

Interference effects and modified Born rule in the presence of torsion

Alexandre Leite Ferreira Junior

PPGCosmo - UFES

alexandre.ferreira@edu.ufes.br

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Eduardo Bittencourt, A.L. Ferreira Junior, Iarley P. Lobo
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Gravity with torsion

- In a general manifold the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}^{\lambda}$ are independent objects, the compass and the ruler.
- Torsion appears naturally in Gauge theories of gravity (Gauging the Poincaré Group).
- Nonetheless, Torsion is useful to model defects and transport properties in condensed matter systems [Gromov2015,Imaki2019].
- The minimal coupling with relativistic fields is well known.
- What is the effect of torsion in non-relativistic quantum matter? Interference effects and Non-Unitary time evolution.

Schrödinger equation in general metric-affine spacetimes

In an arbitrary metric-affine space-time can be done introducing a non-holonomic map [Kleinert2000]

$$dx'^a = e^a{}_\mu(x^\lambda) dx^\mu, \quad (1)$$

where $e^a{}_\mu = \partial x'^a / \partial x^\mu$. Then we have:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}, \quad \Gamma^\mu{}_{\lambda\kappa} = e_a{}^\mu \partial_\lambda e^a{}_\kappa, \quad (2)$$

Torsion is then given by

$$T^\lambda{}_{\mu\nu} = \frac{1}{2} \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \right) = \frac{1}{2} e_a{}^\lambda \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \right), \quad (3)$$

$$R^\lambda{}_{\mu\nu\kappa} = e_a{}^\lambda \left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu \right) e^a{}_\kappa. \quad (4)$$

In order to describe the quantum dynamics of a system in this general metric-affine space, we use the minimal coupling prescription to substitute the partial derivative for the covariant derivative

$$\begin{aligned} i\hbar\partial_t\Psi(x^\lambda) &= -\frac{\hbar^2}{2m}D_iD^i\Psi(x^\lambda) + V(x^\lambda)\Psi(x^\lambda) \\ &= \frac{\hbar^2}{2m}(\partial_i\partial^i + \Gamma^i_{ij}\partial^j)\Psi(x^\lambda) + V(x^\lambda)\Psi(x^\lambda), \end{aligned} \quad (5)$$

which can be rewrote as

$$i\hbar\partial_t\Psi(x^\lambda) = \frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{g}}\partial_ig^{ij}\partial_j\sqrt{g} + T^i_{ij}\partial^j\right)\Psi(x^\lambda) + V(x^\lambda)\Psi(x^\lambda) \quad (6)$$

Torsion generated by a screw dislocation

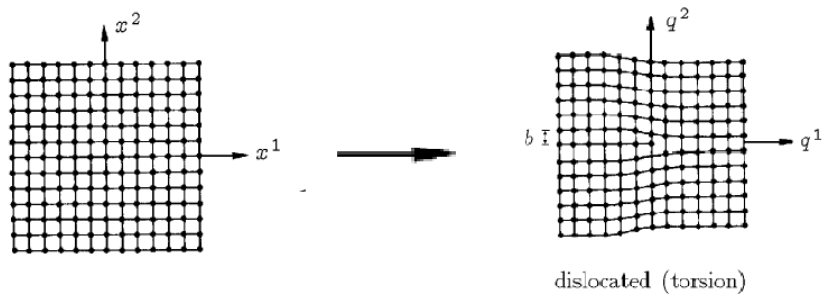


Figure: Figure from Kleinert (2000)

Torsion generated by a screw dislocation

We can map the distorted coordinates $x'^i = (x'^1, x'^2)$ of the atoms in the lattice to the undistorted ones $x^i = (x^1, x^2)$ through the parametrization

$$dx'^1 = dx^1, \quad dx'^2 = dx^2 + \varepsilon \partial_i \phi(x^j) dx^i, \quad (7)$$

with $\varepsilon = nd \ll 1$, where n is the number of planes removed from the crystalline lattice and d being the distance between two of them, so that terms of order ε^2 are negligible.

$$\phi(x^j) = \arctan \left(\frac{x^2}{x^1} \right). \quad (8)$$

So, being C a small closed path encircling the origin, and S the surface enclosed by it, we have:

$$\int_S d^2x (\partial_1 \partial_2 - \partial_2 \partial_1) \phi(x^j) = \oint_C dx^i \partial_i \phi(x^j) = 2\pi. \quad (9)$$

