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CRÉATEURS DE FUTURS DEPUIS 1257

cnrs dépasser les frontières

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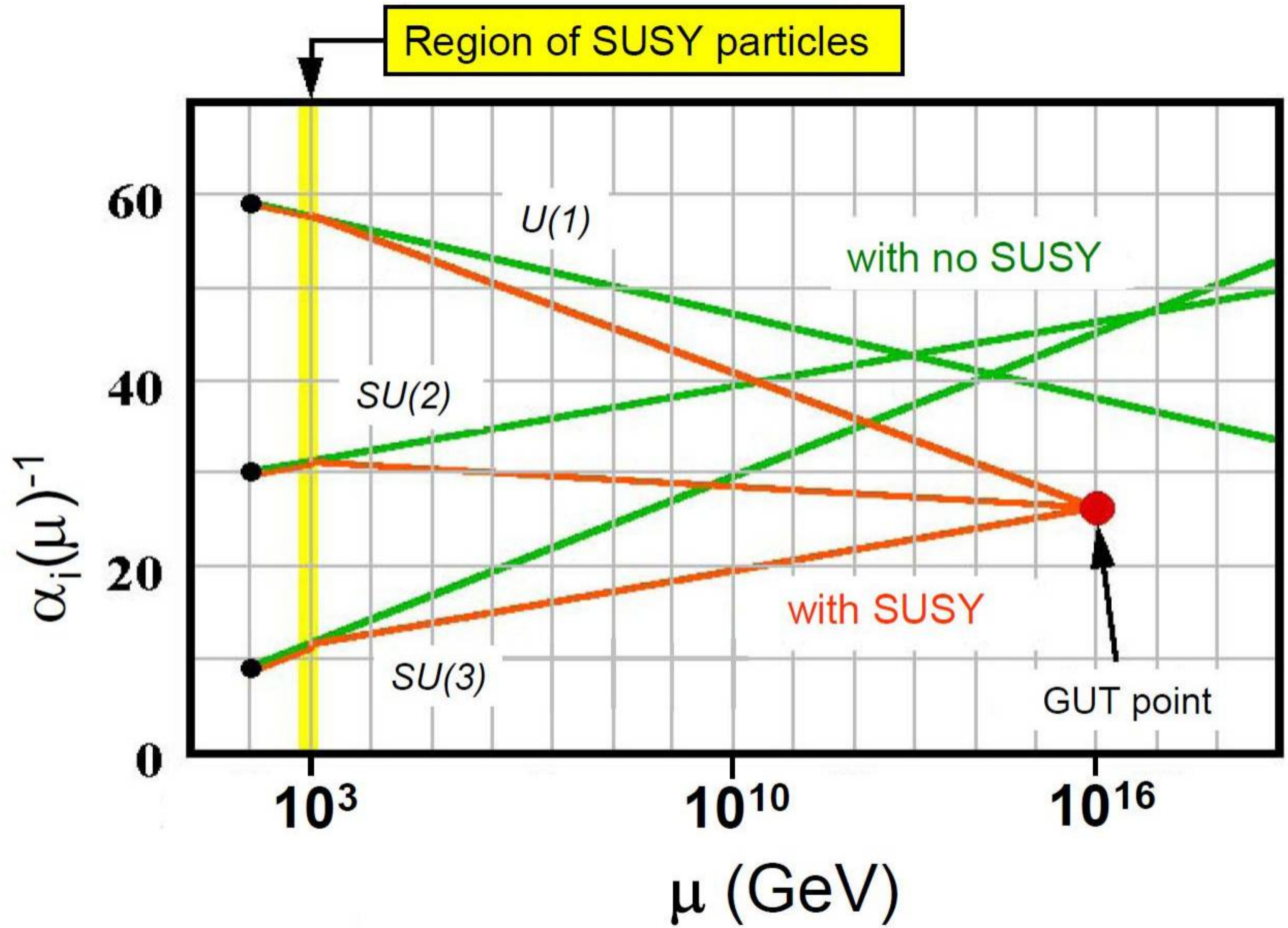
Charge–velocity–dependent one–scale model for cosmic string network evolution

Chacun a son défaut, où toujours il revient
J. de la Fontaine, Fables, 1668

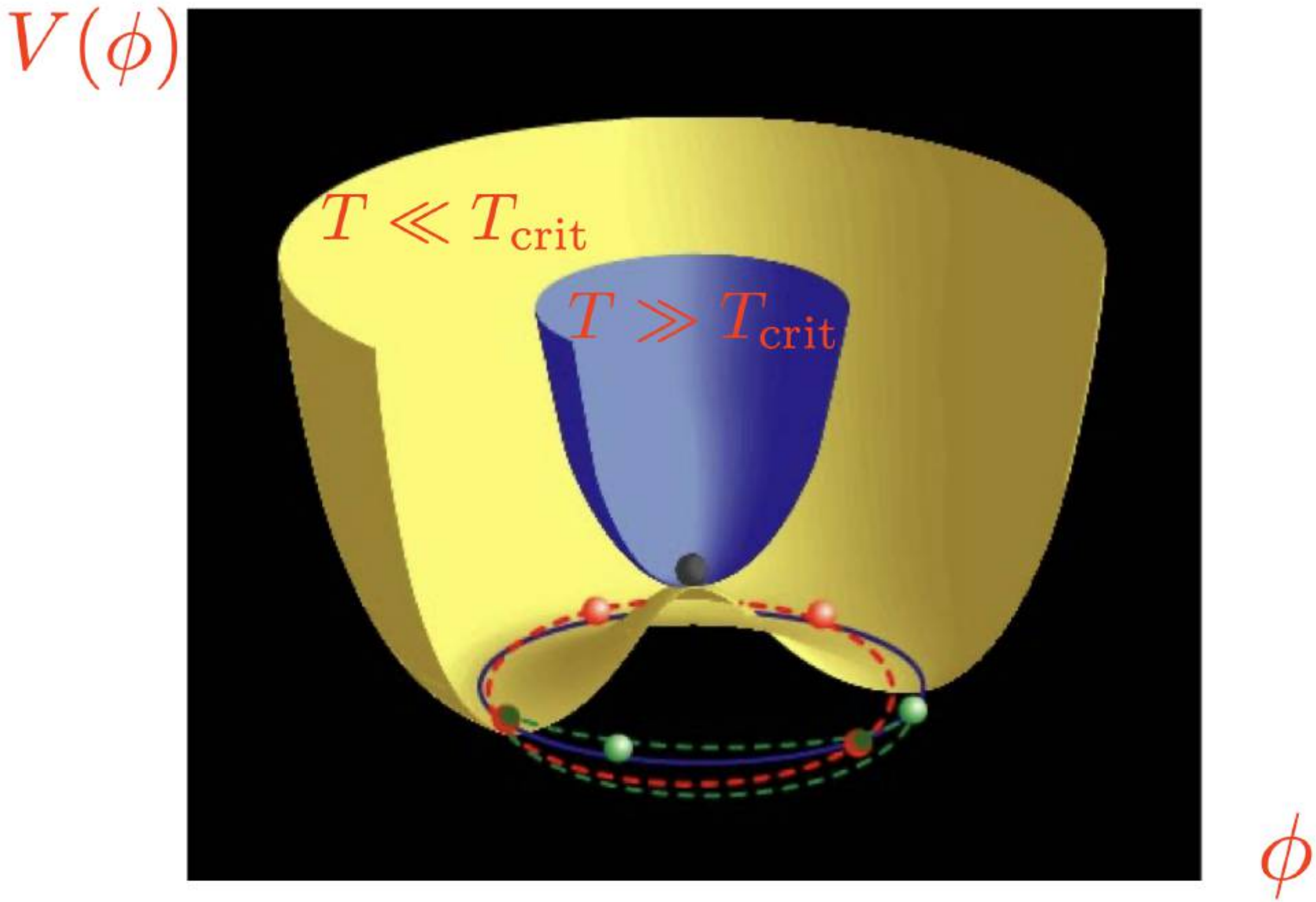
Based on

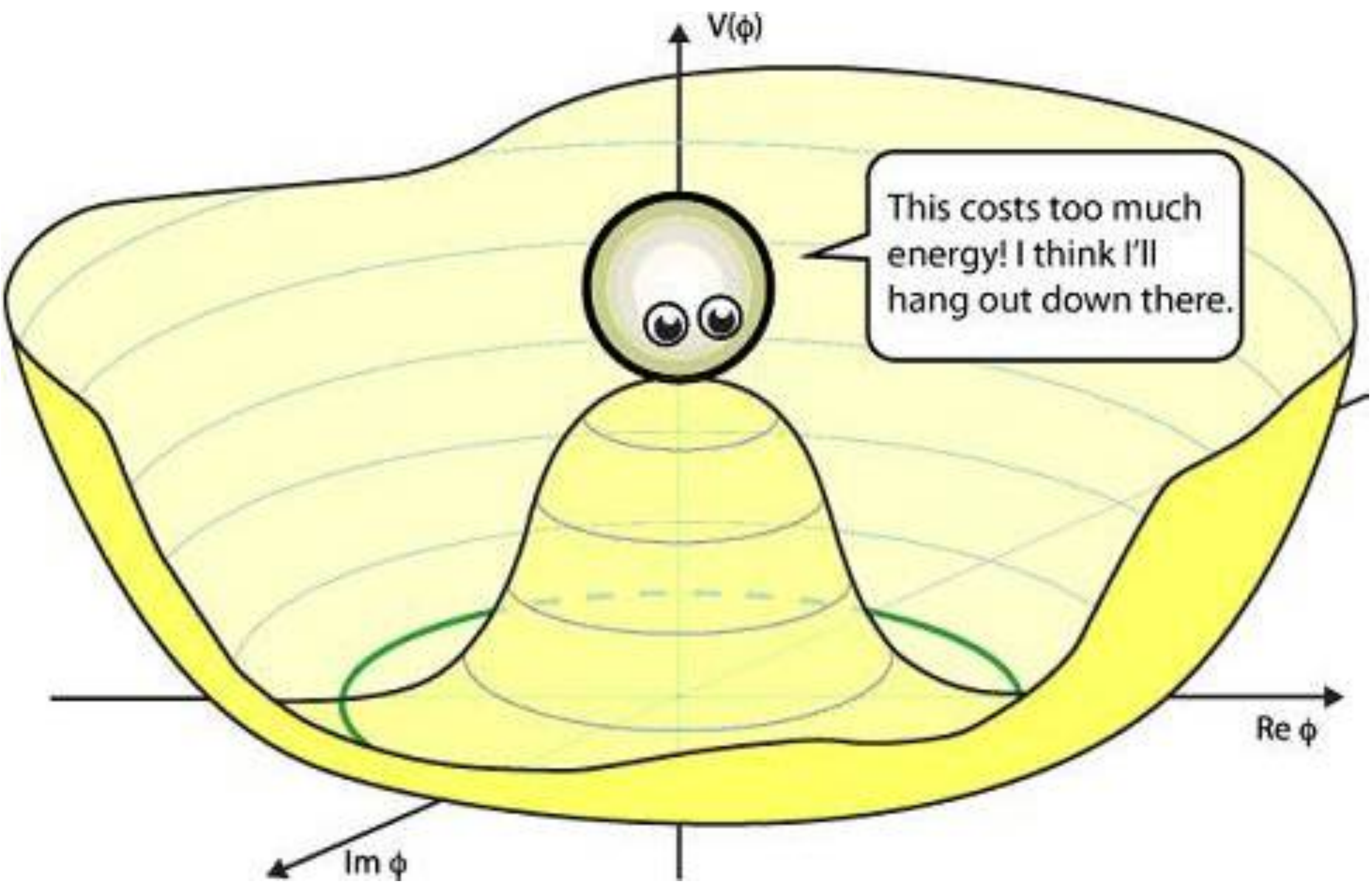
Generalized velocity-dependent one-scale model for current-carrying strings
C. J. A. P. Martins, PP, I. Yu. Rybak and E. P. S. Shellard,
Phys. Rev. **D103**, 043538 (2021) [astro-ph 2011.09700]

Charge-velocity-dependent one-scale linear model
C. J. A. P. Martins, PP, I. Yu. Rybak and E. P. S. Shellard,
Phys. Rev. **D104,** 103506 (2021) [astro-ph 2108.03147]

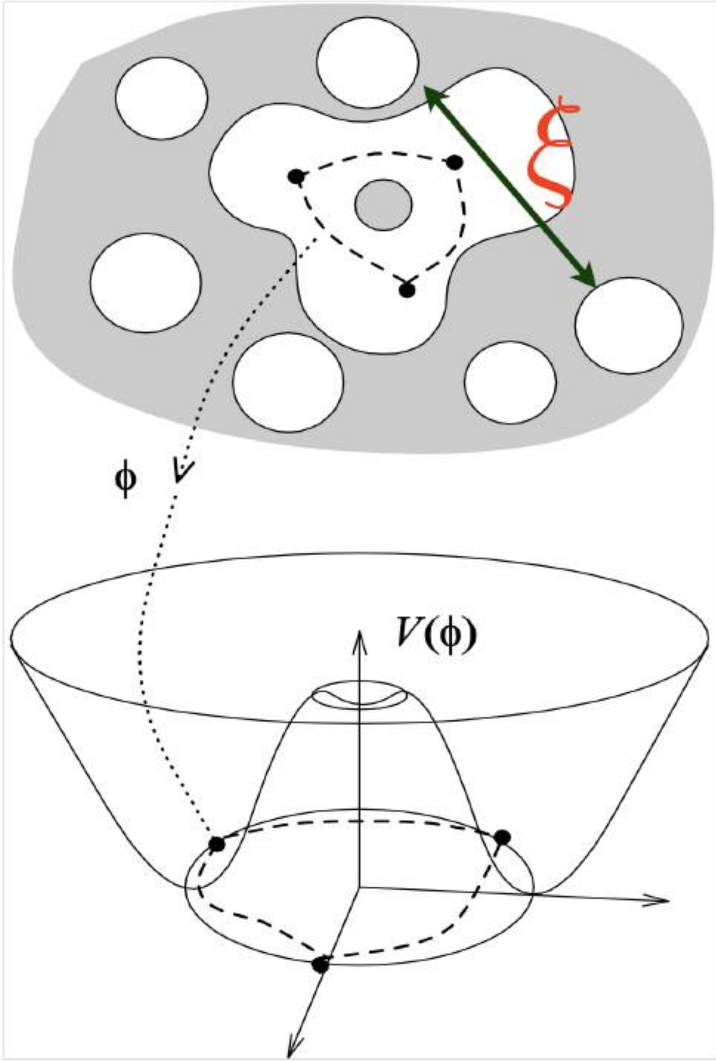

Region of SUSY particles
60
40
20
0
U(1)
with no SUSY
SU(2)
SU(3)
with SUSY
GUT point
$\alpha_i(\mu)^{-1}$
10^3
10^{10}
10^{16}
μ (GeV)

