A Topological-like Gravity Model in 4D

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Abstract

General Relativity (RG) in 3 dimensions space-time with a cosmological constant and in the absence of matter may be described as a topological Chern-Simons theory for the (A)dS gauge group $SO(n, 4-n)$. It has no local degrees of freedom. We present here a model, inicially due to Chamseddine, of gravity in 4 dimensions which may be obtained from a 5-dimensional Chern-Simons theory with de Sitter SO(1,5) as a gauge group. It does have local degrees of freedom even in the absence of matter, and we show cosmological solutions which can be compared with the solutions of standard GR.

Summary

- Topological Field Theory & General Relativity.
- $SO(6-n,n)$ Chern-Simons theory for 5D gravity.
- Chamseddine "Topological" gravity in 4D.
- Cosmological solutions.

About Topological Field Theories and General Relativity

Definition of a background independent theory: a field theory defined in a manifold without a priori metric structure or other external field configuration.
Example: 4D gravity.

Definition of a topological field theory: A field theory which

1. is background independent.
2. has no local degrees of freedom,

Example – 3D Chern-Simons theory:

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– 4D gravity is background independent, but has local degrees of freedom!

– 3D (pure) gravity is topological: it can be described as a Chern-Simons theory for the gauge group ISO(1,2) (3D Poincaré group) or SO(1,3) or SO(2,2) (de Sitter or anti-de Sitter group) *(Witten, 1988)*

– In 5D, Chern-Simons theory has local degrees of freedom *(C. Teitelboim and J. Zanelli, 1987)*

$\rightarrow$ is not a topological theory according to the definition above, although it is often called so ... as I will do in the following.

– Dimensional reduction $5D \rightarrow 4D$ with truncations: a "topological" version of 4D gravity. *(A.H. Chamseddine, 1990)*

(A)dS Chern-Simons theory in 5D

Gauge group (A)dS= SO(1,5) or SO(2,4) (de Sitter or anti-de Sitter) leaves invariant the metric (internal metric!)
$(\eta_{MN}) = \mathrm{diag}(-1, 1, 1, 1, 1, \lambda)$
with $\lambda > 0$ or < 0: the 5D cosmological constant)

15 generators of (A)dS: $M_{MN} = -M_{NM}$ ($M = 0, \cdots, 5$):

$$[M_{MN}, M_{PQ}] = \eta_{MQ} M_{NP} - \eta_{MP} M_{NQ} - \eta_{NQ} M_{MP} + \eta_{NP} M_{MQ}$$

Gauge connection:

$$A = \frac{1}{2} A^{MN} M_{MN} = \frac{1}{2} (A_m^{MN} dx^m) M_{MN}$$

($m = 0, \cdots, 4$: space-time indices)

Gauge transformations (infinitesimal):

$$\delta A = d\varepsilon + [\varepsilon, A] \equiv D\varepsilon$$

Action and equations of motion

(wedge product symbol $\wedge$ is ommitted)

$$S_{\mathrm{CS}_5} = \tfrac{1}{16\kappa}\varepsilon_{M_1\cdots M_6}\int_{\mathcal{M}_5}\Big(A^{M1M2}dA^{M3M4}dA^{M5M6} + (\cdots)A^{M1M2}[A,A]^{M3M4}dA^{M5M6} + (\cdots)A^{M1M2}[A,A]^{M3M4}[A,A]^{M5M6}\Big)$$

($\mathcal{M}_5$: 5-dim. manifold)

Field equation:

$$F \wedge F = 0\,, \quad F \equiv dA + \frac{1}{2}[A,A]$$

Theory has local degrees of freedom *(Zanelli et all)*, in contrast with the 3D case, where the field eq. is $F = 0$, *i.e.*, flat connection $\rightarrow$ no local degrees of freedom.

Chern-Simons gravity in 5D

Let $A, B, \cdot, = 0, \cdots, 4$ be Lorentz SO(1,4) indices, and decompose the generator basis $\{M_{MN}\}$ in Lorentz generators M_{AB} and "translation" generators $P_A = M_{A5}$
$\rightarrow$ commutations rules:

$$[M_{AB}, M_{CD}] = \eta_{AD} M_{BC} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{BC} M_{AD}$$

$$[M_{AB}, P_C] = -\eta_{AC} P_B + \eta_{BC} P_A$$

$$[P_A, P_B] = \lambda\, M_{AB}$$

Defining a spin connection and a 5-bein:

$$\omega^{AB} = A^{AB}\,, \quad e^{A} = A^{A5}\,,$$

as well as curvature and torsion tensors:

$$R^{AB} = d\omega^{AB} + \frac{1}{2}[\omega,\omega]^{AB}\,, \quad T^{A} = de^{A} + [\omega, e]^{A}\,.$$

