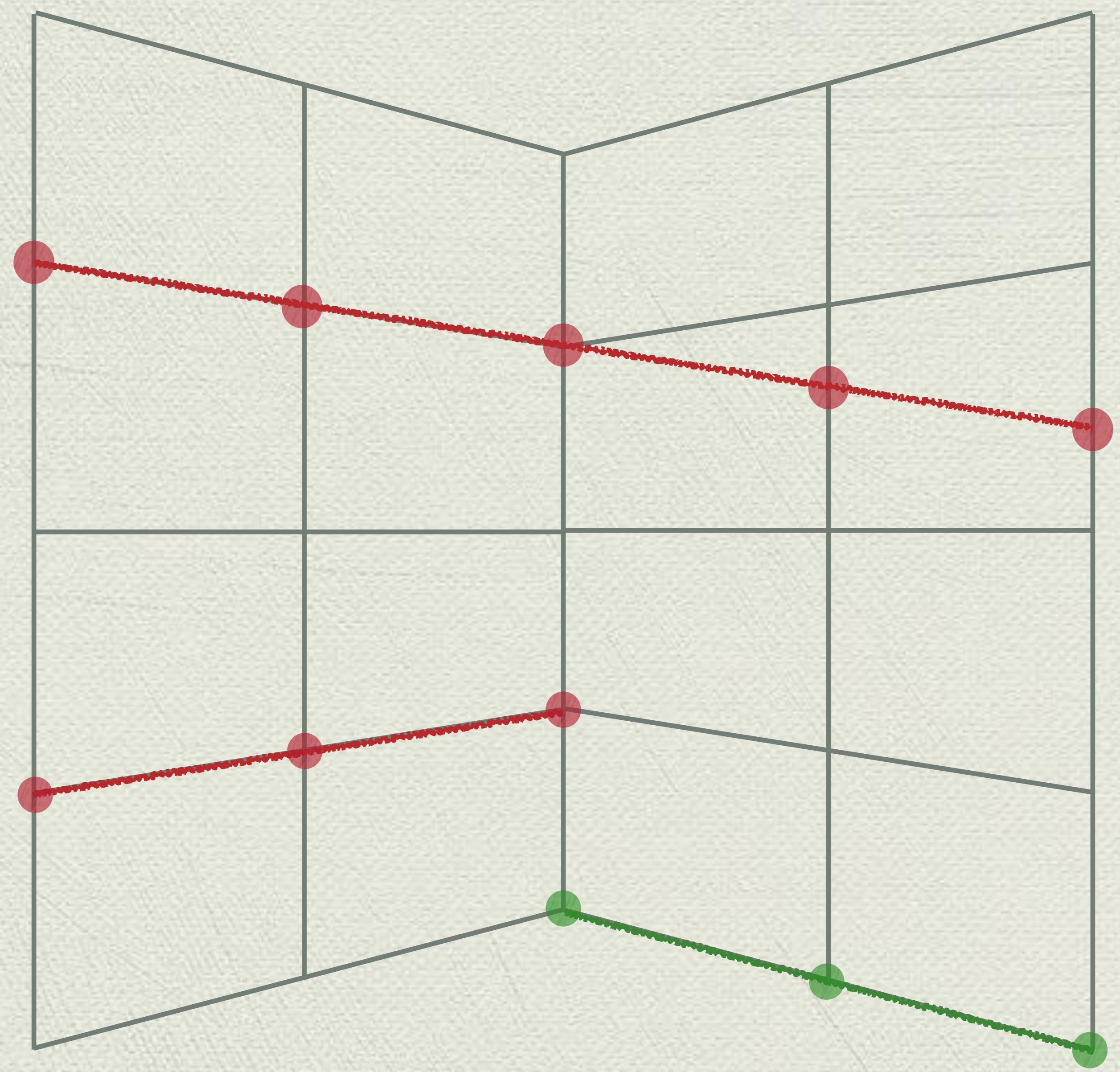
Touching grids



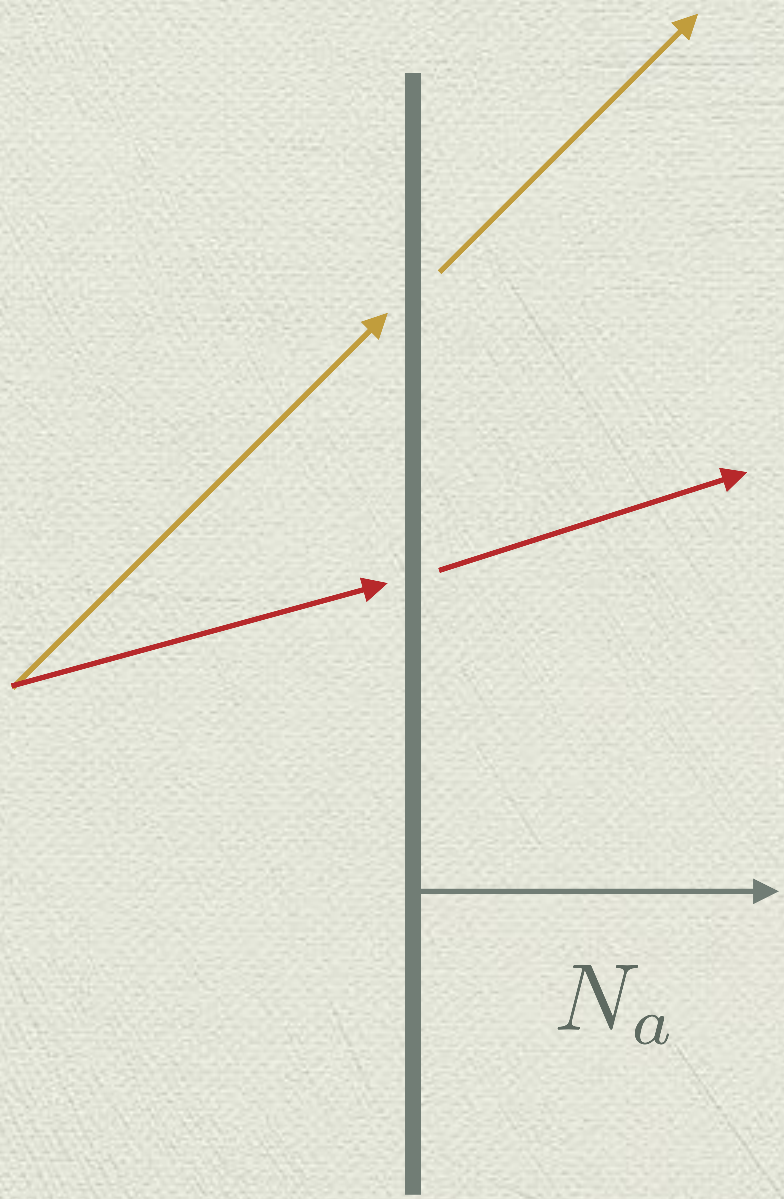
Touching grids



Touching grids

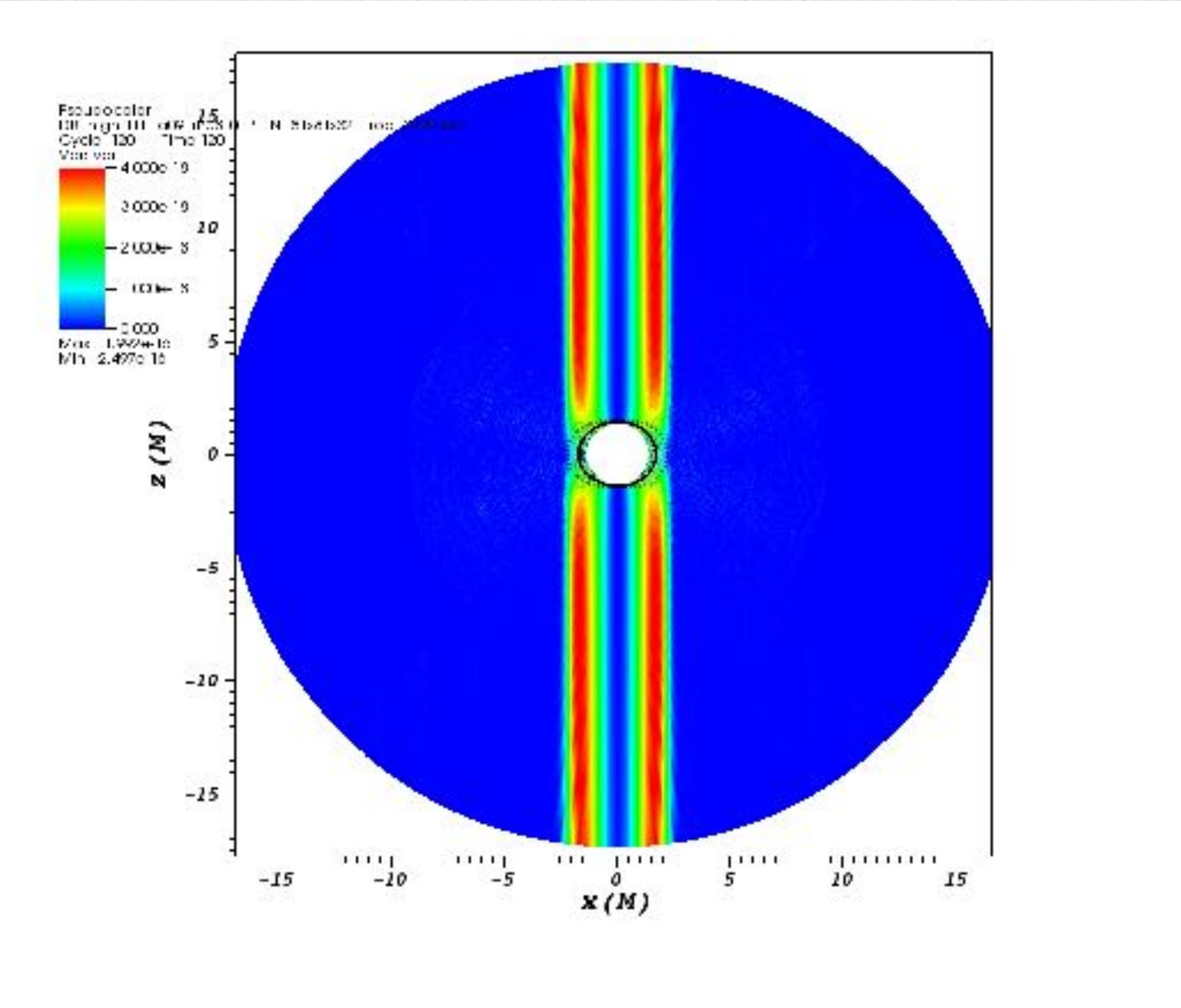


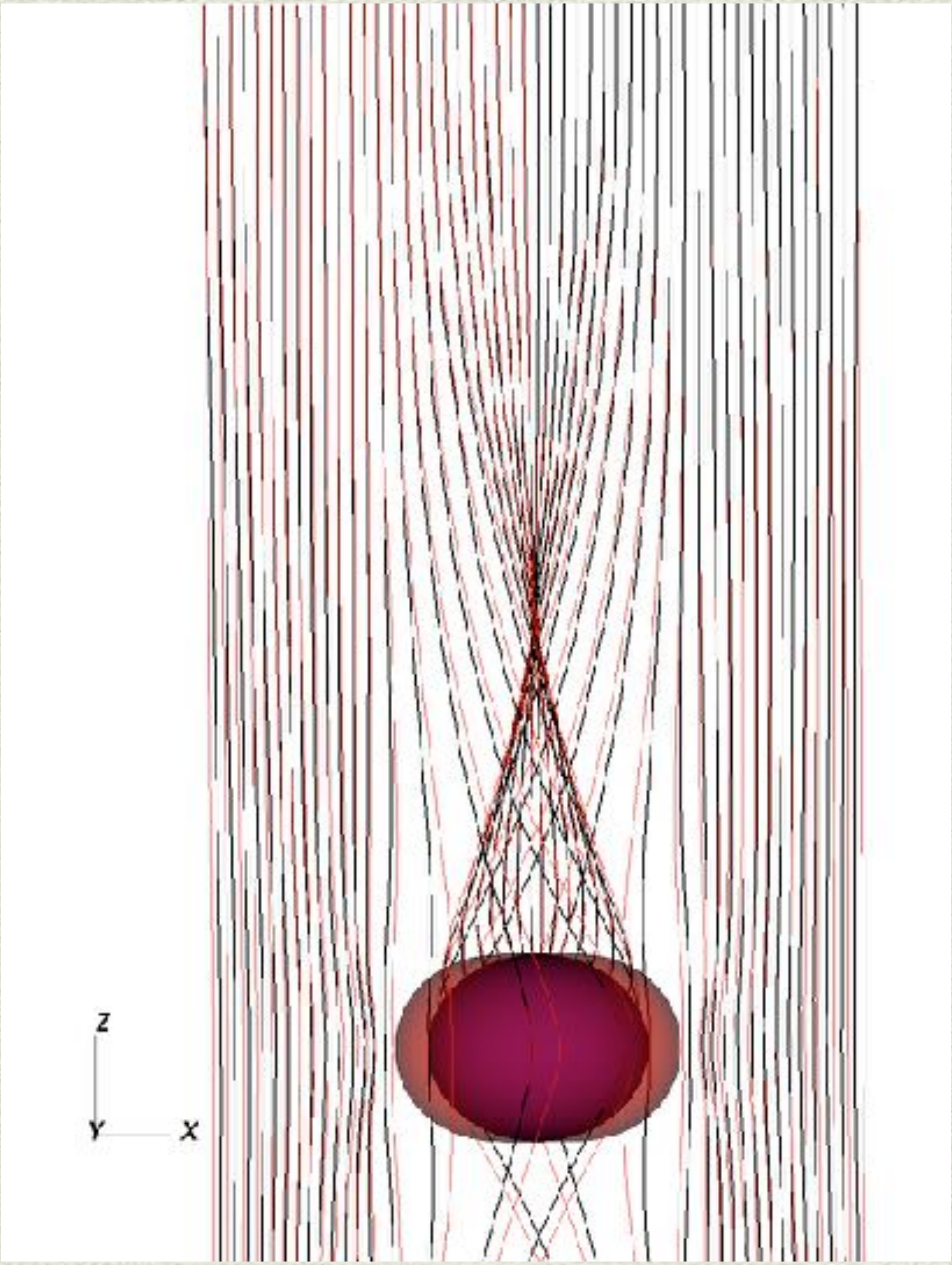