Higgs field and potential




V(φ)
This costs too much energy! I think I'll hang out down there.
Re φ
Im φ

Phase transition



Correlation length

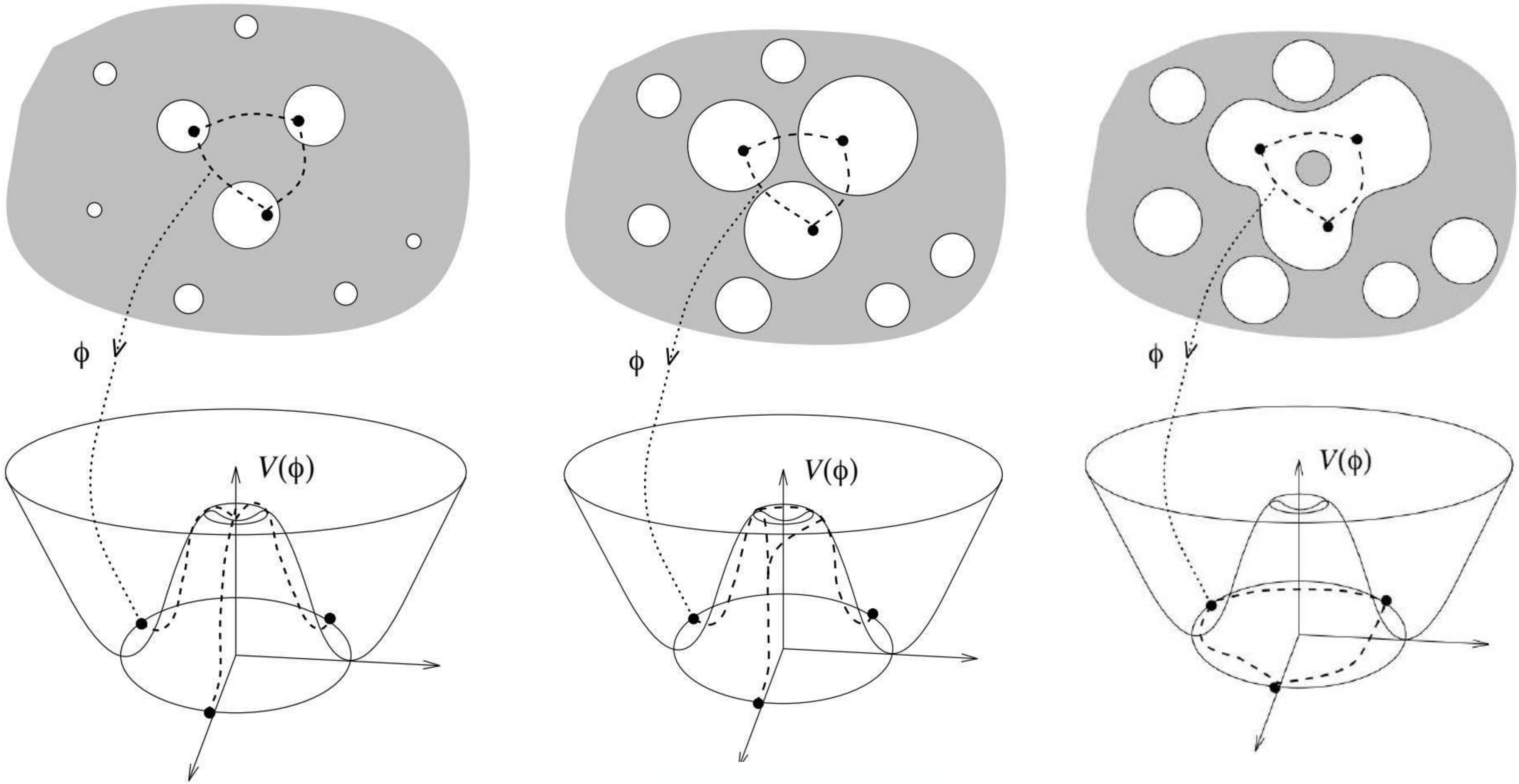
$$\langle 0|\phi(\vec{x})\phi(\vec{x}+\vec{r})|0\rangle \propto \mathrm{e}^{-|\vec{r}|/\xi}$$

Model: cosmic string

T.W.B. Kibble, *J. Phys.* **A9**, 1387 (1976)

M.B. Hindmarsh and T.W.B. Kibble, *Rep. Prog. Phys.* **58**, 477 (1995)

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} D_\mu \Phi \left(D^\mu \Phi \right)^* - V(\Phi)$$



Model:

cosmic string

T.W.B. Kibble, *J. Phys.* **A9**, 1387 (1976)
M.B. Hindmarsh and T.W.B. Kibble, *Rep. Prog. Phys.* **58**, 477 (1995)

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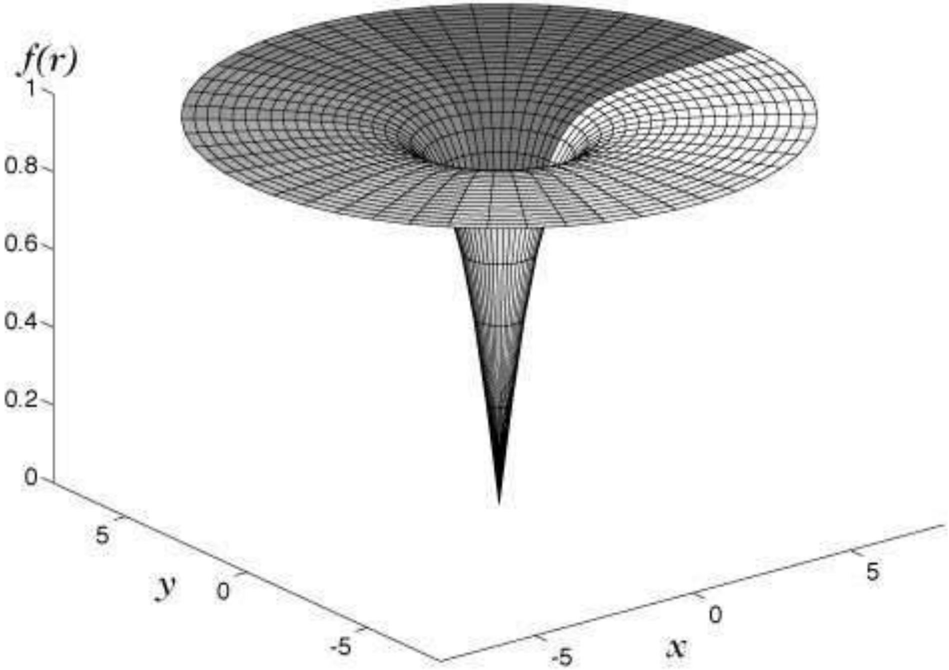
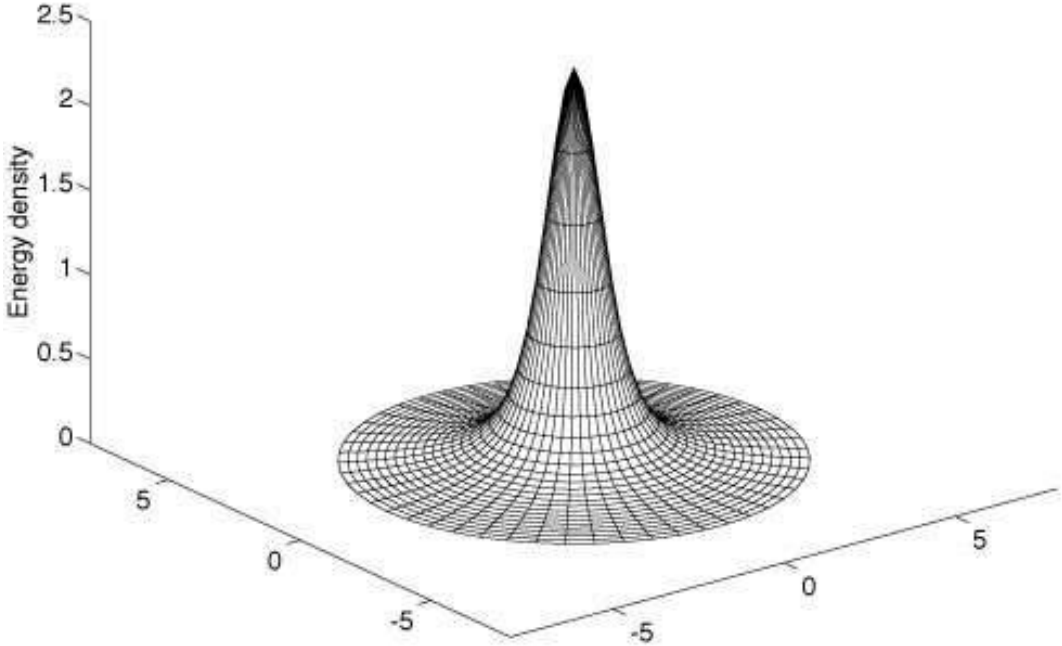
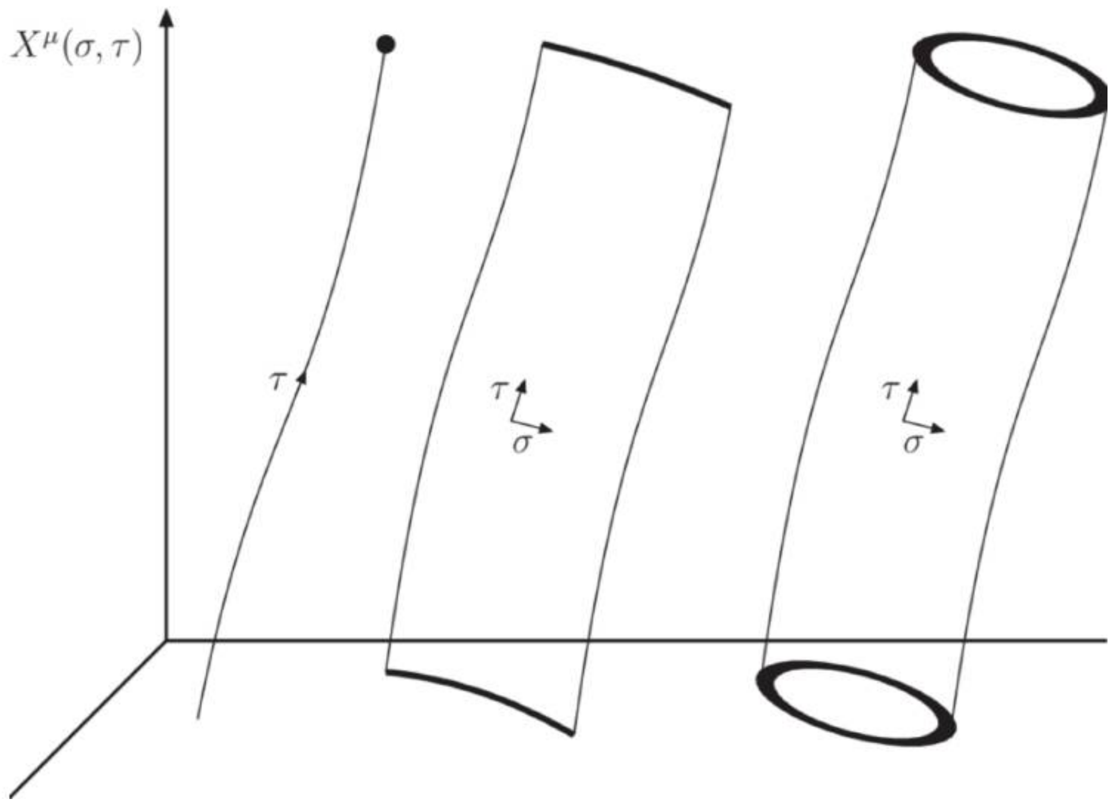
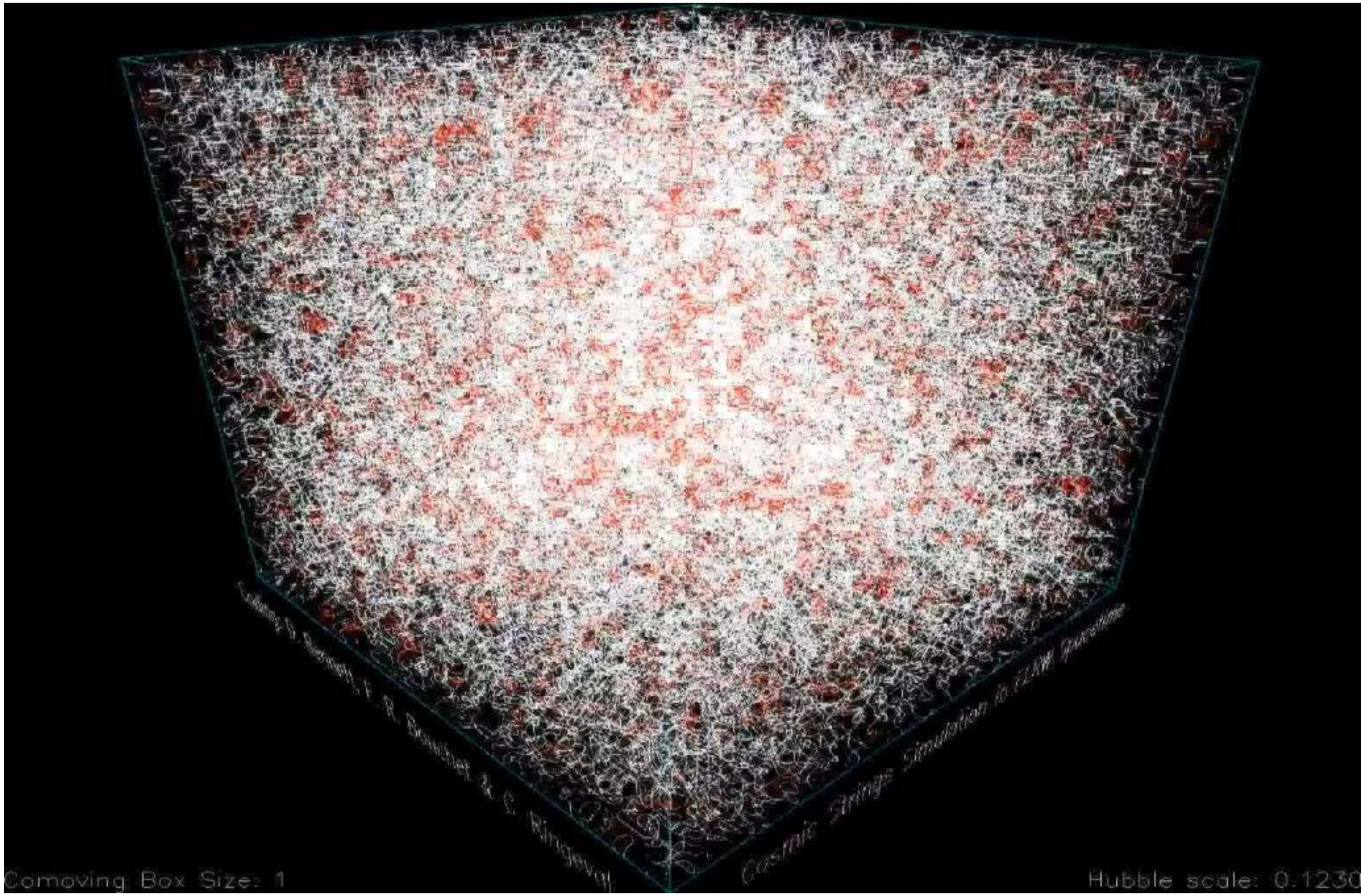