Then

$$S_{\mathrm{CS}_5} = \tfrac{1}{8\kappa}\varepsilon_{A_1\cdots A_5}\int_{\mathcal{M}_5}\Big(e^{A_1}R^{A_2A_3}R^{A_4A_5} - \tfrac{2\lambda}{3}\,e^{A_1}e^{A_2}e^{A_3}R^{A_4A_5} + \tfrac{\lambda^2}{5}\,e^{A_1}e^{A_2}e^{A_3}e^{A_4}e^{A_5}\Big)$$

looks as a (pure) gravity action in 5D. *Chamseddine, 1990*

Why (A)dS in D=5?

Canonical analysis of this theory by *Bañados, Garay and Henneaux (1995)* $\rightarrow$

- Theory is completely constrained (Hamilonian is a sum of constraints) (as it should be for a background invariant theory).
- The independent 1^{rst} class constraints are the generators of the (A)dS gauge invariance and of the space diffeonorphism invariance, all being rather easy to solve at the quantum level, *e.g.*, by loop quantization methods.
  $\rightarrow$ The notoriously difficult constraint – the one which generates the time diffeomorphisms – is automatic (on shell).
- The remaining difficulty is to explicitly separate these 1^{rst} class constraints from the numerous 2^{nd} class ones.

General aim

Quantization of this theory or of a 4D version obtained by dimensional compactification to D=4.

Preliminary task: Test the physical interest of this endeavour by examining a 4D theory obtained from the 5D one by dimensional reduction and truncation.
(A theory which has its own interest, too)

This will be the subject of the rest of my talk.

"Topological" gravity in 4D

- Dimensional reduction 5D *to* 4D (*i.e.*, letting all fields to be independent of x^4),
- Truncation by setting to zero some component fields

$\rightarrow$ 4D topological-like gravitation theory of Chamseddine (1990):

$$S[\Phi, A, \cdots] = \frac{1}{8\kappa} \int_{\mathcal{M}_4} \epsilon_{ABCDE} \Phi^A F^{BC} F^{DE} + S_{\mathrm{matter}}$$

where M_4 is a 4-dimensional manifold, the 2-form F^{AB} is the curvature of the connection 1-form A^{AB},

$$F^{AB} = dA^{AB} + A^{AC} A_C{}^B = \frac{1}{2} F^{AB}_{\mu\nu} dx^\mu dx^\nu \, .$$

Φ^A is a scalar field (0-form) in the vector representation of (A)dS, (A)dS being now reduced to the 4D de Sitter or anti-de Sitter group SO(1,4) or SO(2,3).

Interpretation as a gravitation theory

(A)dS metric: $\mathrm{diag}\,(-1,1,1,1,s)\,, \quad s=\pm$

Decompose the (A)dS generator basis into the Lorentz generators M_{IJ} and the "translation" generators P_I:

$$
\begin{array}{lcl}
\text{Index } A=0,\cdots 4 & \to & \{\{I=0,\cdots 3\},\,4\}\,, \\
\text{Generators } M_{AB} & \to & \{M_{IJ},\, P_I \equiv \sqrt{|\Lambda|}M_{I4}\,, \\
\text{Connection } A^{AB} & \to & \{\omega^{IJ}\equiv A^{IJ},\, e^I \equiv A^{I4}/\sqrt{|\Lambda|}\}\,, \\
\text{Scalar field } \Phi^A & \to & \{\Phi^I,\, \Phi\equiv\Phi^4\}
\end{array}
$$

Algebra:

$$[M_{IJ},M_{KL}] = \eta_{IL}M_{JK}+\eta_{JK}M_{IL}-\eta_{IK}M_{JL}-\eta_{JL}M_{IK}\,,$$

$$[M_{IJ},P_K] = -\eta_{IK}P_J+\eta_{JK}P_I\,, \qquad [P_I,P_J]=\Lambda\,M_{IJ}\,,$$

with $\Lambda = s|\Lambda|$.

Partial gauge fixing

Gauge fixing condition: $\Phi^I = 0\,, \quad I = 0, \cdots, 3$
Local Lorentz invariance remains explicit.