Torsion generated by a screw dislocation

Since the derivative of $\phi(\vec{x})$ is a single-valued function, the curvature tensor is zero. The connection coefficients read

$$\Gamma^i_{jk} = e_A^i \partial_j e_k^A = e_2^i \partial_j e_k^2 = \delta_2^i \varepsilon \partial_j \partial_k \phi(\vec{x}), \quad (10)$$

and finally, the torsion tensor becomes

$$T^1_{ij} = 0, \quad T^2_{ij} = \frac{\varepsilon}{2\pi} (\partial_i \partial_j - \partial_j \partial_i) \phi(\vec{x}) = \epsilon_{ij} \varepsilon \delta(\vec{x}). \quad (11)$$

Carrying out the Schrödinger equation with torsion

Thence, with that the Schrödinger equation becomes

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}(\partial_i\partial^i + (\partial_i\Lambda)\partial^i - T_{12}^2\partial_1)\Psi, \quad (12)$$

with $\Lambda = \varepsilon\partial_2\phi(x^i)$. Substituting $\Psi(x^i, t) = e^{-\frac{\Lambda}{2}}\psi(x^i, t)$ in the equation above and neglecting terms quadratic in ε , we find that

$$i\partial_t\psi = -\frac{\hbar}{2m}\left(\partial_i\partial^i\psi - T_{12}^2\partial_1\psi - \frac{1}{2}\psi\partial_i\partial^i\Lambda\right). \quad (13)$$

Straightforward calculations show that the potential-like term vanishes, that is, $\partial_i\partial^i\Lambda = 0$.

Carrying out the Schrödinger equation with torsion

Thence,

$$\frac{2im}{\hbar} \frac{\partial}{\partial t} \psi + \partial_i \partial^i \psi = -\varepsilon \delta(x^i) \partial_1 \psi. \quad (14)$$

Now, we shall solve this equation perturbatively in powers of ε . Thus, suppose a solution of the form $\psi(x^i, t) = \psi_0(x^i, t) + \varepsilon \psi_1(x^i, t)$. ψ_0 is the solution to the homogeneous equation away from the origin. So, ψ_1 must satisfy

$$\frac{2im}{\hbar} \frac{\partial}{\partial t} \psi_1 + \nabla^2 \psi_1 = -\delta(x^i) \partial_1 \psi_0. \quad (15)$$

Integrating this equation over a small region S enclosing the origin, we have

$$\frac{2im}{\hbar} \frac{\partial}{\partial t} \int_S d^2x \psi_1 + \oint_{\partial S} dn^i \partial_i \psi_1 = -\partial_1 \psi_0|_{\vec{x}=0}. \quad (16)$$

Taking the side η of S approaching zero, we find

$$\lim_{\eta \rightarrow 0} \left[\int_{-\eta/2}^{\eta/2} dx^1 \partial_2 (\psi_u - \psi_d) + \int_{-\eta/2}^{\eta/2} dx^2 \partial_1 (\psi_r - \psi_l) \right] = -\partial_1 \psi_0|_{\vec{x}=0}. \quad (17)$$

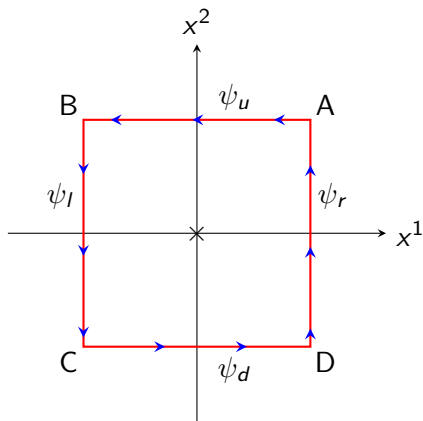


Figure: Oriented squared region of side η enclosing the origin in the (x^1, x^2) -plane.

Plane wave solution

Here, we will choose the right-going wave to stay the same, $\psi_R = \psi_0$, while the left-going one is modified due to the effective torsion, according to

$$\lim_{x^1 \rightarrow 0^+} \psi_L(x^i, t) - \lim_{x^1 \rightarrow 0^-} \psi_L(x^i, t) = \varepsilon A B_L. \quad (18)$$

Then, from Eq. (17), we find that $B_L = -i k_1^2 k_2 / k^2$, where $k^2 = k_1^2 + k_2^2$. So, after crossing the defect (region $x^1 > 0$), the left-going wave is

$$\psi_L(x^i) = A(1 + \varepsilon B_L) e^{i k_j x^j} = A e^{i(k_j x^j - \theta)} + \mathcal{O}(\varepsilon^2), \quad (19)$$

with $\theta = \arctan(\varepsilon k_1^2 k_2 / k^2)$.

