Revisiting the analytic continuation method for computing quasinormal modes

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Based on **Phys. Rev. D 103, 045001 (2021)**

In collaboration with J.Fabris and A. Saa

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Quasinormal Modes-I

1. The quasinormal mode analysis consists in looking for solutions of the form $\Psi(t,x) = e^{-i\omega t}u(x)$ of an $(1+1)$-dimensional wave equation ($\Box_\eta \Psi = V\Psi$), with $u(x)$ obeying a Schrödinger-like equation [Berti +, '09], [Konoplya& Zhidenko, '11]:

$$\left(-\epsilon^2 \frac{d^2}{dx^2} + V(x)\right) u = \omega^2 u. \qquad (1)$$

2. The domain of $u(x)$ depends on the specific problem, e.g., $\mathbb{R}$, $\mathbb{R}_+$.

3. Here ϵ is an intrinsic scale parameter of the problem.

4. The potential $V(x)$ represents a positive barrier which vanishes at $x \to \pm\infty$ $[Dom(u) = \mathbb{R}]$.

Quasinormal Modes-II

1. The QNM frequencies $\omega \in \mathbb{C}$ such that the solutions of (1) behave as outgoing waves at infinity and ingoing ones at $-\infty$:

$$u = \begin{cases} e^{\frac{i\omega x}{\epsilon}}, & x \to +\infty \\ e^{-\frac{i\omega x}{\epsilon}}, & x \to -\infty, \end{cases} \tag{2}$$

2. The QNM modes will be exponentially suppressed over time for $\mathbb{I}m(\omega) < 0$.

3. Physically speaking, the QNMs depend only on the parameters of the system and the information of the initial perturbation is washed away. Hit gently a glass of wine with a knife!!!

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Quasinormal Modes-III

1. There are several strategies to solve a QNM problem in pratice. Just to mention a few of them:
   1.1 The WKB approach-[Iyer, ′87], [Konoplya, ′03]
   1.2 The continued fraction method (CFM) [Leaver ′85]
   1.3 Numerical integration of the non-linear differential equation [Chandrasekhar& Detweiler ′75], [Cardoso, Lemos, Yoshida ′03], [Berti&Kokkotas,′03].
   1.4 The asymptotic iteration method (AIM) [Cho +, ′10].
   1.5 The **analytic continuation method** [Blome&Mashhoon ′84, Ferrari&Mashhoon, ′84].
   1.6 Pseudospectral methods [Jaramillo+ ′04, Jansen ′17]
   1.7 Monodromies/Stokes lines [Natário + ′04]

ACM-I

1. The analytical continuation method was introduced by Blome-Ferrari-Mashhoon several decades ago [Blome&Mashhoon '84, Ferrari&Mashhoon, '84].

2. The method consists in a formal map between the QNM solutions of (1) and the bounded states of the quantum mechanical problem corresponding to the case of the inverted potential barrier $-V(x)$ with the following Schrödinger operator,

$$\mathcal{H} = -\hbar^2 \frac{d^2}{dx^2} - V(x). \tag{3}$$

3. The bounded states of $\mathcal{H}$ will decay exponentially, $\psi \propto e^{\mp\sqrt{-E}\frac{x}{\hbar}}$ as $x \to \pm\infty$, where $\mathcal{H}\psi = E\psi$.

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ACM-II

1. We can map the quantum mechanical bound states of $\mathcal{H}$ in the QNM of (1) by demanding that

$$\omega^2 = -E(\hbar = i\epsilon). \tag{4}$$

2. Starting with the QNM modes of a potential barrier, one can obtain the associated bounded states for the inverted potential with the Schrödinger operator $\mathcal{H}$ by setting $\epsilon = -i\hbar$ in (1) and reversing (4).

3. We cannot assure that all solutions of the QNM problem may be found by the analytical continuation in the parameters ϵ and $\hbar$. **This needs to be verified!!!**

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An exact example-I

1. Our starting point is to consider the infinite potential barrier on $\mathbb{R}_+$ given by

$$V(x) = \frac{V_0}{\sinh^2 x} + \frac{V_1}{\cosh^2 x}, \tag{5}$$

where $V_0 > 0$.