With $\Phi \equiv \Phi^4$, the action reads now (divided by Λ):

$$S = \frac{1}{8\kappa}\varepsilon_{IJKL}\int \Phi\left(\frac{1}{\Lambda}R^{IJ}R^{KL} - 2e^I e^J R^{KL} - \Lambda\, e^I e^J e^K e^L\right) + S_{\mathrm{matter}}$$

Curvature 2-form: $\quad R^{IJ} = d\omega^{IJ} + \omega^I{}_K\omega^{KJ}$

Torsion 2-form: $\quad T^I = de^I + \omega^I{}_K e^K$

(A similar action, but not directly related to the 5D Chern-Simons theory has been investigated by *A. Tolosi and J. Zanelli (2013)*.)

Field equations

$$\delta S/\delta e^I = 0: \quad \epsilon_{IJKL}\Phi(e^J R^{KL} - \tfrac{\Lambda}{3} e^J e^K e^L) = 2\kappa\tau_I$$

$$\delta S/\delta \omega^{IJ} = 0: \quad \epsilon_{IJKL}(\tfrac{1}{\Lambda} d\Phi(R^{KL} - \tfrac{\Lambda}{3} e^K e^L) + \tfrac{2}{3}\Phi e^K T^L) = 0,$$

$$\delta S/\delta \phi^4 = 0: \quad \epsilon_{IJKL}\left(\Lambda e^I e^J R^{KL} - \tfrac{\Lambda^2}{6} e^I e^J e^K e^L - \tfrac{3}{2} R^{IJ} R^{KL}\right) = 0,$$

where $\tau_I = \frac{1}{6}\varepsilon_{MJKL}\,\tau^M{}_I e^J e^K e^L = \delta S_{\rm matter}/\delta e^I =$ energy-momentum 3-form.
The $\tau^M{}_I$ are the energy-momentum tensor components in the vierbein basis.

N.B. If Φ constant: First 2 eqs. are Einstein eqs. in the "first order" (or Palatini) formulation of RG with cosmological constant.
But there is the the third equation...

Cosmological solutions

The assumptions, as usual (perturbations being neglected), are that space is homogeneous and isotropic: Physical objects are described by tensors $t_{\mu\cdots}^{\nu\cdots}$ obeying 6 conditions of invariance:

$$\mathcal{L}_{\xi_\alpha} t_{\mu\cdots}^{\nu\cdots} = 0\,, \quad \alpha = 1, \cdots, 6\,,$$

with the $\mathcal{L}_{\xi_\alpha}$ are the Lie derivatives along the 6 Killing vectors ξ_α which generate rotations and "translations".

In spherical coordinates $\rightarrow$ FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right) \;(k = 0, \pm 1)\,.$$

But we have also torsion, $= T^I = de^I + \omega^I{}_K e^K$

A vierbein reproducing the FLRW metric can be choosen as:

$$\begin{aligned} e^0 &= dt\,, & e^1 &= \frac{a(t)}{\sqrt{1-kr^2}}dr, \\ e^2 &= a(t)rd\theta\,, & e^3 &= a(t)r\sin\theta d\varphi \end{aligned}$$

Killing conditions on the torsion *(Tolosi and Zanelli, 2013)*

The Killing vectors generating the translations and rotations are, in spherical coordinates:

$$\xi_{(a)} = \sqrt{1 - kr^2}\partial_a \,, \quad \text{and} \quad \xi_{(ab)} = x_a\partial_b - x_b\partial_a$$

where the x^a, $a = 1, 2, 3$ are the space Cartesian coordinates corresponding to the spherical coordinates r, θ, ϕ. Applying these vectors to the scalar field Φ and to the torsion components $T^{\mu}_{\nu\rho}$, we get

$$\Phi = \Phi(t) \,,$$

$$T^r{}_{\theta\varphi} = 2f(t)a(t)r^2\sqrt{1 - kr^2}\,\sin\theta, \qquad T^\varphi{}_{r\theta} = \frac{2f(t)a(t)}{\sqrt{1-kr^2}\,\sin\theta},$$

$$T^\varphi{}_{r\theta} = -\frac{2f(t)a(t)\sin\theta}{\sqrt{1-kr^2}}, \qquad T^r{}_{rt} = T^\theta{}_{\theta t} = T^\varphi{}_{\varphi t} = h(t)$$

(All other components $= 0$)

We are left with 4 independent functions of t: the scale $a(t)$, the scalar field $\Phi(t)$ and the torsion parameters $f(t)$, $h(t)$.