Nambu-Goto strings

$$\mathcal{S} = \int \sqrt{-\gamma}\, \mathrm{d}\sigma \mathrm{d}\tau \quad \Longrightarrow \quad \ddot{\boldsymbol{x}} = \boldsymbol{x}''$$



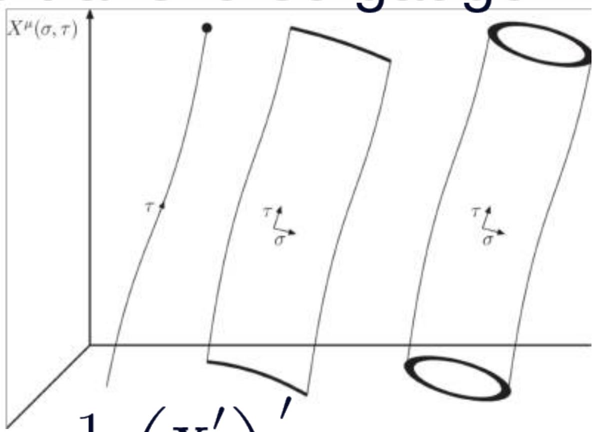


https://commons.wikimedia.org/wiki/File:Cosmic_strings_evolution_during_the_expansion_of_the_Universe.webm

Velocity one-scale model

Preliminaries: Uniformly expanding universe, the string EoM are:

Using conformal time $\sigma^0 = \tau$ and the transverse gauge



$$\dot{\mathbf{x}} \cdot \mathbf{x}' = 0$$

the string equations become

acceleration

$$\ddot{\mathbf{x}} + \left(2\frac{\dot{a}}{a} + \frac{a}{\ell_{\mathrm{f}}}\right)\left(1 - \dot{\mathbf{x}}^2\right)\dot{\mathbf{x}} = \frac{1}{\epsilon}\left(\frac{\mathbf{x}'}{\epsilon}\right)',$$

curvature

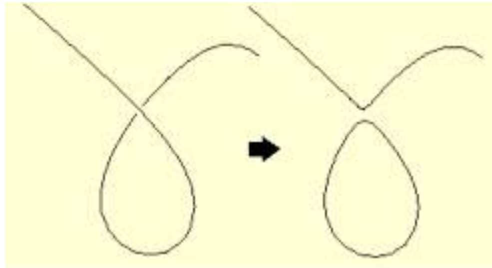
damping

$$\dot{\epsilon} + \left(2\frac{\dot{a}}{a} + \frac{a}{\ell_{\mathrm{f}}}\right)\dot{\mathbf{x}}^2\epsilon = 0\,,$$

where ϵ is the energy per unit σ is $\epsilon^2 = \dfrac{\mathbf{x}'^2}{1 - \dot{\mathbf{x}}^2}$,

Velocity One-scale Model (VOS)

Thermodynamic approach: Total network energy



$$E = \mu a(\tau) \int \epsilon d\sigma \,,$$

Time derivative - energy conservation equation

$$\frac{d\rho}{dt} + \left(2H\left(1+v^2\right) + \frac{v^2}{\ell_{\rm f}} \right) \rho = 0 \,.$$

where the averaged rms velocity is defined by

$$v^2 \equiv \langle \dot{\mathbf{x}}^2 \rangle = \frac{\int \dot{\mathbf{x}}^2 \epsilon d\sigma}{\int \epsilon d\sigma} \,.$$

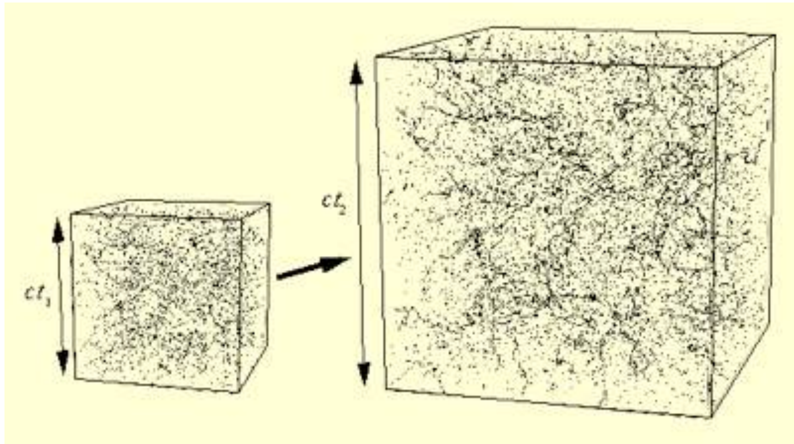
Kibble 1980; Vilenkin 1982

Martins & EPS 1995,6

Brownian network 'correlation length'

1 string μ*L* per correlation volume L^3

$$\rho_\infty \equiv \frac{\mu}{L^2} \,.$$



The VOS Model

Governs large-scale network evolution over cosmic history

Martins & EPS 1995,6

$$2\frac{dL}{dt} = 2HL(1+v_\infty^2) + \frac{Lv_\infty^2}{\ell_{\rm f}} + \tilde{c}v_\infty\,.$$

Hubble stretching & redshifting *damping* *loop creation*

$$\frac{dv}{dt} = (1-v^2)\left[\frac{k}{R} - v\left(2H + \frac{1}{\ell_{\rm f}}\right)\right]\,.$$

friction

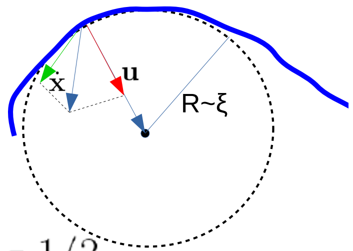
acceleration *curvature* *Hubble damping*

where the curvature term is

*NB: only one free parameter **c***

$$k(v) = \frac{2\sqrt{2}}{\pi}\,\frac{1-8v^6}{1+8v^6}$$

$$k(v) \equiv \frac{\left\langle (1-\dot{\boldsymbol{X}}^2)(\dot{\boldsymbol{X}}\cdot\boldsymbol{u})\right\rangle}{v(1-v^2)}$$



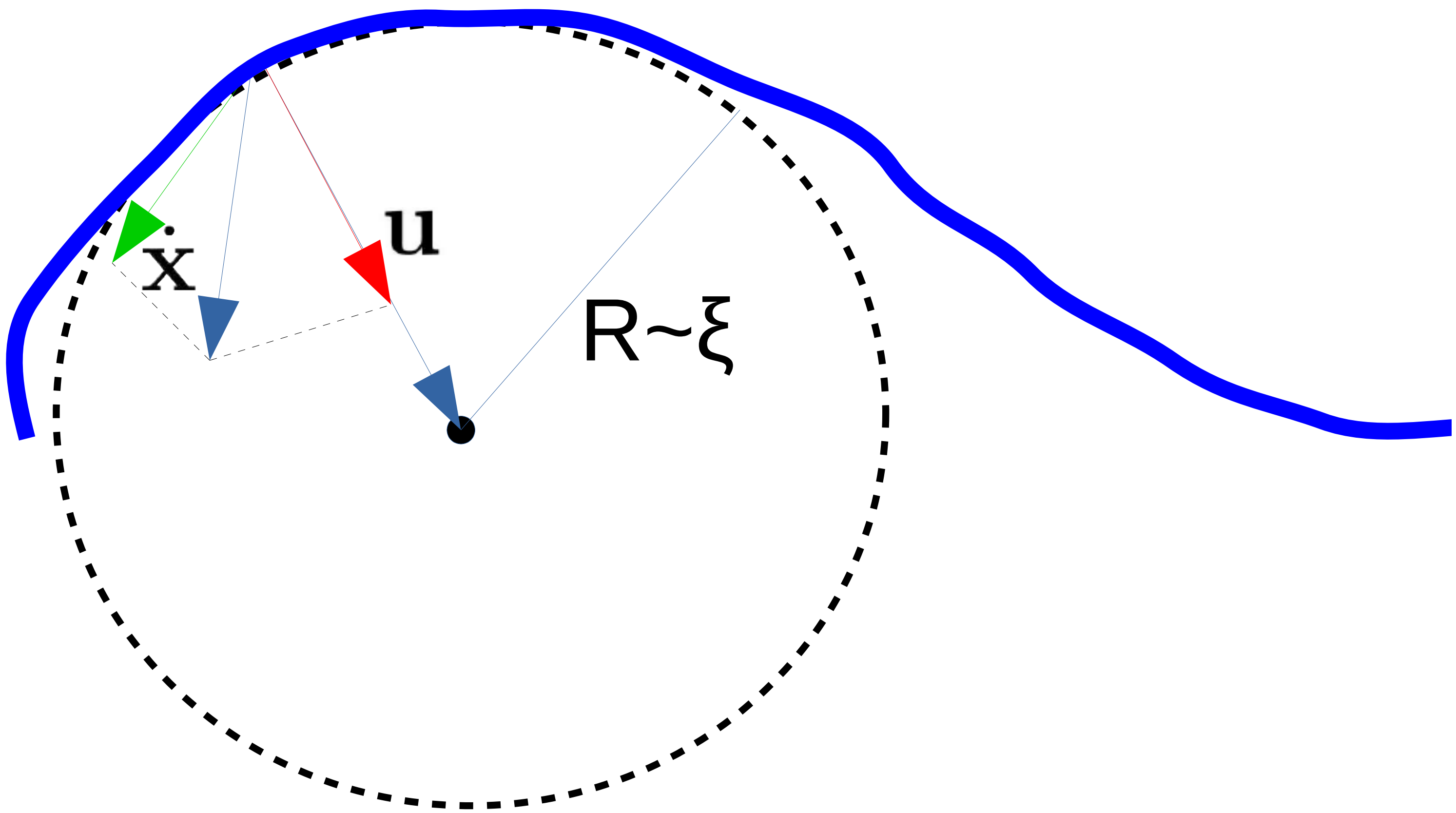
For scale factor $a(t) \propto t^\beta$ we have

Solution has linear scaling:

$$L = \left[\frac{k(k+\tilde{c})}{4\beta(1-\beta)}\right]^{1/2} t\,, \qquad v = \left[\frac{k(1-\beta)}{\beta(k+\tilde{c})}\right]^{1/2}$$

that is, $L \propto t$ and $v = \text{const.}$ with $\rho_\infty \propto t^{-2}$.