From the usual definition of the probability density

$$\begin{aligned}\mathcal{P}(x^i, t) \sqrt{g(x^i)} d^2x &= |\Psi(x^i, t)|^2 \sqrt{g(x^i)} d^2x \\ &= e^{-\Lambda(x^i)} |\psi(x^i, t)|^2 \sqrt{g(x^i)} d^2x \approx |\psi(x^i, t)|^2 d^2x,\end{aligned}\quad (20)$$

with $e^{-\Lambda} \sqrt{g(x^i)} \approx (1 - \Lambda)(1 + \Lambda) \approx 1$ at first order in ε , one would expect to measure on the screen

$$|\psi|^2 = |\psi_R(x_{(1)}^i) + \psi_L(x_{(2)}^i)|^2 = 4A^2 \cos^2 \left[\frac{k_i(x_{(2)}^i - x_{(1)}^i)}{2} - \frac{\theta}{2} \right], \quad (21)$$

where $x_{(1)}^i$ and $x_{(2)}^i$ denote the distances from the slits on the right and on the left, respectively.

Wave packet solution

The homogeneous solution for the wave packet

$$\begin{aligned}\psi_0(x^i, t) &= \frac{2a}{\sqrt{2\pi}} \frac{e^{-i\xi}}{(a^4 + 4\hbar^2 t^2/m^2)^{\frac{1}{2}}} e^{i(k_j)_0(\Delta x^j)} \\ &\times \exp \left[-\frac{(\Delta x^j - \frac{\hbar t}{m} k_0^j)^2}{(a^2 + 2i\hbar t/m)} \right],\end{aligned}\quad (22)$$

with $\xi = \hbar t k_0^2/(2m) + \arctan[2\hbar t/(a^2 m)]$.

$$\begin{aligned}\psi_1(x^i, t) &= -\frac{ia}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} \frac{k_1^2 k_2}{k^2} e^{-\frac{a^2}{4} \Delta k_i^2} e^{i(k_j x^j - \omega t)} d^2 k \\ &= -iB_L(\vec{x}, t) \psi_0(\vec{x}, t),\end{aligned}\quad (23)$$

With

$$\begin{aligned} B_L(x^i, t) &= \frac{\alpha}{\pi} \int d^2 k \frac{k_1^2 k_2}{k^2} e^{-\alpha(k^i - \beta^i)^2} \\ &= \frac{\beta_2}{\alpha^2 \beta^6} \left[(\beta^2 - 4\beta_1^2) \left(e^{-\alpha\beta^2} + \alpha\beta^2 - 1 \right) + 2\alpha^2 \beta^4 \beta_1^2 \right], \end{aligned} \quad (24)$$

where we defined the auxiliary quantities

$$\alpha = \frac{1}{4} \left(a^2 + \frac{2i\hbar t}{m} \right), \quad \beta_j = \frac{1}{2\alpha} \left(\frac{a^2}{2} (k_j)_0 + ix_j \right), \quad (25)$$

and $\beta^2 \equiv \beta_1^2 + \beta_2^2$.

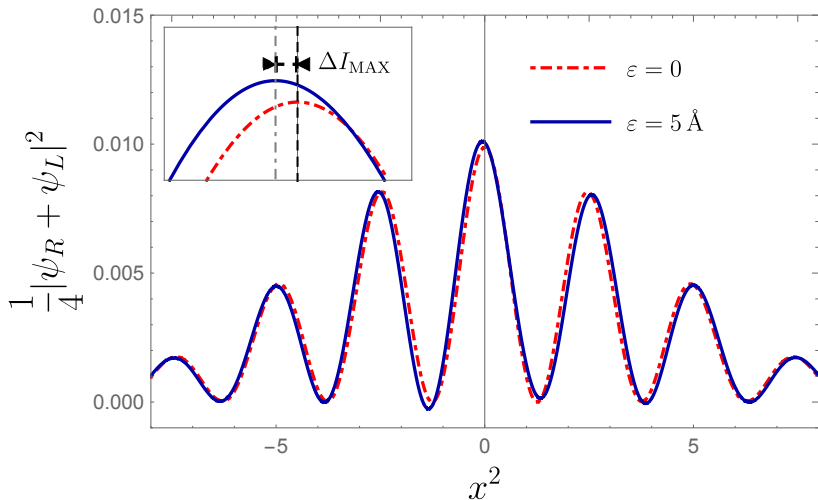


Figure: Interference fringes pattern of a double-slit experiment in the presence of torsion generated by a screw dislocation (solid line) compared with the same setup in the absence of torsion (dot-dashed line).

What about the non-unitarity?