Touching Grids



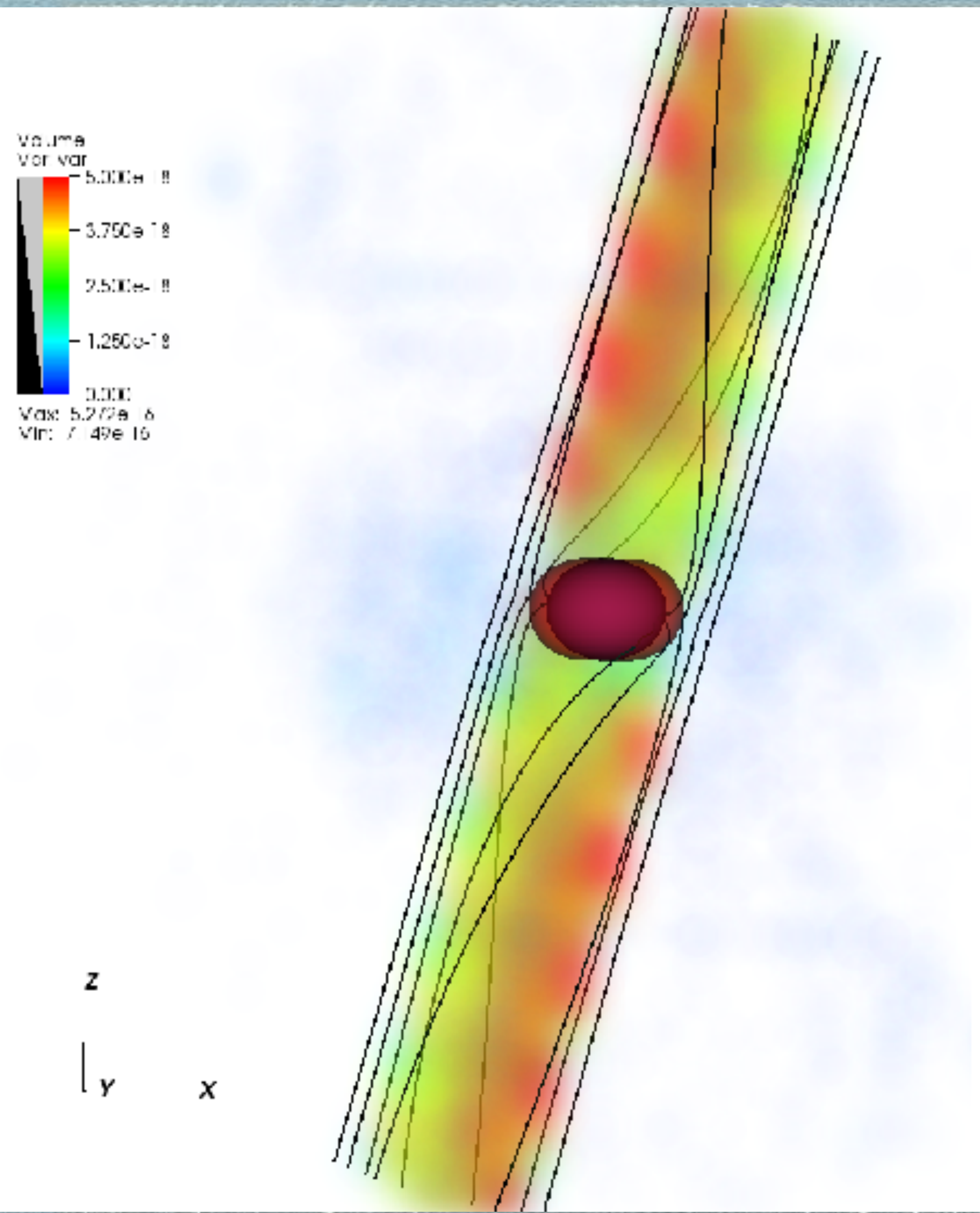
Incoming-Outgoing mode decomposition

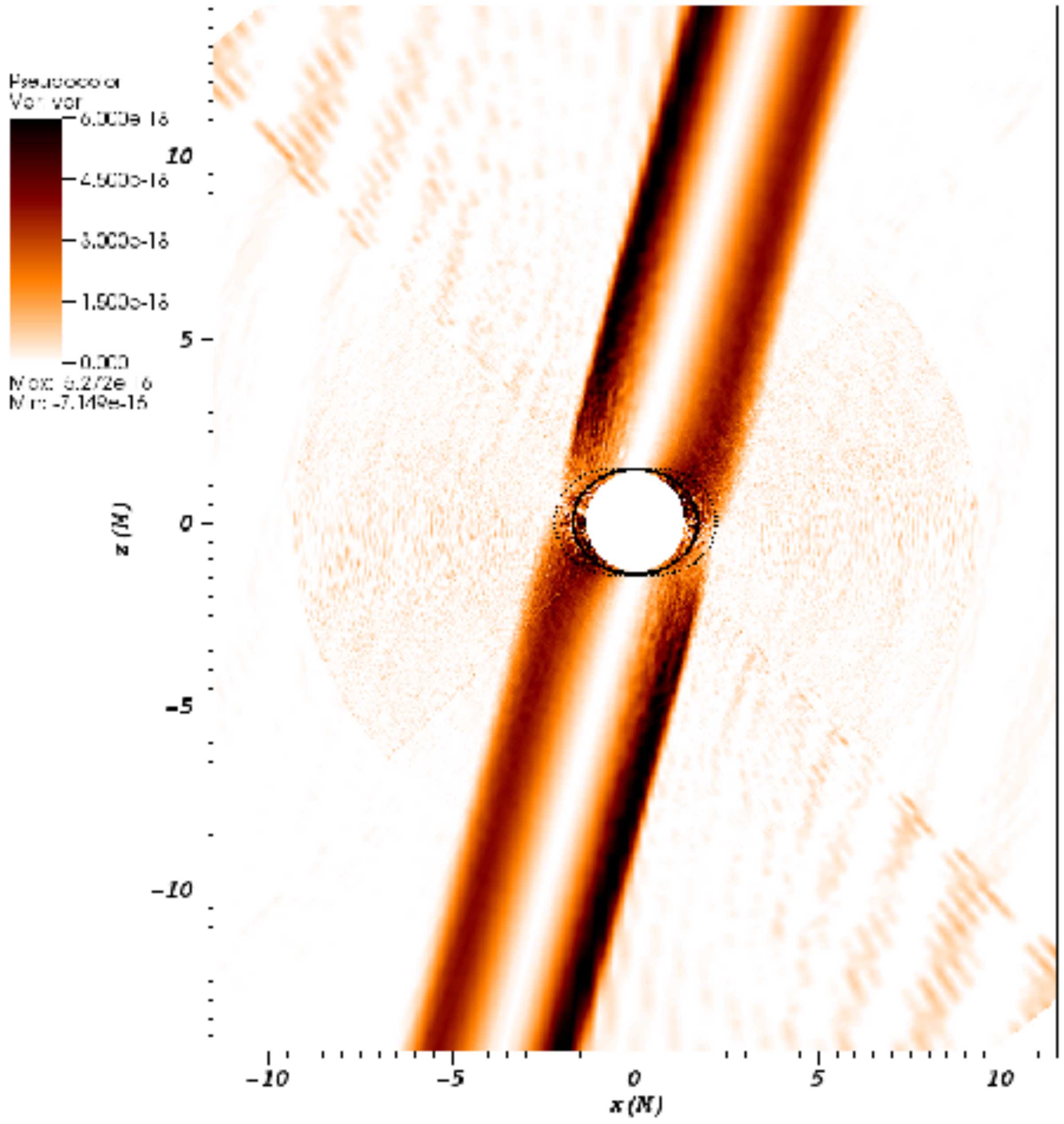
Jets in Force Free

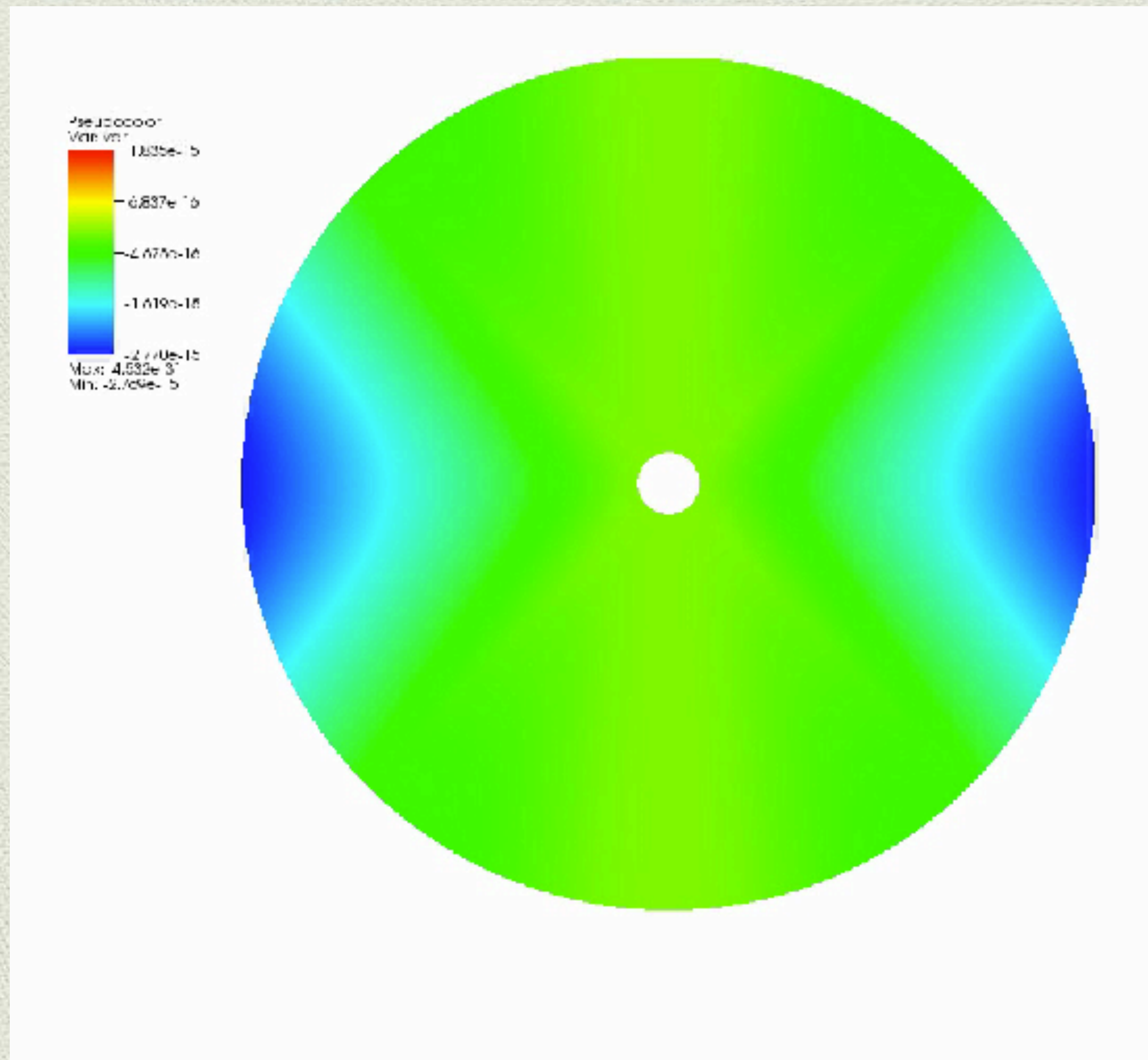