2. The master equation of the QNMs (1) for the generalized Pöschl-Teller potential (5) is an exactly integrable system [Alhassid+, '83 PRL], [Du+, '04], [Cardona& Molina '17].

An exact example-II

1. For $c \notin \mathbb{Z}$, the general solution is

$$u = z^{\alpha}(1-z)^{\beta}\,{}_2F_1\left(a, b; c; z\right), \tag{6}$$

where $z \in (0,1]$ such that $z = 1/\cosh^2 x$. The coeffients are given by

$$\begin{aligned} a &= \alpha+\beta+\gamma_{+}, \quad b = \alpha+\beta+\gamma_{-}, \quad c = 2\alpha+1, && (7)\\ \alpha &= \pm\frac{i\omega}{2}, \qquad \beta = \frac{1}{4}\left(1 \pm \sqrt{1+4V_0}\right), && (8)\\ \gamma_{\pm} &= \frac{1}{4}\left(1 \pm \sqrt{1-4V_1}\right). && (9) \end{aligned}$$

2. As $z \to 0^{+}$ we select $\alpha = -\frac{i\omega}{2}$ to get an outgoing wave at infinity.

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An exact example-III

1. To impose the second QNM boundary condition, $\lim_{x\to 0^+} u = 0$, we need to determine the behavior of our solution for $z \to 1^-$.

2. We make use of the following identity [Abramowitz],

$$
{}_2F_1\left(a,b;c;z\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left(a,b;a+b-c+1;1-z\right) \qquad (10)
$$
$$
+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\,{}_2F_1\left(c-a,c-b;c-a-b+1;1-z\right).
$$

3. Near $x=0$, the asymptotic form of $u(x)$ reduces to

$$
u = Ax^{2\beta}\left[1+\mathcal{O}(x^2)\right] + Bx^{1-2\beta}\left[1+\mathcal{O}(x^2)\right], \qquad (11)
$$

where

$$
A = \frac{\Gamma(1-i\omega)\Gamma\left(\frac{1}{2}-2\beta\right)}{\Gamma\left(1-\beta-\gamma_+-\frac{i\omega}{2}\right)\Gamma\left(1-\beta-\gamma_--\frac{i\omega}{2},\right)}, \qquad (12)
$$

$$
B = \frac{\Gamma(1-i\omega)\Gamma\left(2\beta-\frac{1}{2}\right)}{\Gamma\left(\beta+\gamma_+-\frac{i\omega}{2}\right)\Gamma\left(\beta+\gamma_--\frac{i\omega}{2}\right)} = T^{-1}. \qquad (13)
$$

An exact example-IV

1. To keep the solution regular as $x \to 0^+$, the Gamma function in the second term of (11) must have some poles at the QNM frequencies,

$$\omega_\pm = -i\left(2n + 1 + \sqrt{V_0 + \frac{1}{4}} \pm \sqrt{\frac{1}{4} - V_1}\right), \tag{14}$$

with $n \in \mathbb{Z}_{\geq 0}$. This equivalent to find the singular points in the transfer matrix–or transmission coefficient– $T(\omega_\pm) = \infty$.

2. If that is the case, the QNMs must meet the following effective boundary condition:

$$\lim_{x \to 0^+} x^{-\delta}\left(\sqrt{x}u'(x) - \frac{1}{\sqrt{x}}\left(\frac{1}{2} + \delta\right)u(x)\right) = 0. \tag{15}$$

where $\delta = \sqrt{V_0 + \frac{1}{4}}$.