From this, and from
torsion $= T^I = de^I + \omega^I{}_K e^K$,

we deduce the general form of the connection components:

$$\omega^{0i} = (H+h)e^i, \qquad \omega^{12} = -\frac{\sqrt{1-kr^2}}{ar} e^2 - fe^3\,,$$

$$\omega^{31} = \frac{\sqrt{1-kr^2}}{ar} e^3 - fe^2, \qquad \omega^{23} = -\frac{\cot\theta}{ar} e^3 - fe^1,$$

(All other components $= 0$), where

$$H(t) = \dot{a}(t)/a(t) \quad \text{(Hubble parameter)}$$

Inserting these vierbein and connection expression in the field equations $\rightarrow$

A system of differential equations for the functions $a(t)$, $\Phi(t)$, $f(t)$, $h(t)$:

$$\frac{\Lambda}{3}\Phi(U^2 + \frac{k}{a^2} - f^2 - \frac{\Lambda}{3}) = -\frac{\kappa\rho_m}{3},$$

$$\frac{\Lambda}{3}\Phi[U^2 + \frac{k}{a^2} - f^2 - \Lambda + 2(\dot{U} + HU)] = \kappa p_m,$$

$$\dot{\Phi}(U^2 + \frac{k}{a^2} - f^2 - \frac{\Lambda}{3}) + \frac{2\Lambda}{3}\Phi h = 0,$$

$$f(\dot{\Phi}U + \frac{\Lambda}{3}\Phi) = 0,$$

$$(U^2 + \frac{k}{a^2} - f^2 - \frac{\Lambda}{3})(\dot{U} + HU - \frac{\Lambda}{3}) - 2fU(\dot{f} + Hf) = 0,$$

$$\text{where} \quad U(t) = H(t) + h(t)$$

Continuity equation

ρ_m and p_m in the differential equations : energy density and pressure of a fluid with energy-momentum tensor

$$\tau^I_J = \mathrm{diag}(-\rho_m, p_m, p_m, p_m)$$

N.B. With the redefinitions

$$\rho = \tfrac{3}{\Lambda\Phi}\rho_m + \tfrac{3}{\kappa}(2Hh + h^2 - f^2)\,,$$
$$p = \tfrac{3}{\Lambda\Phi}p_m - \tfrac{1}{\kappa}(2\dot{h} + 4Hh + h^2 - f^2)\,,$$

follow the continuity equation for the "energy" ρ and "pressure" p (to which the torsion parameter $f(t)$ and $h(t)$ contribute!):

$$\frac{d}{dt}(\rho a^3) + p\frac{d}{dt}a^3 = 0\,.$$

General solution of the differential equations

We take the flat space case, $k = 0$, and $\Lambda > 0$.

General solution (involves 3 integration constants C_1, C_2, C_3, C_4):

$$a(t) = C_3 \cosh(t\sqrt{\Lambda/3} - C_1)^{2/3} \Big(-C_2\sqrt{\Lambda} \cosh(t\sqrt{\Lambda/3} - C_1) + 3\sqrt{3} \sinh(t\sqrt{\Lambda/3} - C_1) \Big)^{1/3}$$

$$H(t) = \sqrt{\Lambda} \operatorname{sech}(t\sqrt{\Lambda/3} - C_1) \Big(-C_2\sqrt{\Lambda} + \cosh(t\sqrt{\Lambda/3} - C_1) + 3\sqrt{3} \sinh(t\sqrt{\Lambda/3} - C_1) \Big)^{-1} + \sqrt{\Lambda/3} \tanh(t\sqrt{\Lambda/3} - C_1)\,,$$

$$\Phi(t) = C_4 \operatorname{cosech}(t\sqrt{\Lambda/3} - C_1)$$

Choice: $C_3 = 1,\ C_4 = 1$ (normalization constants)

C_2 fixed in terms of C_1 and Λ such that $a(0) = 0$.

Continuity erquation for the matter

One checks that

$$\rho_m(t) = \frac{\text{const}}{a(t)^3} \,,$$

which is the density continuity equation for matter (we have $p_m = 0$).

N.B. This continuity equation does not need to be postulated.