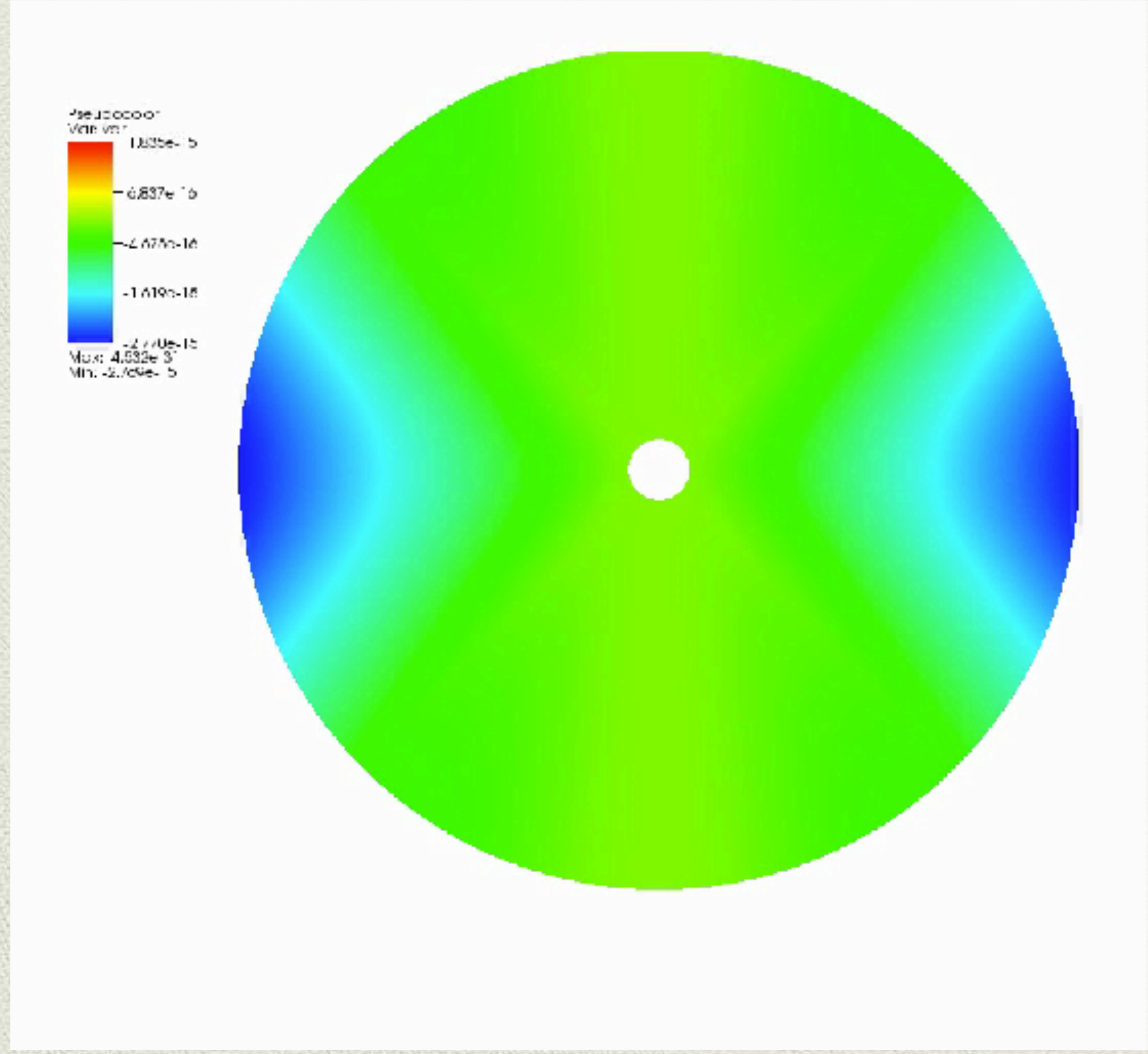


Jets in Force Free





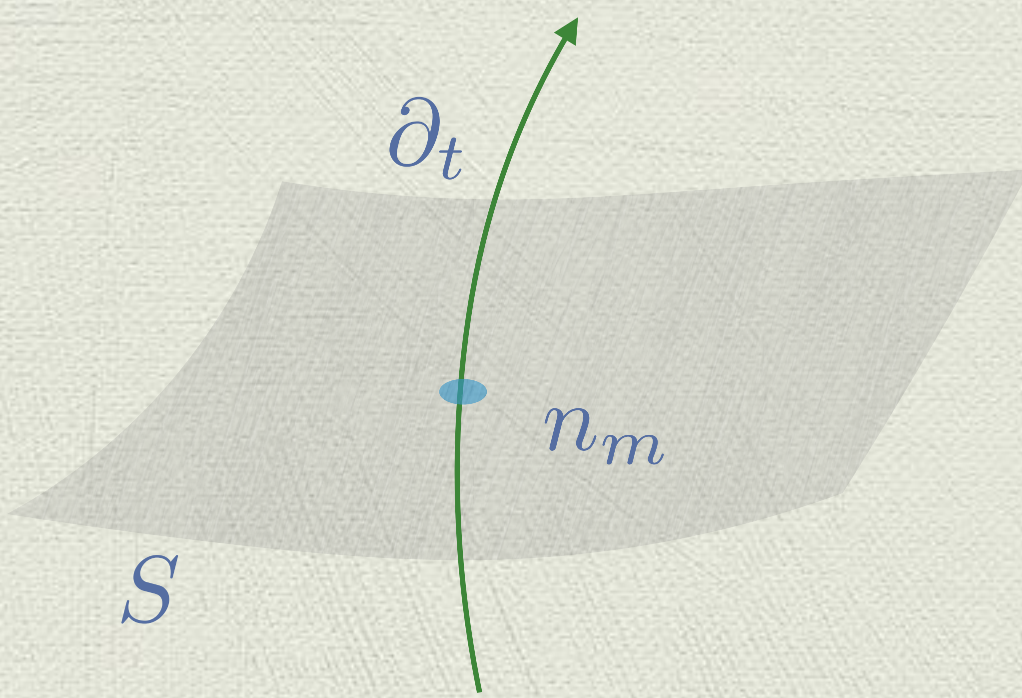




Constraints

$$A^{An}{}_{\alpha} \nabla_n \varphi^{\alpha} = J^A$$

$$C^{\Gamma m}{}_{A} \quad \text{such that} \quad C^{\Gamma(m}{}_{A} A^{An)}{}_{\alpha} = 0$$



$$n_m \partial_t^m = 1$$

$$\nabla_n \varphi = n_n \partial_t \varphi + D_n \varphi$$

$$n_m \, C^{\Gamma m}{}_{A} A^{An}{}_{\alpha} \nabla_n \varphi^{\alpha}$$

The above linear combination of equations only have derivatives at S

Open Problems:

- Find a proof of constraint propagation (strongly hyperbolic) in a covariant setting. Enlarged systems hyperbolicity.
- Find criteria to decide when a general first order system admits a symmetrizer
  - There is already a criteria which ensures in certain cases there is no symmetrizer (F. Ábalos) is it the other way around?
  - There is a conjecture on a formula relating the dimension of certain kernels, there is a symmetrize if and only if it is satisfied.
- Which are the most general well posed systems in normed spaces? There are weakly hyperbolic systems which are trivially well posed, can we characterize them?
- Relation between Leray's and Fridrich's hyperbolicity

Gracias por su atención!