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An exact example-V

1. The simple exact spectrum (14) illustrates clearly all qualitative behaviors that QNMs can exhibit.

   - For $V_0 > 0$ and $V_1 > \frac{1}{4}$, QNMs have the usual exponentially suppressed oscillatory behavior,

   $$\omega_+ = \sqrt{|V_1 - \frac{1}{4}|} - i\left(2n + 1 + \sqrt{V_0 + \frac{1}{4}}\right), \tag{16}$$

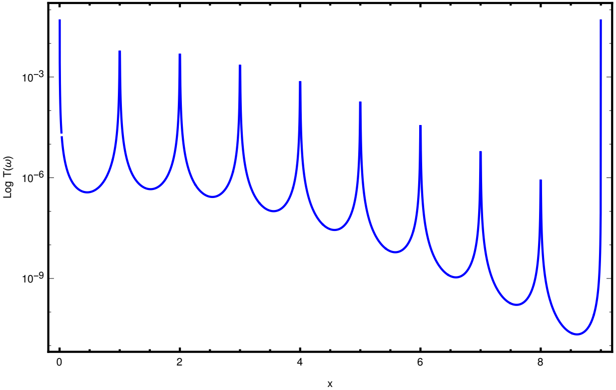
   - For $V_1 \leq \frac{1}{4}$, the so-called algebraically special QNMs are recovered with $\omega_\pm \in -i\mathbb{R}_+$ [Chandrasekhar, '84].

   - For V_1 is small enough, one could effectively attain regimes with $\mathbb{I}m\,(\omega_-) > 0$, denouncing the presence of unstable modes.

2. Since $\beta > \frac{1}{2}$, the functional energy always converges as $x \to 0^+$,

$$\mathcal{E} = \int \left[(\partial_t \Psi)^2 + (\partial_x \Psi)^2 + V(x)\Psi^2\right] dx. \tag{17}$$

Case: $V_0 > 0$ and $V_1 > \frac{1}{4}$



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The QM problem-I

1. Is is true that the QNM and bounded states of the Schrödinger operator (3) are related by an AC in the parameters ϵ and $\hbar$? [Blome&Mashhoon '84, Ferrari&Mashhoon, '84, Hatsuda, '20].

2. Take $\hbar = 1$, the AC will map the QNM u of the potential barrier in eigenstates ψ of the potential well corresponding to the inverted potential barrier according to

$$\psi = u\left(V \to -V, \omega \to -i\omega\right). \tag{18}$$

3. The associated eigenvalues (energies) will be given by $E = \omega^2\left(V \to -V\right)$:

$$E_{\pm} = -\left(2n + 1 + \sqrt{\frac{1}{4} - V_0} \pm \sqrt{\frac{1}{4} + V_1}\right)^2. \tag{19}$$

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The QM problem-II

1. In contrast with the QNM analysis of the potential barrier, we now have to deal with two distinct cases accordingly with the attractiveness of the inverted potential at $x \to 0^+$ [Essin& Griffiths,'07, Gitman+,'12]:
   - $0 < V_0 \leq \frac{1}{4}$
   - $V_0 > \frac{1}{4}$
2. Since we can have complex values of E for $V_0 > \frac{1}{4}$ or $V_1 < \frac{1}{4}$, the self-adjointness of $\mathcal{H}$ is clearly an issue.
3. We use the von Neumann theorem to check the self-adjointness of $\mathcal{H}$ [Simon&Reed, '75].
4. The so-called deficiency subspaces $N_\pm \subset D(\mathcal{H}^\dagger)$ are defined by

$$N_\pm = \left\{\phi \in D(\mathcal{H}^\dagger), \quad \mathcal{H}\phi = \pm i\phi\right\}, \tag{20}$$

where $\mathcal{H}^\dagger$ is the adjoint of $\mathcal{H}$. $D(\mathcal{H}^\dagger) \subset L^2[0, \infty)$.

The QM problem-III

1. We obtain that $n_\pm = \dim N_\pm = 1$. The local solution near $x = 0$ is

$$\phi_\pm = \sqrt{x}\left(A_\pm x^{\delta} + B_\pm x^{-\delta}\right), \tag{21}$$

with $\delta(V_0 \to -V_0) = \bar{\delta}$.