Momentum parameter: $k(v) \equiv \dfrac{\left\langle (1-\dot{\boldsymbol{X}}^2)(\dot{\boldsymbol{X}}\cdot\boldsymbol{u}) \right\rangle}{v(1-v^2)}$

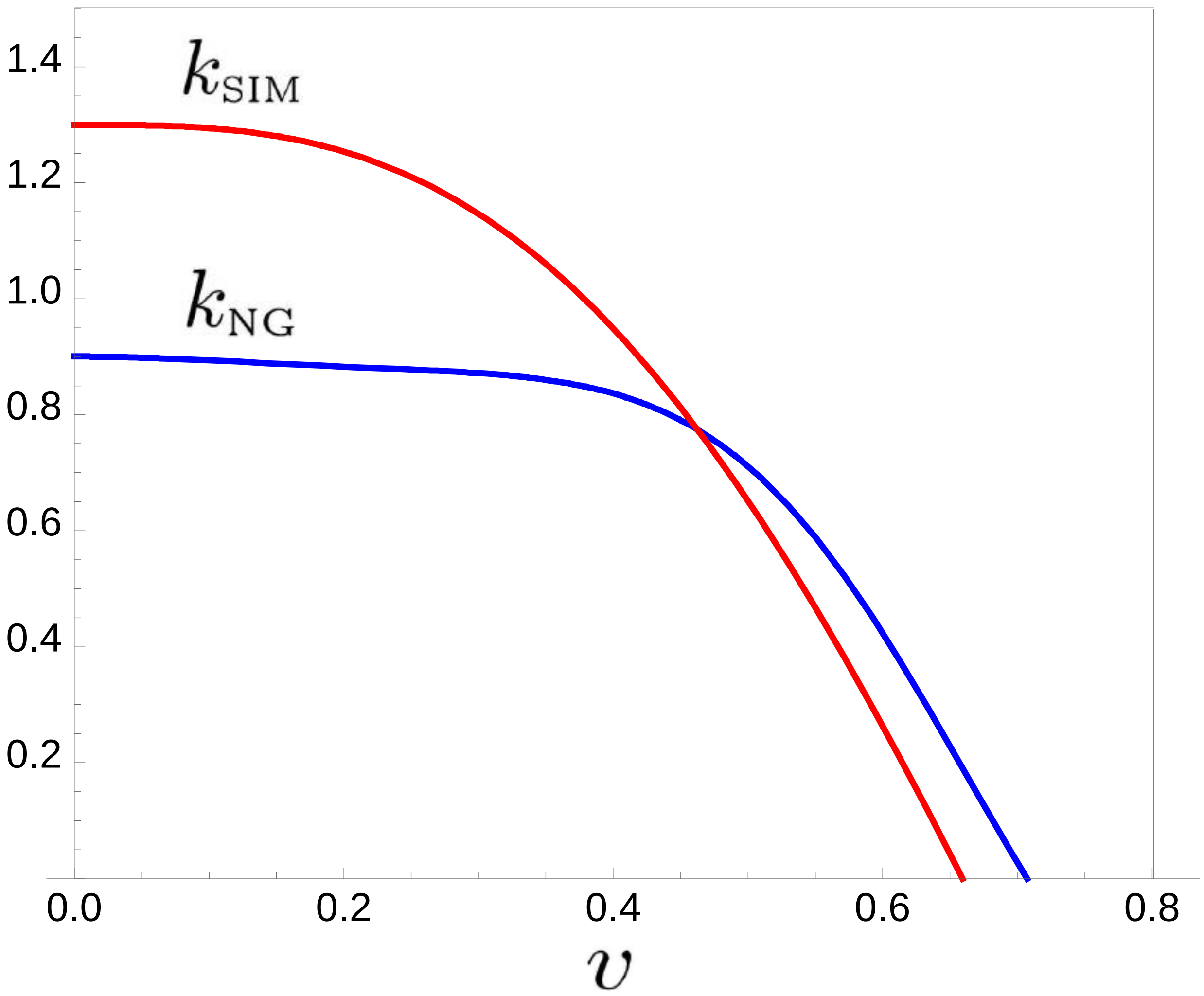


Momentum parameter: $$k(v) \equiv \frac{\left\langle (1-\dot{\boldsymbol{X}}^2)(\dot{\boldsymbol{X}}\cdot\boldsymbol{u}) \right\rangle}{v(1-v^2)}$$

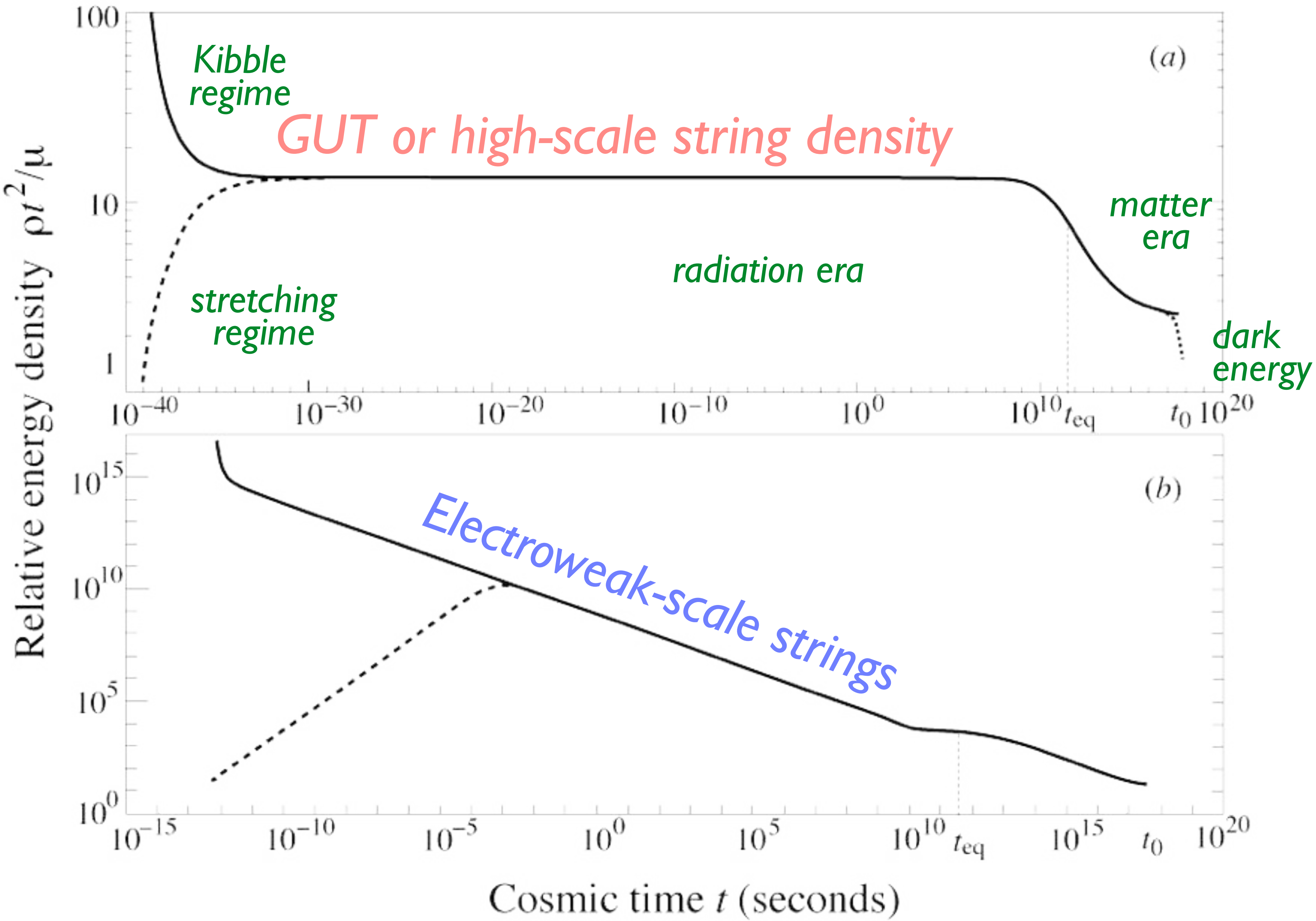
From Nambu-Goto network simulations $$k_{\rm NG}(v) = \frac{2\sqrt{2}}{\pi}\frac{1-8v^6}{1+8v^6}\left(1-v^2\right)\left(1+2\sqrt{2}v^3\right)$$

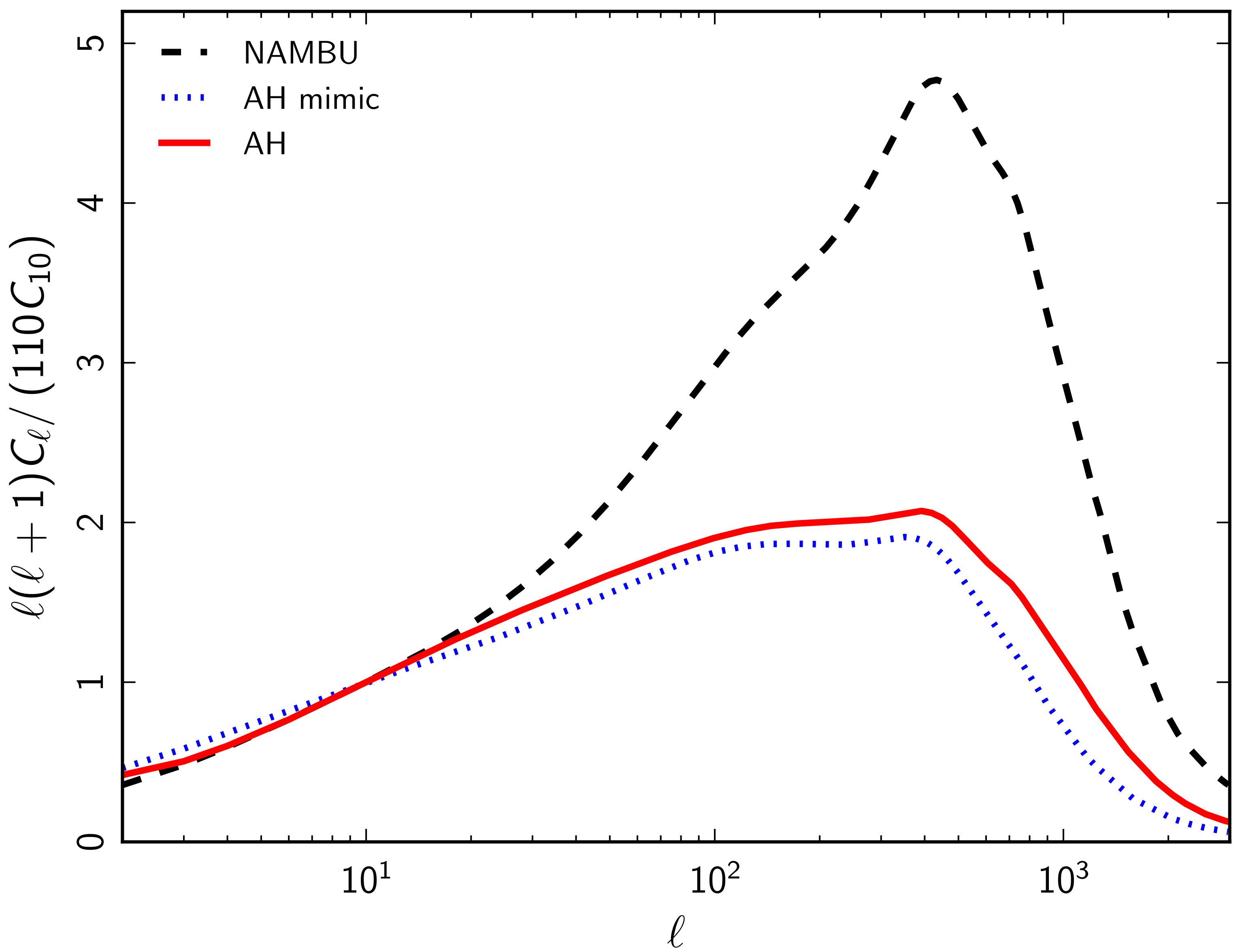
From Abelian-Higgs simulations $$k_{\rm SIM}(v) = k_0\frac{1-(\alpha v^2)^\beta}{1+(\alpha v^2)^\beta}$$

Momentum parameter: $k(v) \equiv \dfrac{\left\langle (1-\dot{\boldsymbol{X}}^2)(\dot{\boldsymbol{X}}\cdot\boldsymbol{u}) \right\rangle}{v(1-v^2)}$



Complete history of strings

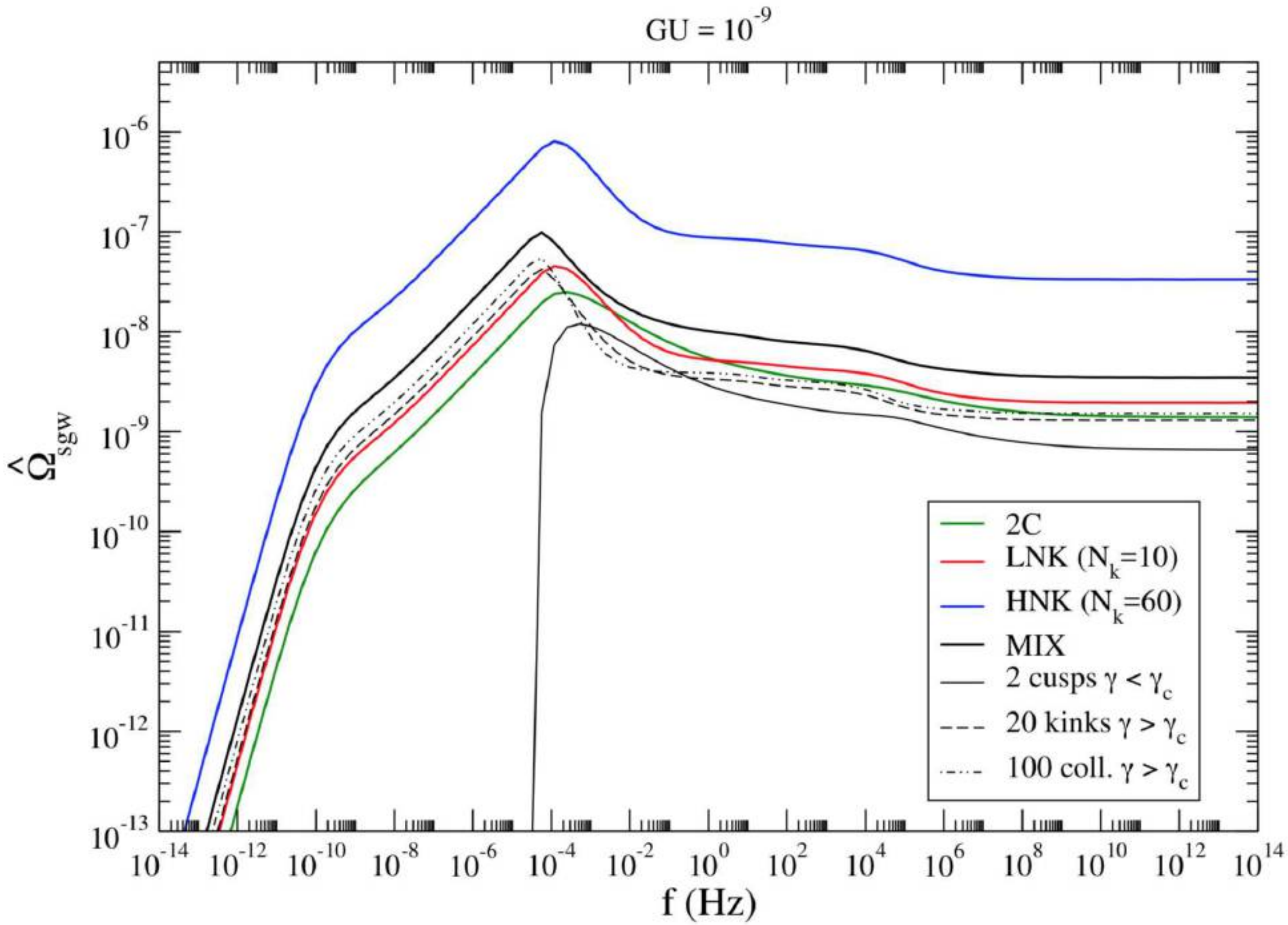




NAMBU
AH mimic
AH
$\ell(\ell+1)C_\ell/(110C_{10})$
ℓ
0
1
2
3
4
5
10^1
10^2
10^3

GW stochastic background

C. Ringeval & T. Suyama, *JCAP* **17,** 12 (2017)
J. J. Blanco-Pillado & K. D. Olum, *Phys. Rev.* **D96**, 104046 (2017)
J. J. Blanco-Pillado K. D. Olum & X. Siemens, *Phys. Lett.* **B778,** 392 (2018)



