Brans-Dicke Unimodular Gravity

Alexandre Magno Rodrigues Almeida

Coauthors: Júlio C. Fabris, Mahamadou H. Daouda, Richard Kerner, Hermano Velten and Willian S. Hipólito- Ricaldi

Abstract

We propose a unimodular version of the Brans-Dicke theory designated with a constrained Lagrangian formulation. The resulting field equations are traceless. The vacuum solutions in the cosmological background reproduce the corresponding solutions of the usual Brans-Dicke theory but with a cosmological constant term. A perturbative analysis of the scalar modes is performed and unstable configuration appears, in contrast with the Brans-Dicke case for which only stable configurations occur. On the other hand, tensorial modes in this theory remain the same as in the traditional Brans-Dicke theory.

Introduction

The unimodular gravity is a possible alternative formulation of a geometric theory of gravity that appeared soon after the general relativity theory. The unimodular condition fixes the determinant of the metric $g_{\mu\nu}$ equal to one. The main goal of this work is to explore the unimodular condition in the Brans-Dicke theory of gravity . This is done by adding a **l**agrange multiplier (χ) to the Brans-Dicke lagrangian with the condition $g = \xi$:

$$L = \sqrt{-g}\left\{\phi R - \omega\frac{\phi_{;\rho}\phi^{;\rho}}{\phi}\right\} + \phi\chi(g-\xi) + L_m. \quad (1)$$

One consequence of adding this constraint is that the unimodular field equations are traceless, and thus we lose information. Also, the conservation of the energy-momentum tensor it is not a consequence of the Bianchi identities but actually is an imposition. With this, we obtain the usual Brans-Dicke theory with cosmological constant.

In the unimodular Brans-Dicke the cosmology background solutions for the vacuum are the same as in the original Brans-Dicke with the cosmological constant term. However in the perturbative level the equivalence is lost. In particular, an unstable solution appears, while in the Brans-Dicke case only stable solutions are present in the vacuum case, with or without the cosmological constant.

Field Equations

·The field equations from the Lagrangian (**1**) are,

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = \frac{8\pi}{\phi}\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right) + \frac{\omega}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{4}g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\right) + \frac{1}{\phi}\left(\phi_{;\mu\nu} - \frac{1}{4}g_{\mu\nu}\Box\phi\right), \quad (2)$$

$$\Box\phi = \frac{1}{2}\frac{\phi_{;\rho}\phi^{;\rho}}{\phi} - \frac{\phi}{2\omega}R. \quad (3)$$

With the Bianchi identities, we obtain the following,

$$(\phi R)^{;\nu} = \omega\left(\frac{\phi_{;\rho}\phi^{;\rho}}{\phi}\right)^{;\nu} + 32\pi\left(T^{\mu\nu}{}_{;\mu} - \frac{1}{4}T^{;\nu}\right) + 3(\Box\phi)^{;\nu}. \quad (4)$$

If we impose that the energy-momentum tensors conserves, then we can integrate the equation (**4**) and substitute the result in (**2**) and (**3**), obtaining the original Brans-Dicke theory with the addition of the integration constant which can be interpreted as the cosmological constant. The study is focused to the option in which the tensor is not conserved.

Cosmology

The cosmological flat model described by the FLRW metric to the Brans-Dicke unimodular theory led us to the following field equations,

$$\dot{H} = -\frac{4\pi}{\phi}(\rho + p) - \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - \frac{1}{2}\left(\frac{\ddot{\phi}}{\phi} - H\frac{\dot{\phi}}{\phi}\right), \quad (5)$$

$$\frac{\ddot{\phi}}{\phi} + 3H\frac{\dot{\phi}}{\phi} = \frac{1}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{3}{\omega}\left(\dot{H} + 2H^2\right). \quad (6)$$

These equations are sensitive to the combination of $\rho + p$, which is related to the enthalpy, so the same solutions obtained for $\rho = -p$ are obtained in the vacuum case $\rho = p = 0$, in contrast for what happens in the usual Brans-Dicke case. However, these equations have the same solutions as in the original Brans-Dicke Theory.

Cosmological Perturbations

Because of the unimodular condition, when we turn to the perturbative analysis with the linear perturbations, such that $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ we have $h_{kk} = 0$. This condition leads to the impossibility of using the Newtonian gauge. The synchronous gauge, however, can be used since the generalized conservation law (**4**) is considered. The analysis is made restricting to the synchronous gauge in the unimodular context implying $h_{\mu 0} = 0, h_{kk} = 0.$

The difference between the original theory and the unimodular case is seen in the scalar perturbations, which lead to the following solutions,

$$\delta\phi = x^{3/2}\{c_1 J_\nu(\tilde{k}x) + c_2 J_{-\nu}(\tilde{k}x)\}, \quad (7)$$

where $c_{1,2}$ are constants and $\tilde{k} = \frac{\sqrt{\bar{c}_s^2}k}{p}$, with $\bar{c}_s^2 = -\frac{1+2\omega}{3-2\omega}\frac{1}{a_0^2}$.
For $\bar{c}_s^2 < 0$ the solutions are expressed in terms of modified Bessel functions,

$$\delta\phi = x^{3/2}\{c_1 K_\nu(|\tilde{k}|x) + c_2 I_\nu(|\tilde{k}|x)\}. \quad (8)$$

The solutions (**8**) display instabilities at small scales as $k \to \infty$, with an exponential growth of the perturbations at this limit. If we inspect the cases where $\omega = -1/2$ and $\omega = 3/2$ it is clear from the solutions that in this interval the instabilities appear.

Conclusion

We constructed the unimodular version of the Brans–Dicke theory using an action with its constraint expressed by a Lagrangian multiplier. On the cosmological setup, when we do not consider the usual conservation law, all equations are sensitive only to the combination $\rho + p$ which makes the solutions for $\rho = -p$ equivalent to the vacuum solutions in presence of a cosmological constant term.

However, this equivalence is broken at perturbative level. The vacuum perturbed equations can be solved analytically, and they are not the same as in the usual Brans–Dicke theory. In particular, the perturbations reveal instabilities in the small scale limit if the BD parameter is in the interval $-1/2 < \omega < 3/2$. For all other values of ω, the perturbations are stable.

References

[1] Almeida, A.M.R.; Fabris, J.C.; Daouda, M.H.; Kerner, R.; Velten, H.; Hipólito-Ricaldi, W.S. Brans–Dicke Unimodular Gravity. *Universe* **2022**, *8*, 429. https://doi.org/10.3390/universe8080429

[2] Fabris, J.C.; Alvarenga, M.H.; Hamani-Daouda, M.; Velten, H. Nonconservative unimodular gravity: A viable cosmological scenario? Eur. Phys. J. C **2022**, 82, 522.

[3] Fabris, J.C.; Alvarenga, M.H.; Hamani-Daouda, M.; Velten, H. Nonconservative Unimodular Gravity: Gravitational Waves. Symmetry **2022**, 14, 87.