2. The self-adjoint extensions of $\mathcal{H}$ are determined by the boundary conditions on $\psi \in D(\mathcal{H}^\dagger)$ such that

$$\langle \phi, \mathcal{H}\psi\rangle - \langle \mathcal{H}\phi, \psi\rangle = \lim_{x\to 0^+}[\bar{\phi}(x)\psi'(x) - \bar{\phi}'(x)\psi(x)] = 0, \tag{22}$$

3. The solution is $\phi = \phi_+ + \bar{\lambda}\phi_-$ with λ a free complex parameter. The self-adjoint extensions are parametrized by $\mathcal{U}(1)$ group.

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The QM problem-IV

- We have that the self-adjoint extensions of $\mathcal{H}$ will satisfy the following boundary conditions on the eingenstates ψ,

$$\lim_{x\to 0^+}\left(\sqrt{x}\left[\mathcal{A}_\lambda x^{\bar{\delta}}+\mathcal{B}_\lambda x^{-\bar{\delta}}\right]\psi'(x)-\frac{1}{\sqrt{x}}\left[\left(\frac{1}{2}+\bar{\delta}\right)\mathcal{A}_\lambda x^{\bar{\delta}}+\left(\frac{1}{2}-\bar{\delta}\right)\mathcal{B}_\lambda x^{-\bar{\delta}}\right]\psi(x)\right)=0, \tag{23}$$

- The condition (23) coincides wit the usual self-adjoint boundary condition of $1/x^2$ attractive potentials, the so-called **Calogero problem** [Essin& Griffiths,'07, Gitman+,'12].

The QM problem-V

1. Let us consider the QNM boundary condition (15), which the analytically continued eigenstates would also be expected to obey in the QM problem.
2. The first the case is $0 < V_0 < \frac{1}{4}$. The QNM boundary condition requires a value for λ such that $\mathcal{A}_\lambda = 0$.
3. However, in this case the effective boundary condition reads

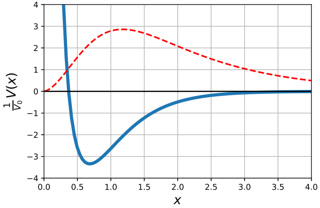
$$\lim_{x\to 0^+} x^{-\delta}\left(\sqrt{x}u'(x) - \frac{1}{\sqrt{x}}\left(\frac{1}{2}-\delta\right)u(x)\right) = 0. \tag{24}$$

4. **This does not coincide with (15), meaning that no self-adjoint extension of $\mathcal{H}$ can accommodate the QNM boundary condition for the case of real nonzero δ.**

The AC method does not always work !

The QM problem-VI

1. The so-called critical case $\delta = 0$ could be analyzed in a similar way, but in this case the deficiency subspaces $N_{\pm}$ are generated by the solutions
$$\phi_{\pm} = \sqrt{x}\,(A_{\pm} + B_{\pm}\ln x)\,, \tag{25}$$
2. We will end up with (24) once more, implying again that no self-adjoint extension of $\mathcal{H}$ will be able to accommodate the QNM boundary condition.
3. The curious observation is that the unstable QNM with a purely imaginary QNM frequency (algebraically special) corresponds, in fact, to a bounded state of the potential barrier.



Summary and open questions

1. We have shown that the QNM solutions u of the potential barrier V given by (5) can be analytically continued in eigenstates ψ of the Schrödinger operator $\mathcal{H}$ associated with the inverted potential $-V$.

2. However, the eigenstates ψ are not in the domain of any self-adjoint extension of $\mathcal{H}$, challenging the practical use of the analytical continuation method for this kind of potential barrier.

3. If we had tried to solve this problem starting with the states of $\mathcal{H}$, we would not have any physical or mathematical criterion to select the states ψ which would given origin to the QNM u via the analytical continuation.

4. This result poses a **serious challenge** on how to extend the analytic continuation method in several spacetimes (naked singularity, pure dS, Ads, etc,) to obtain the proper QNMs. Exploring different spacetimes and/or different kinds of matter fields (spinor, vector, tensor)—Work in progress—

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Thank you for listening.Any question?