Model: Witten superconducting cosmic string

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}D_\mu\Phi\,(D^\mu\Phi)^* + \frac{1}{2}D_\mu\Sigma\,(D^\mu\Sigma)^* - V(\Phi,\Sigma)$$

Field strength tensors

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Covariant derivatives

$$D_\mu\Phi = (\partial_\mu - \mathrm{i}\,e_\phi B_\mu)\,\Phi$$

$$D_\mu\Sigma = (\partial_\mu - \mathrm{i}\,e_\sigma A_\mu)\,\Sigma$$

Potential

$$V(\Phi,\Sigma) = \frac{\lambda_\phi}{4}(|\Phi|^2 - \eta^2)^2 + f(|\Phi|^2 - \eta^2)|\Sigma|^2 + \frac{\lambda_\sigma}{4}|\Sigma|^4 + \frac{m_\sigma^2}{2}|\Sigma|^2$$

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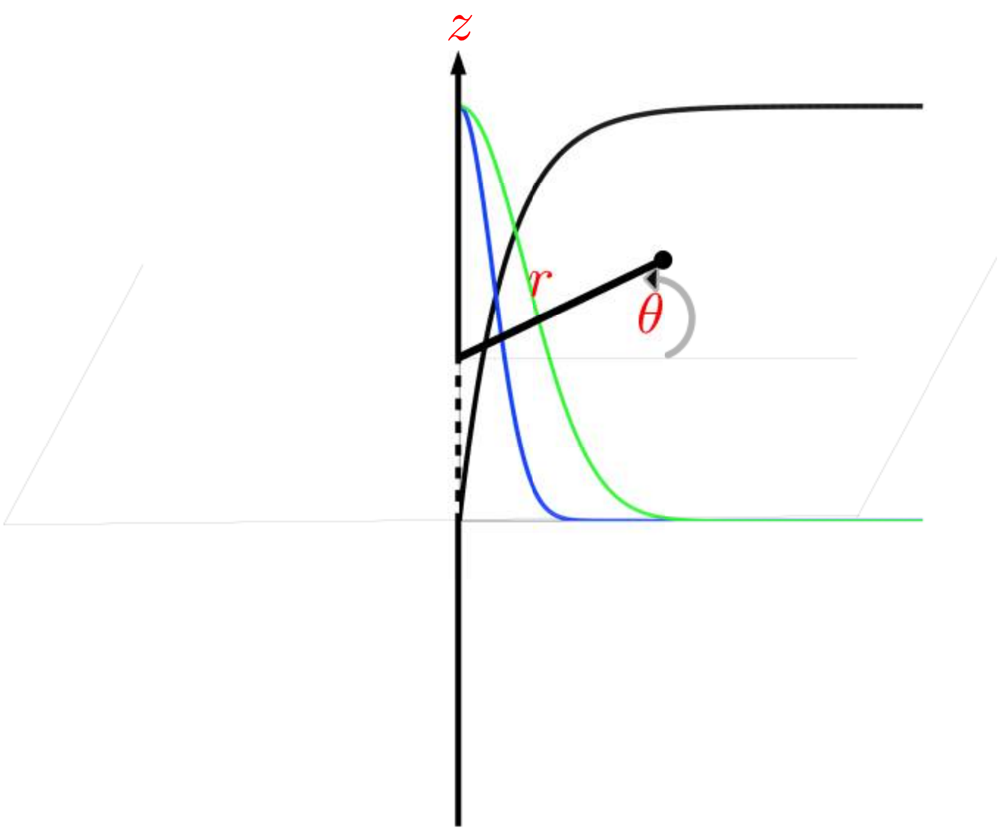
Cylindrical coordinates (t, r, θ, z)



Ansatz

$$B_\mu \mathrm{d}x^\mu = \frac{1}{e_\phi}\left[n - P(r)\right]\mathrm{d}\theta$$

$$\Phi(r,\theta) = \eta\phi(r)\mathrm{e}^{\mathrm{i}n\theta}$$

$$A_\mu \mathrm{d}x^\mu = A_z(r)\mathrm{d}z + A_t(r)\mathrm{d}t$$

$$\Sigma(t,r,z) = \eta\sigma(r)\mathrm{e}^{\mathrm{i}(\omega t - kz)}$$

Model: Witten superconducting cosmic string

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}D_\mu\Phi\left(D^\mu\Phi\right)^* + \frac{1}{2}D_\mu\Sigma\left(D^\mu\Sigma\right)^* - V(\Phi,\Sigma)$$

Cylindrical coordinates (t, r, θ, z)

Background and condensate

$$\Phi(r,\theta) = \eta\phi(r)\mathrm{e}^{\mathrm{i}n\theta}$$

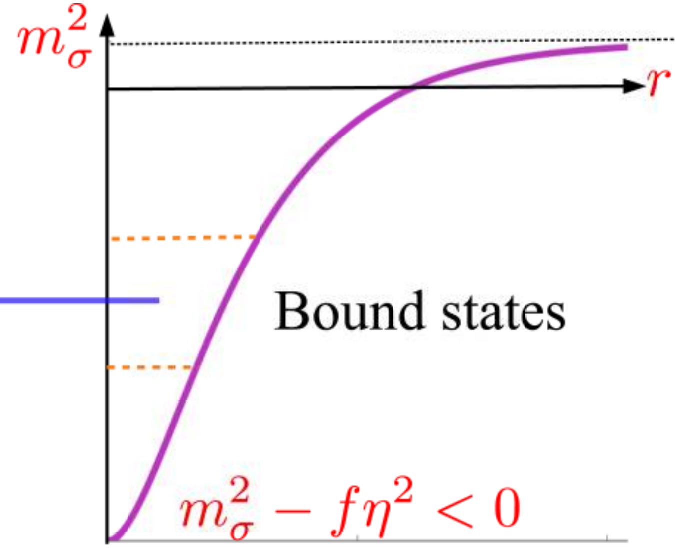
$$\Sigma(t,r,z) = \eta\sigma(r)\mathrm{e}^{\mathrm{i}Et}$$

$\Longrightarrow$

2D Schrödinger equation

$$\left[-\Delta + V(r)\right]\sigma = E^2\sigma$$

$$V(r) = f\left[\phi^2(r) - \eta^2\right] + m_\sigma^2$$



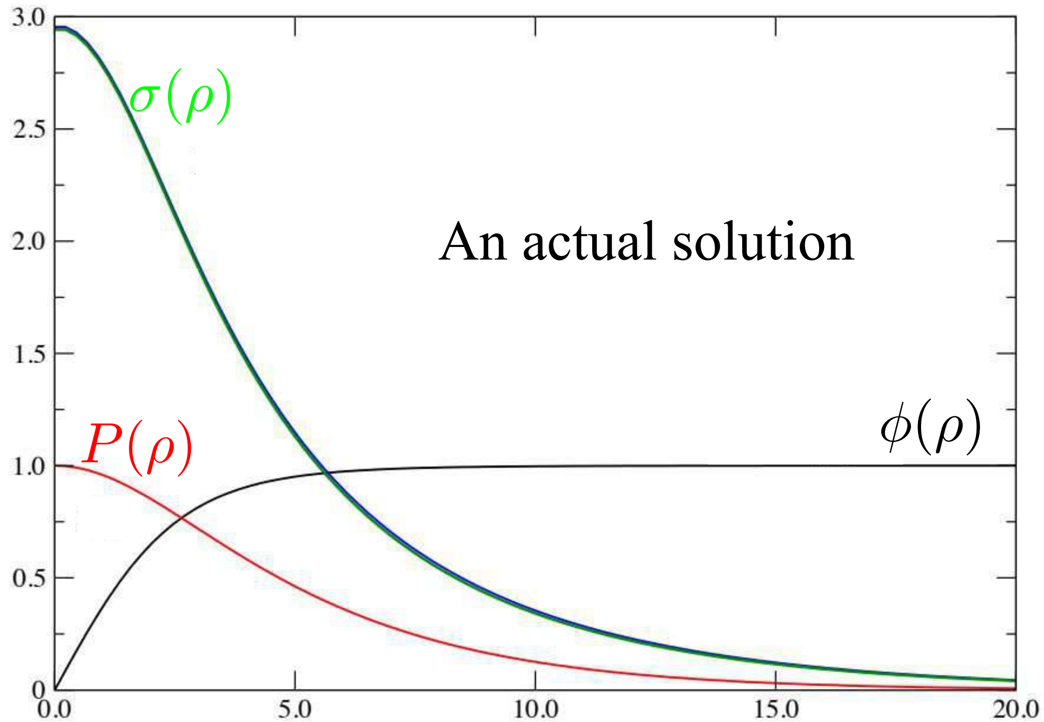
Negative energy solutions
$\Longrightarrow$ instabilities

E. Witten, *Nucl. Phys.* **B249**, 557 (1985)

Model: Witten superconducting cosmic string

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}D_\mu\Phi\left(D^\mu\Phi\right)^* + \frac{1}{2}D_\mu\Sigma\left(D^\mu\Sigma\right)^* - V(\Phi,\Sigma)$$

Cylindrical coordinates (t, r, θ, z)



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$$\Sigma(t,r,z) = \eta\sigma(r)\mathrm{e}^{\mathrm{i}(\omega t - kz)}$$

$$\rho \equiv \sqrt{\lambda_\phi}\eta r$$

PP, *Phys. Rev.* **D45**, 1091 (1992)

Model: Witten superconducting cosmic string

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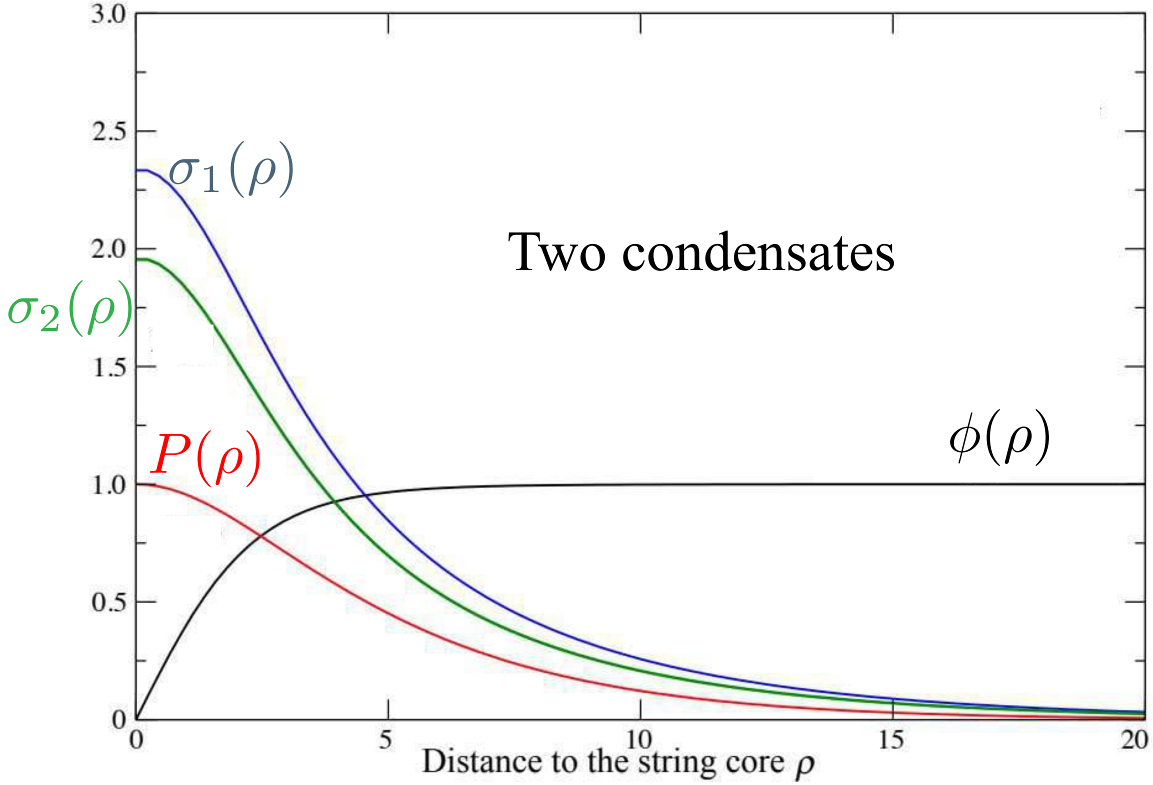
Cylindrical coordinates (t, r, θ, z)