The continuity equation in the presence of a screw dislocation reads

$$\int \left(\frac{\partial}{\partial t} |\psi|^2 + \partial_k j^k \right) d^2 \vec{x} = - \int \delta(\vec{x}) \frac{\varepsilon \hbar}{m} \text{Im}(\psi_0^* \partial_1 \psi_0) |_{\vec{x}=0} d^2 \vec{x}. \quad (26)$$

On the left-hand side, we have the continuity equation terms as the integrand, but the right-hand side is not zero whenever there is torsion $\varepsilon \neq 0$.

Now, the wave functions discussed before can be written down as

$$\psi(x^i, t) = \begin{cases} \psi_0(x^i, t) & \text{if } x^1 < 0, \\ \psi_0(x^i, t) \left[1 - \frac{i\varepsilon}{2} B_L(x^i, t)\right] & \text{if } x^1 > 0, \end{cases} \quad (27)$$

for which the modulus squared is trivial when $x^1 < 0$ and for $x^1 > 0$, to first order in ε , we have

$$|\psi(x^i, t)|^2 \approx |\psi_0|^2 (1 + \varepsilon \text{Im} B_L) . \quad (28)$$

For a Gaussian wave packet produced in the negative x^1 -axis away from the defect, and initially normalized, the source term of Eq. (26) reads

$$\begin{aligned} \text{Im}(\psi_0^* \partial_1 \psi_0)|_{\vec{x}=0} \\ = \frac{2a^2 \left(a^4 k_0^1 - \frac{4\hbar t}{m} x_0^1 \right)}{\pi \left(a^4 + \frac{4\hbar^2 t^2}{m^2} \right)^2} \exp \left[-2a^2 \frac{\left(x_0^i + \frac{\hbar t}{m} k_0^i \right)^2}{a^4 + \frac{4\hbar^2 t^2}{m^2}} \right]. \end{aligned} \quad (29)$$

This is always positive for the wave function given by Eqs. (22)-(23), ensuring a draining of probability in the defect. However, if we reflect the problem with respect to the x^2 -axis, so that $k_0 < 0$ and $x_0 > 0$, the system gains probability from the dislocation.

We realize that we can preserve a probabilistic interpretation by redefining the probability density function as

$$\bar{\mathcal{P}}(x^i, t) = \mathcal{P}(x^i, t) + \frac{\varepsilon \hbar}{m} \delta(x^i) \int_{t_0}^t \text{Im}(\psi^* \partial_1 \psi) dt, \quad (30)$$

where t_0 is the instant of time that normalizes the new probability distribution as $\int_{\Omega} \bar{\mathcal{P}} d^2x = 1$, for an arbitrarily large integration region Ω . This guarantees that somewhere in space the particle described by the wave function must be detected.

What about a fundamental torsion?

In a fundamental setting, one would have to abandon the probabilistic interpretation in favour of some other theory, such as the De Broglie-Bohm approach. Just to have a hint we write the eq. in polar form

$$\Psi(\vec{x}, t) = R(\vec{x}, t)e^{iS(\vec{x}, t)/\hbar},$$

where $R(\vec{x}, t)$ is the amplitude and $S(\vec{x}, t)$ is the phase, the Schrödinger equation is equivalent to a modified Hamilton-Jacobi

$$-\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \pm \frac{\hbar^2}{2m} \frac{\Gamma \cdot \nabla R}{R}$$

and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \frac{\nabla S}{m} \right) = \pm \rho \frac{\Gamma \cdot \nabla S}{m}$$

In Summary

- The coupling of torsion with non-relativistic quantum mechanics change the pattern of interference phenomena.
- In the effective picture, the non-unitary evolution of the system suggests that the system is open, and a modification of the Born rule is needed.
- In a fundamental setting, the presence of Torsion is incompatible with the probabilistic interpretation of QM. Luckily, we have other approaches, such as the De Broglie-Bohm theory of QM.

References

-  A. Gromov and A. G. Abanov, “Thermal Hall Effect and Geometry with Torsion,” Phys. Rev. Lett. **114** (2015), 016802 [arXiv:1407.2908].
-  S. Imaki and A. Yamamoto, “Lattice field theory with torsion,” Phys. Rev. D **100** (2019) no.5, 054509 [arXiv:1906.02406].
-  H. Kleinert, “Nonholonomic Mapping Principle for Classical and Quantum Mechanics in Spaces with Curvature and Torsion,” General Relativity and Gravitation **32**, 769–839 (2000) [arXiv:gr-qc/0203029]..
-  F. Cabral, F. S. N. Lobo and D. Rubiera-Garcia, “Fundamental Symmetries and Spacetime Geometries in Gauge Theories of Gravity—Prospects for Unified Field Theories,” Universe **6** (2020) no.12, 238 [arXiv:2012.06356].

Thank you!