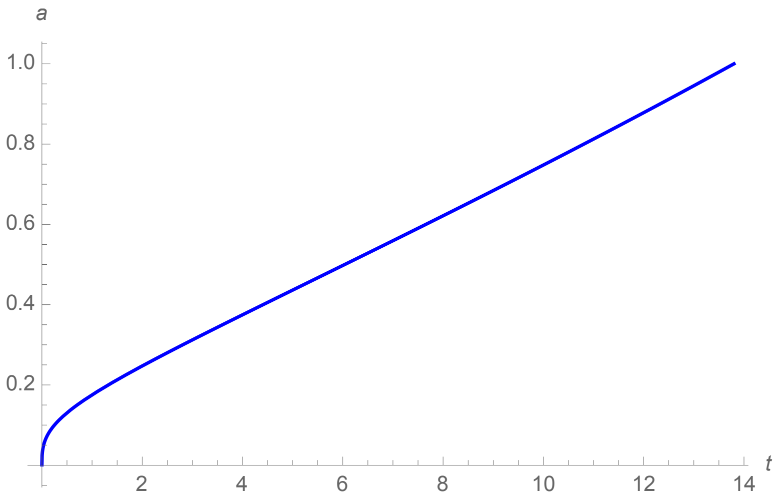
Two remaining parameters Λ, C_1 are determined from the observed present values of the:

- Hubble constant $H = a'/a|_{t=t_0} = 67.3$ km/s/Mpsec (*Planck*),
- Deceleration parameter $q = -a''/aH^2|_{t=t_0}$:
  $-31 > q > -53$ *(R. Giostri, M. Vargas dos Santos, I. Waga, R. R. R. Reis, M. O. Calvão, B. L. Lago, 2012)*

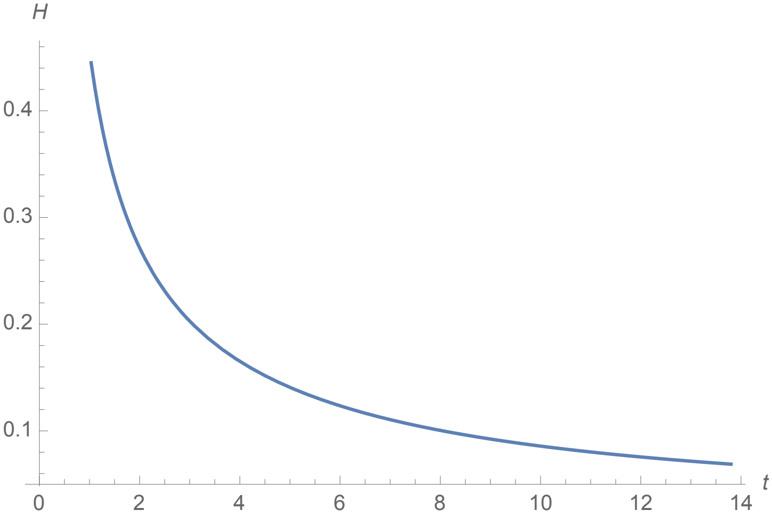
  Age of the universe: $t_0 = 13.8$ Gyear (*Planck*).

Results:

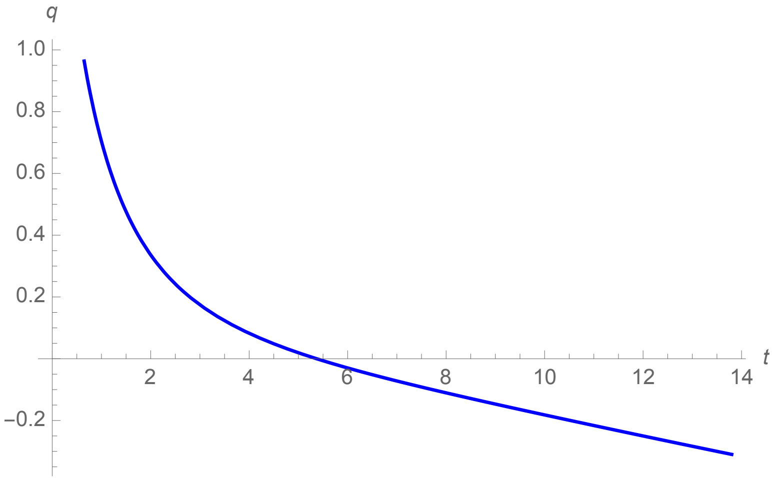
Normalized scale parameter $a(t)/a(t_0)$