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M. Lilley, X. Martin & PP, *Phys. Rev.* **D79**, 103514 (2009)

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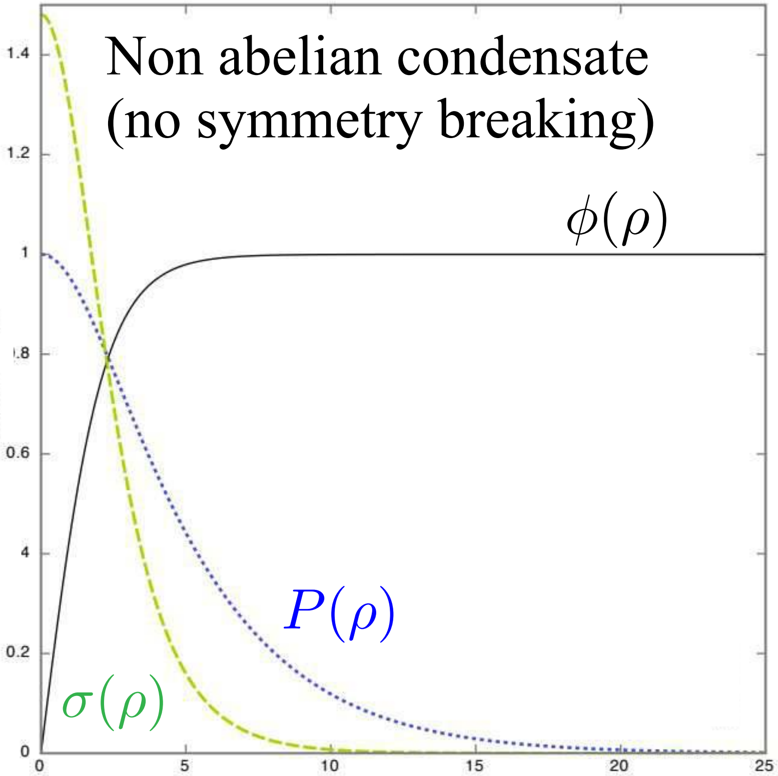
Cylindrical coordinates (t, r, θ, z)



M. Lilley, F. Di Marco, J. Martin & PP, *Phys. Rev.* **D82**, 023510 (2010)

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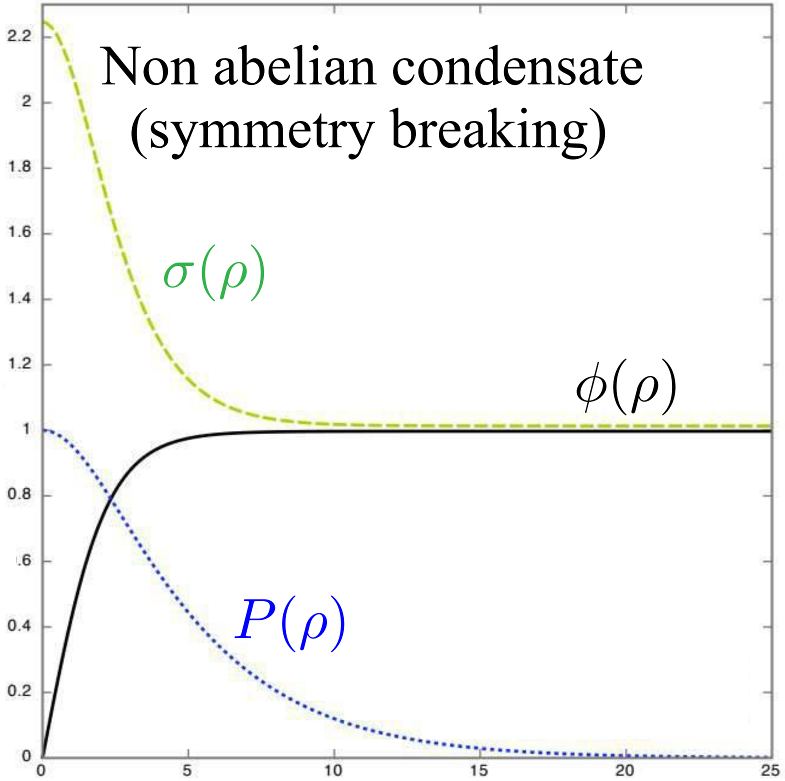

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M. Lilley, F. Di Marco, J. Martin & PP, *Phys. Rev.* **D82**, 023510 (2010)

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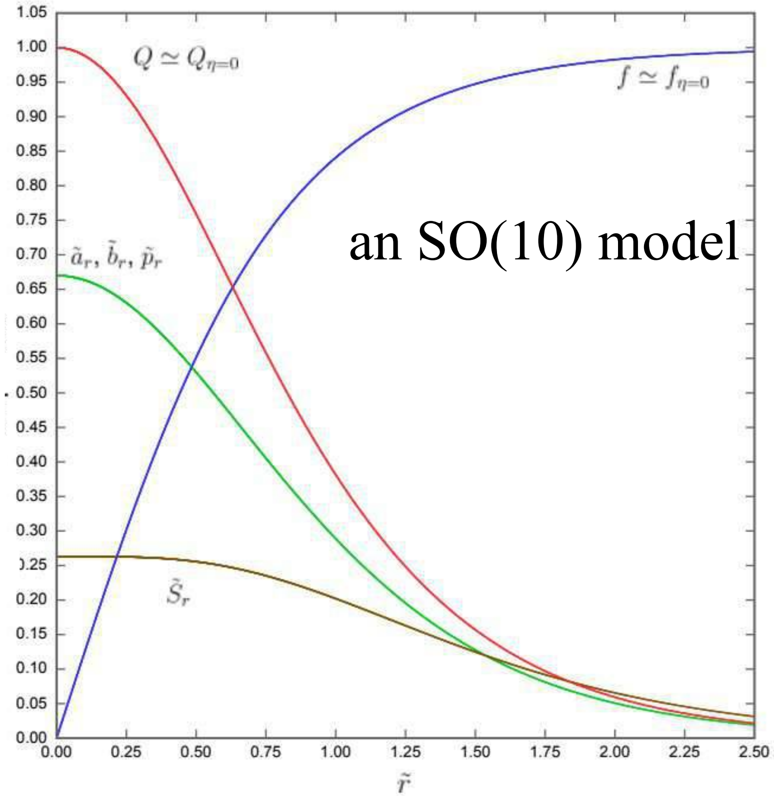
Cylindrical coordinates (t, r, θ, z)



A.-C. Davis & S. C. Davis, *Phys. Rev.* **D55**, 1879 (1997)
E. Allys, *Phys. Rev.* **D93**, 105021 (2016)

$$\rho \equiv \sqrt{\lambda_\phi}\eta r$$

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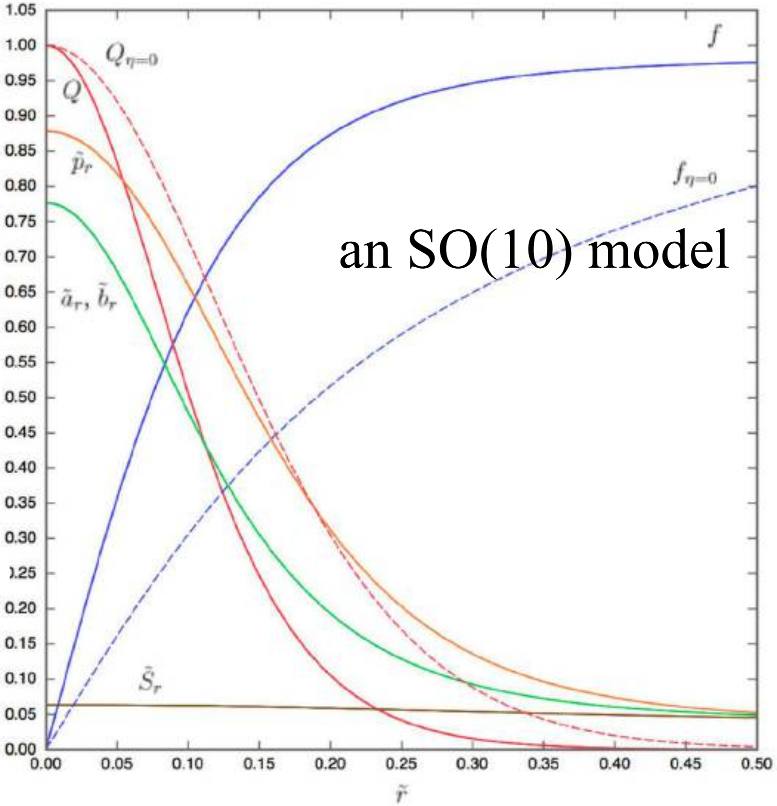
Cylindrical coordinates (t, r, θ, z)



$$\rho \equiv \sqrt{\lambda_\phi}\eta r$$

A.-C. Davis & S. C. Davis, *Phys. Rev.* **D55**, 1879 (1997)
E. Allys, *Phys. Rev.* **D93**, 105021 (2016)

Neutral limit: (avoid complications due to long-range interaction...) PP, *Phys. Rev.* **D46**, 3335 (1992)
Phys. Rev. **D47**, 3169 (1993)

$e_\sigma \to 0$

$A_\mu \to 0$

$$\Sigma(x^\alpha) = \sigma(x^\perp)\mathrm{e}^{\mathrm{i}\psi(\xi_a)}$$

Conserved current $\mathcal{J}_\mu = \sigma^2(r)\partial_\mu\psi$

$$\omega t_\mu - k z_\mu = \omega \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - k \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Integrated (macroscopic) current $c_\mu \equiv \int \mathrm{d}^2x^\perp \mathcal{J}_\mu = 2\pi\partial_\mu\psi \int \sigma^2(r) r\mathrm{d}r$

Scalar current $\mu = \sqrt{|c_\mu c^\mu|}$

Scalar state parameter $w = g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \to k^2 - \omega^2$

$$\longrightarrow \quad \mu = 2\pi\sqrt{|w|}\int \sigma^2(r) r\mathrm{d}r$$

ν k

$-\omega$

Stress energy tensor $T^{\mu\nu} = -2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} + g^{\mu\nu}\mathcal{L}$

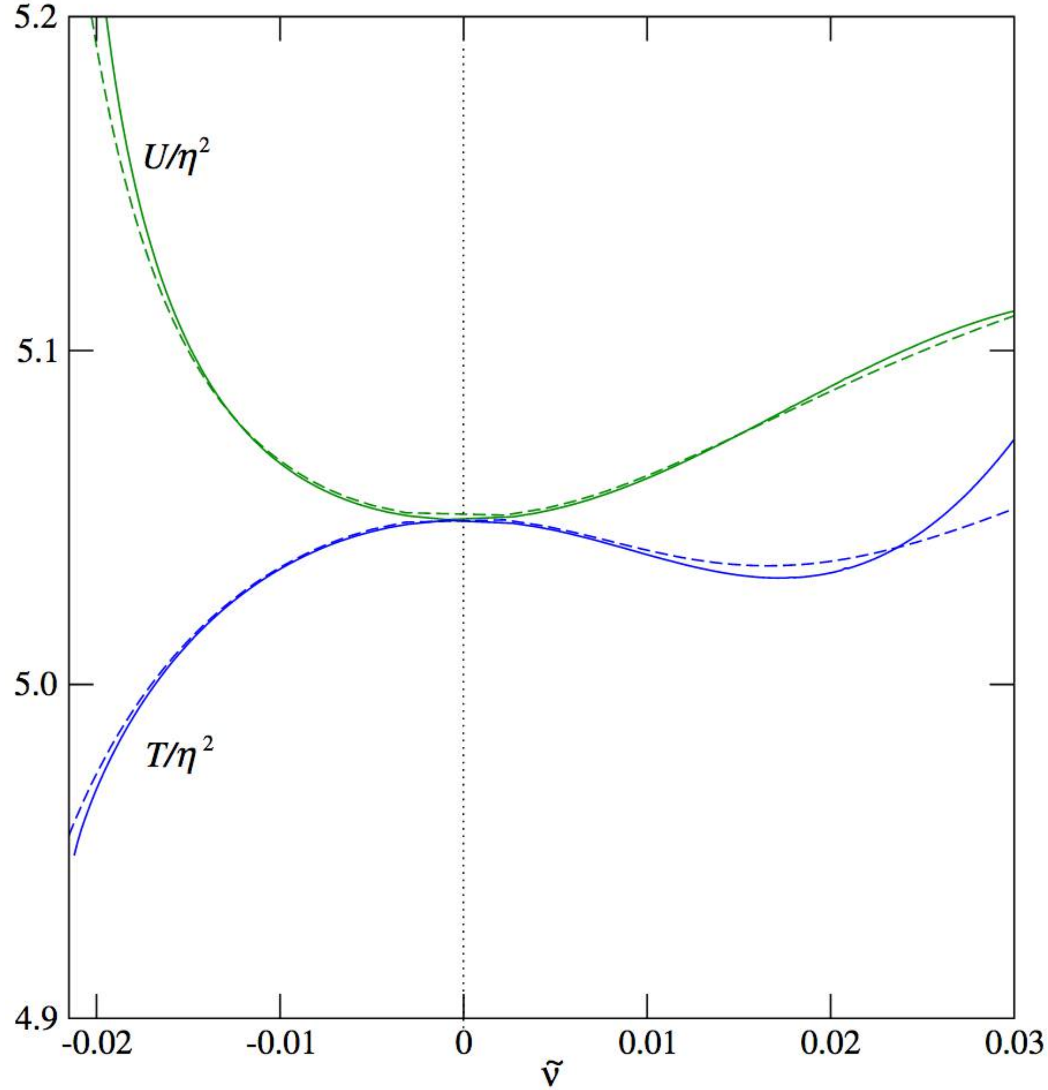
Integrated for macroscopic treatment

$$\mathcal{T}^{\mu\nu} \equiv \int T^{\mu\nu}\mathrm{d}^2x^{\perp}$$
$$= Uu^{\mu}u^{\nu} - Tv^{\mu}v^{\nu}$$
$$= (U-T)u^{\mu}u^{\nu} - T\eta^{\mu\nu}$$