Hubble parameter $H = a'/a$

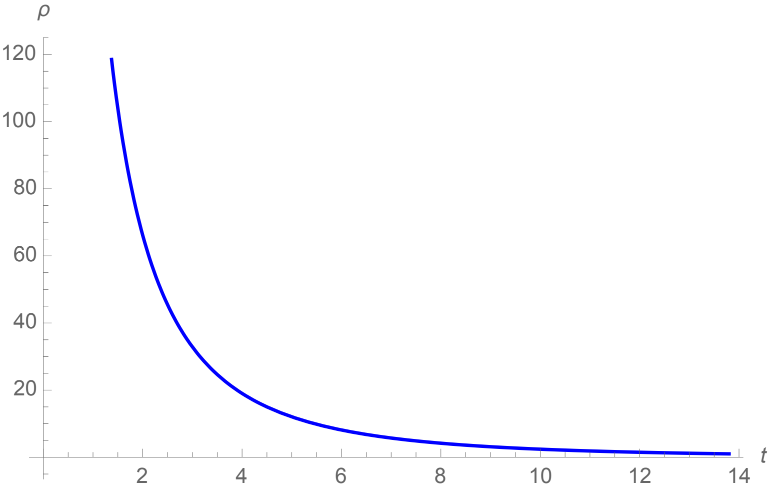


Deceleration parameter $q = -a''/(aH^2)$

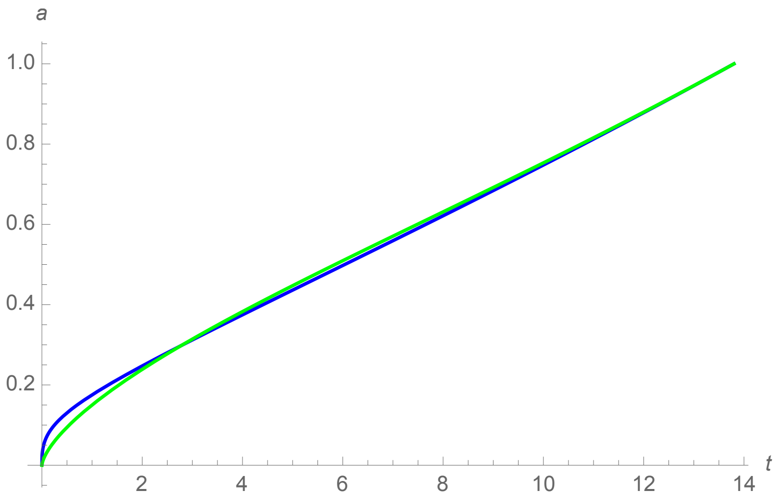


$q(t = 0) = 2$

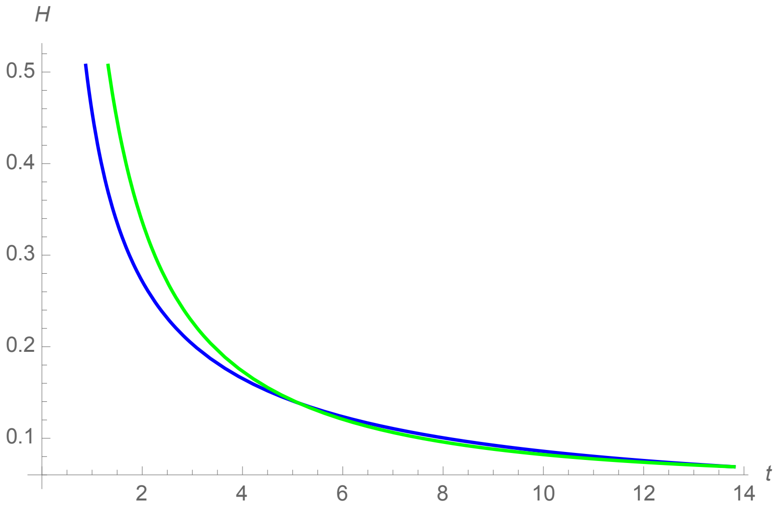
Normalized matter density $\rho_m(t)/\rho_m(t_0)$



Comparison with ΛCDM scale



Comparison with ΛCDM Hubble parameter



Conclusions

- This preliminary work shows results which are, at least qualitatively, compatible with the data.
- The results look rather similar to those of Tolosi and Zanelli, but the actual behaviour of $a(t)$ is very different. *E.g.*, for $\Lambda > 0$, they have no inicial singularity: $a(0) \neq 0$. For $\Lambda < 0$, their expansion is accelerated, too; ours is not.

Prospectives:

- Take a full, untruncated action derived from the 5D Chern-Simons – which is the theory of real interest in view of the quantization.
- Complete the canonical constraints analysis (Along the line of Bañados, Garay and Henneaux's papers).
- Quantize.

In 3D $\rightarrow$ 2D: *Rodrigo Barbosa, Clisthenis Constantinidis, Zui Oporto and O.P., 2013*.