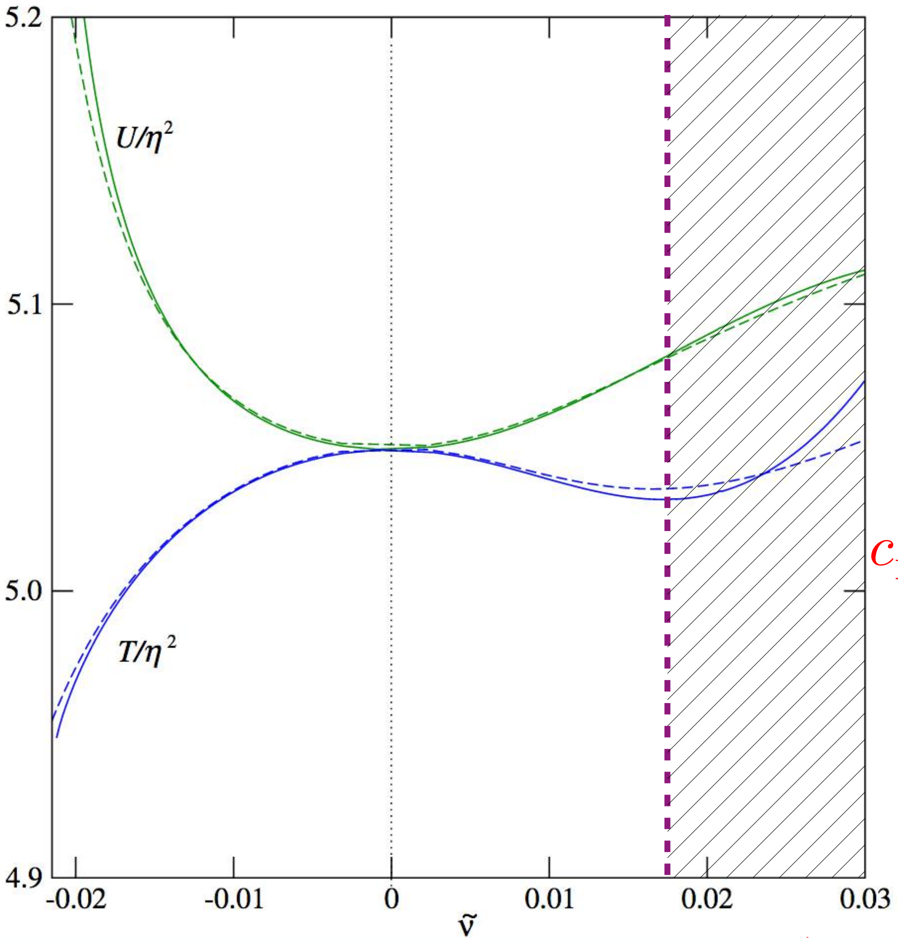
$U(\nu)$ $T(\nu)$

$$\boxed{U - T = \nu\mu}$$

Legendre transform $\Longrightarrow$ dual formalism



Phase frequency threshold

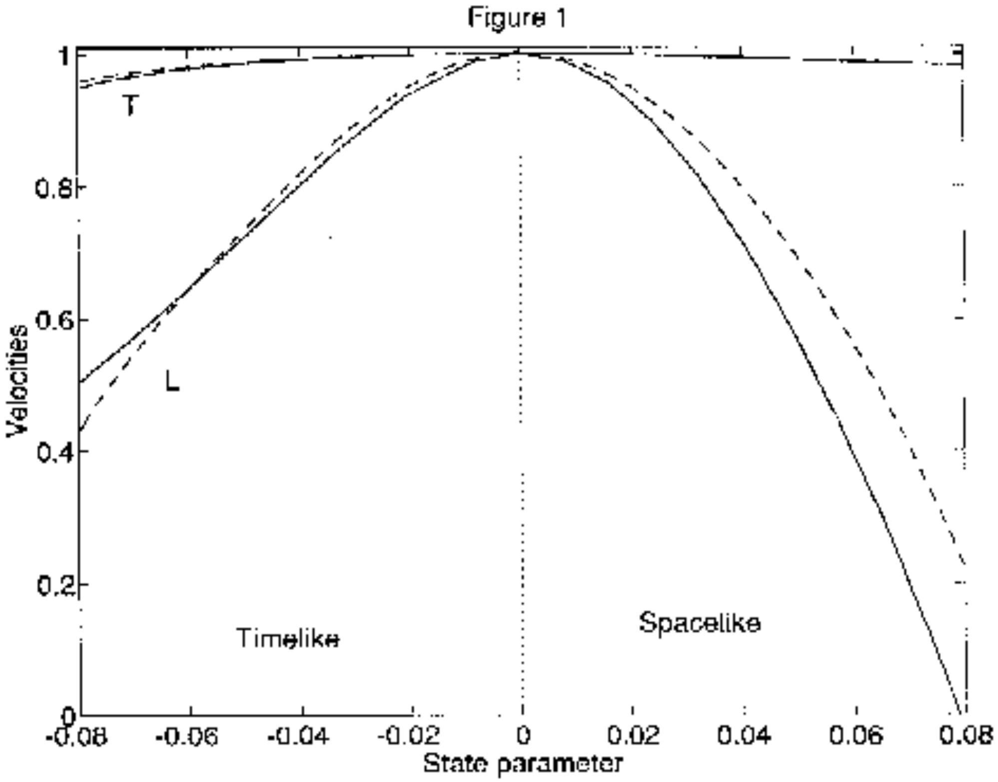


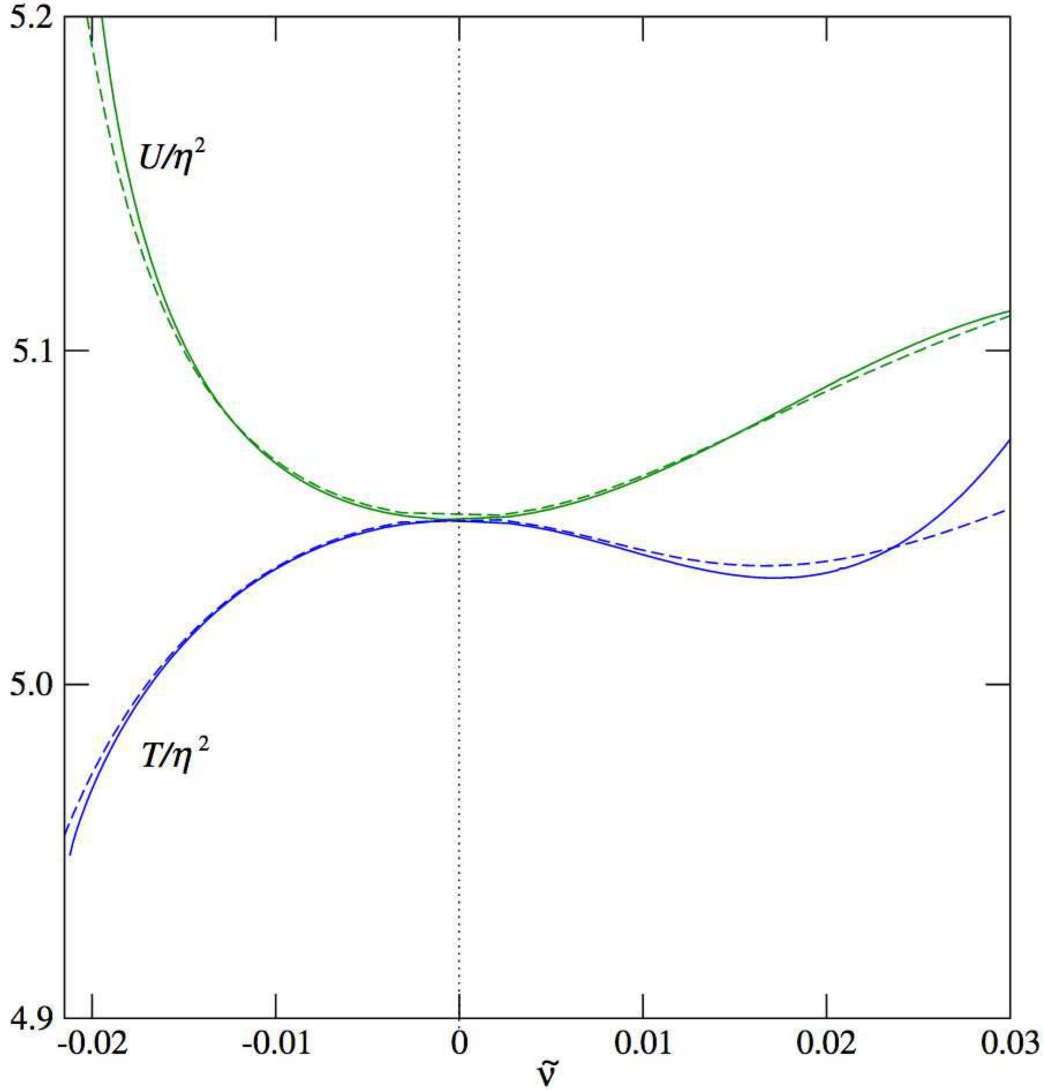
$c_{\mathrm{L}} \le 0$

Macroscopic perturbations (stability constraint)

$$c_{\mathrm{T}}^2 = \frac{T}{U} > 0 \qquad c_{\mathrm{L}}^2 = -\frac{\mathrm{d}T}{\mathrm{d}U} = \frac{\nu}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\nu} > 0$$

(no spring)





$$\boxed{U = T + \mu\nu}$$

Legendre transform $\Longrightarrow$ dual formalism

B. Carter & PP, *Phys. Rev.* **D52**, R1744 (1995)
B. Carter, PP & A. Gangui, *Phys. Rev.* **D55**, 4647 (1997)

Macroscopic formalism

State parameter $w \Longrightarrow$ worldsheet lagrangian $\mathcal{L}(w)$ and $w = \kappa_0 \gamma^{ab} \partial_a \varphi \partial_b \varphi$

$$\nu^2 = |w|$$

$$\mathcal{S}_{\mathcal{L}} = -m^2 \int \mathrm{d}^2\xi \sqrt{-\gamma} \mathcal{L}(w)$$

$$\mu^2 = |\chi|$$

Master function (dual to lagrangian) $\Lambda(\chi)$: set $\chi = \tilde{\kappa}_0 \gamma^{ab} \partial_a \psi \partial_b \psi$

$\longrightarrow$ 2 conserved (orthogonal:$\gamma_{ab} n^a z^a = 0$) currents

$$z^a = -\frac{\partial \mathcal{L}}{\partial \partial_a \varphi} \qquad n^a = -\frac{\partial \Lambda}{\partial \partial_a \psi}$$

Equivalent alternative dynamical description

$$\mathcal{S}_{\mathcal{L}} \Longleftrightarrow \mathcal{S}_{\Lambda} = -m^2 \int \mathrm{d}^2\xi \sqrt{-\gamma} \Lambda(\chi)$$

Current amplitudes $\quad \mathcal{K} z^a = \kappa_0 \partial^a \varphi \quad \Longleftrightarrow \quad \tilde{\mathcal{K}} n^a = \tilde{\kappa}_0 \partial^a \psi$

$$\mathcal{K}^{-1} = -2\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}w} \quad \Longleftrightarrow \quad \tilde{\mathcal{K}}^{-1} = -2\frac{\mathrm{d}\Lambda}{\mathrm{d}\chi}$$

Equivalent formulations

$$\tilde{\mathcal{K}} = -\mathcal{K}^{-1} \quad \Longrightarrow \quad w = \mathcal{K}^2 \chi$$

Legendre transformation $\quad \Lambda = \mathcal{L} + \mathcal{K}\chi$

$$U = T + \mu\nu$$

$$\mu = \frac{\mathrm{d}U}{\mathrm{d}\nu}$$

$$\nu = -\frac{\mathrm{d}T}{\mathrm{d}\mu}$$

Identification for the macroscopic description

Regime	U	T	w and χ	current
Electric	$-\Lambda$	$-\mathcal{L}$	≤ 0	timelike
Magnetic	$-\mathcal{L}$	$-\Lambda$	≥ 0	spacelike

Specific models

- Nambu-Goto (structureless) $\mathcal{L} = -m^2 \implies U = T$
- Small field expansion $\mathcal{L} = -m^2 + \frac{\kappa}{2} \implies \Lambda = -m^2 + \frac{\chi}{2}$ self-dual

Subsonic equation of state $U + T = 2m^2 \implies c_{\mathrm{T}} < c_{\mathrm{L}} = 1$

- Kaluza-Klein $\mathcal{L} = -m\sqrt{m^2 - \kappa} \implies \Lambda = -m\sqrt{m^2 - \chi}$ self-dual

Transonic equation of state $UT = m^4 \implies c_{\mathrm{T}} = c_{\mathrm{L}} \leq 1$

- Small field expansion (higher order) $\mathcal{L} = -m^2 + \frac{\kappa}{2} + \frac{\kappa^2}{4m_*^2} + \mathcal{O}(\kappa^3)$

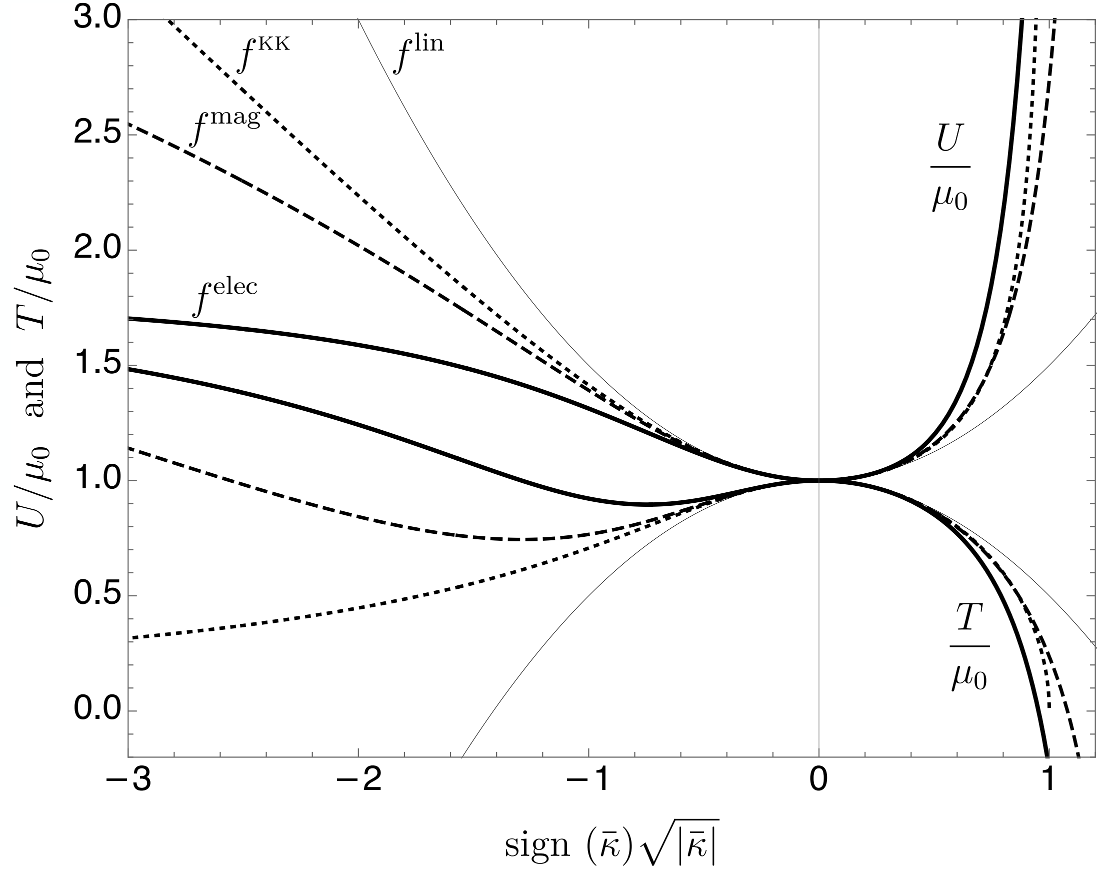
- Witten magnetic model $\mathcal{L} = -m^2 + \frac{\kappa}{2}\left(1 - \frac{\kappa}{m_*^2}\right)^{-1}$
- Witten electric model $\mathcal{L} = -m^2 - \frac{m_*^2}{2}\ln\left(1 - \frac{\kappa}{m_*^2}\right)$

Supersonic equation of state

$c_{\mathrm{L}} \leq c_{\mathrm{T}} \leq 1$

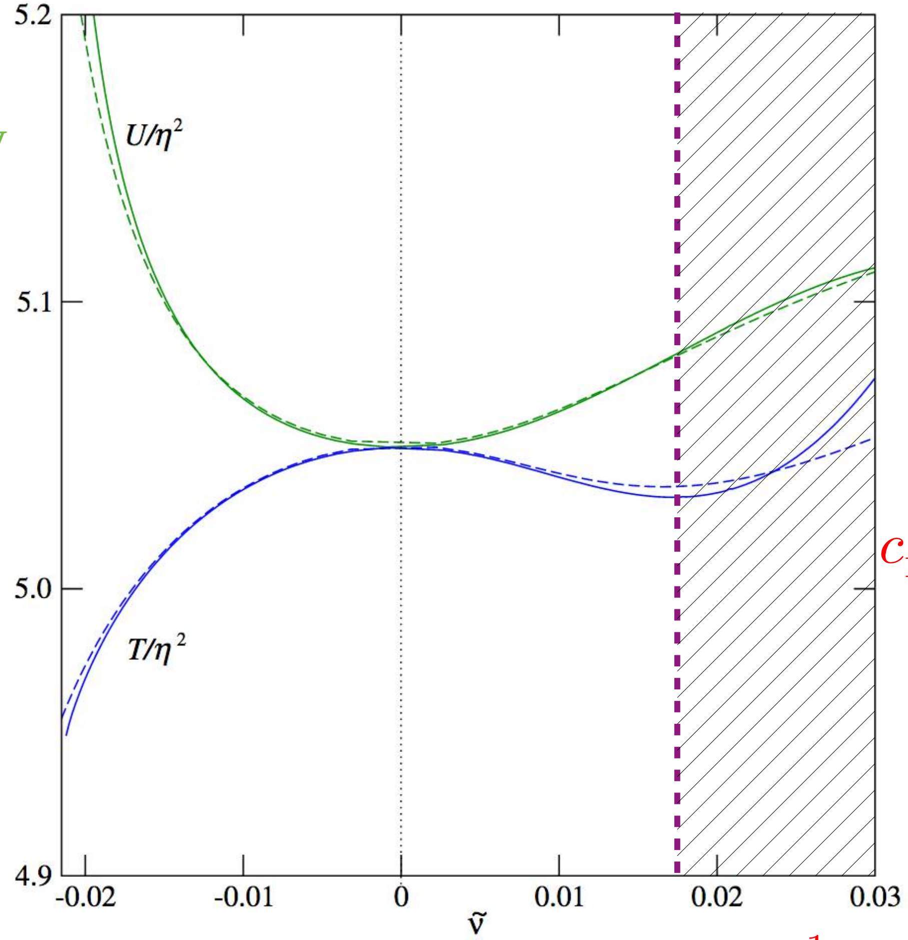
Phase frequency threshold

$\lim_{\nu \to -m_*} \mu = \infty$



3.0
2.5
2.0
1.5
1.0
0.5
0.0
−3
−2
−1
0
1
U/μ_0 and T/μ_0
sign $(\bar{\kappa})\sqrt{|\bar{\kappa}|}$
f^{KK}
f^{lin}
f^{mag}
f^{elec}
$\frac{U}{\mu_0}$
$\frac{T}{\mu_0}$

Phase frequency threshold



$c_{\mathrm{L}} \leq 0$

- Witten magnetic model $\mathcal{L} = -m^2 + \frac{w}{2}\left(1 - \frac{w}{m_*^2}\right)^{-1}$
- Witten electric model $\mathcal{L} = -m^2 - \frac{m_*^2}{2}\ln\left(1 - \frac{w}{m_*^2}\right)$

Dynamics of current-carrying cosmic strings in expanding universes

Effective action $S = \int \mathcal{L}(\kappa)\sqrt{-\gamma}\,\mathrm{d}^2\sigma = -\mu_0 \int f(\kappa)\sqrt{-\gamma}\,\mathrm{d}\sigma^0\mathrm{d}\sigma^1$

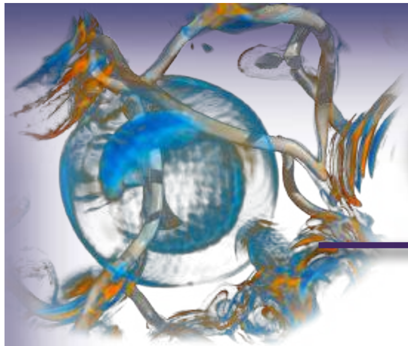
Induced metric $\gamma_{ab} \equiv g_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b} = g_{\mu\nu}X^\mu_{,a}X^\nu_{,b}$

Worldsheet $X^\mu(\sigma^a) = \left\{X^0(\sigma^0,\sigma^1), \boldsymbol{X}(\sigma^0,\sigma^1)\right\}$ Gauge choice 1: $X^0 = \tau$

Background metric $\mathrm{d}s^2_{\mathrm{FLRW}} = a^2(\tau)\left(\mathrm{d}\tau^2 - \mathrm{d}\boldsymbol{x}^2\right) = \mathrm{d}t^2 - a^2(t)\mathrm{d}\boldsymbol{x}^2$

Gauge choice 2: $\frac{\partial \boldsymbol{X}}{\partial \tau}\cdot\frac{\partial \boldsymbol{X}}{\partial \sigma} \equiv \dot{\boldsymbol{X}}\cdot\boldsymbol{X}' = 0 \implies \gamma_{ab} = \mathrm{diag}\left[a^2\left(1-\dot{\boldsymbol{X}}^2\right), -a^2\boldsymbol{X}'^2\right]$

$\implies$ State parameter $\kappa = \frac{\dot{\varphi}^2}{a^2\left(1-\dot{\boldsymbol{X}}^2\right)} - \frac{\varphi'^2}{a^2\boldsymbol{X}'^2} \equiv q^2 - j^2$



Current/Charge VOS Model

Use formalism to obtain the superconducting string equations of motion

State parameter $\kappa = \frac{\dot{\varphi}^2}{a^2\left(1-\dot{\boldsymbol{X}}^2\right)} - \frac{\varphi'^2}{a^2\boldsymbol{X}'^2} \equiv q^2 - j^2$

Equations of motion

$$\partial_\tau\left(\epsilon\bar{U}\right) + \frac{\dot{a}}{a}\epsilon\left[\left(\bar{U}+\bar{T}\right)\dot{\boldsymbol{X}}^2 + \bar{U} - \bar{T}\right] = \partial_\sigma\Phi$$

$$\ddot{\boldsymbol{X}}\epsilon\bar{U} + \frac{\dot{a}}{a}\epsilon\left(\bar{U}+\bar{T}\right)\left(1-\dot{\boldsymbol{X}}^2\right)\dot{\boldsymbol{X}} = \partial_\sigma\left(\frac{\bar{T}}{\epsilon}\boldsymbol{X}'\right) + 2\Phi\dot{\boldsymbol{X}}' + \boldsymbol{X}'\left(\dot{\Phi} + 2\frac{\dot{a}}{a}\Phi\right.$$

$$\partial_\tau\left(f_\kappa a\sqrt{q^2\boldsymbol{X}'^2}\right) = \partial_\sigma\left[f_\kappa a\sqrt{j^2(1-\dot{\boldsymbol{X}}^2)}\right]$$

$$\boxed{\epsilon^2 = \frac{\boldsymbol{X}'^2}{1-\dot{\boldsymbol{X}}^2}}$$

Averaging with string currents

Thermodynamic approach: Total network energy now including current/charges

$$E = a\mu_0 \int \bar{U}\epsilon\, \mathrm{d}\sigma$$ Energy

or just the strings

$$E_0 = a\mu_0 \int \epsilon\, \mathrm{d}\sigma$$ Bare energy

Total charge $Q^2 \equiv \langle q^2 \rangle$ and current $J^2 \equiv \langle j^2 \rangle$

RMS velocity $v \equiv \sqrt{\langle \dot{\boldsymbol{X}}^2 \rangle}$

Integrated state parameter

$$K = Q^2 - J^2$$

Averaging assumptions:

Uncorrelated variables

$$\langle \mathcal{F}(\mathcal{O}) \rangle \approx \mathcal{F}(\langle \mathcal{O} \rangle)$$

+ Brownian string network $E = \dfrac{\mu_0 V}{L_\mathrm{c}^2 a^2} \iff E_0 = \dfrac{\mu_0 V}{\xi_\mathrm{c}^2 a^2}$

$$E = E_0 \langle f - 2q^2 f_\kappa \rangle \implies \frac{E}{E_0} = F - 2Q^2 F' \qquad F(K) \equiv \langle f(\kappa) \rangle \quad F' \equiv \langle f_\kappa \rangle \quad F'' \equiv \langle f_{\kappa\kappa} \rangle$$

Averaging process

$$E = a\mu_0 \int \bar{U}\epsilon \, \mathrm{d}\sigma$$ Energy

$$E_0 = a\mu_0 \int \epsilon \, \mathrm{d}\sigma$$ Bare energy

$$\boxed{\langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{O}\epsilon \, \mathrm{d}\sigma}{\int \epsilon \, \mathrm{d}\sigma}}$$ General variable

Integrated state parameter

$$K = Q^2 - J^2$$

Total charge $Q^2 \equiv \langle q^2 \rangle$ and current $J^2 \equiv \langle j^2 \rangle$

RMS velocity $v \equiv \sqrt{\langle \dot{\boldsymbol{X}}^2 \rangle}$

General assumptions / hypothesis

Uncorrelated variables

$$\langle \mathcal{F}(\mathcal{O}) \rangle \approx \mathcal{F}(\langle \mathcal{O} \rangle)$$

Vanishing boundary terms

$$\int \partial_\sigma \{\mathcal{F}[\boldsymbol{X}(\sigma,\tau)]\} \,\mathrm{d}\sigma \;\to\; \oint \partial_\sigma \{\mathcal{F}[\boldsymbol{X}(\sigma,\tau)]\} \,\mathrm{d}\sigma \approx 0$$

\+ Brownian string network $E = \dfrac{\mu_0 V}{L_\mathrm{c}^2 a^2} \iff E_0 = \dfrac{\mu_0 V}{\xi_\mathrm{c}^2 a^2}$

$$E = E_0 \langle f - 2q^2 f_\kappa \rangle \implies \frac{E}{E_0} = F - 2Q^2 F' \qquad F(K) \equiv \langle f(\kappa) \rangle \quad F' \equiv \langle f_\kappa \rangle \quad F'' \equiv \langle f_{\kappa\kappa} \rangle$$

CVOS evolution equations

Equations of motion from formalism with general master equation (EoS)

$$\frac{dL_c}{d\tau} = \frac{\dot{a}}{a}\frac{L_c}{F - 2Q^2F'}\left\{v^2\left[F - \left(Q^2 - J^2\right)F'\right] - \left(Q^2 + J^2\right)F'\right\}$$

$$\frac{dv}{d\tau} = \frac{\left(1 - v^2\right)}{F - 2Q^2F'}\left\{\frac{k(v)}{L_c\sqrt{F - 2Q^2F'}}\left(F + 2J^2F'\right) - 2v\frac{\dot{a}}{a}\left[F - \left(Q^2 - J^2\right)F'\right]\right\}$$

$$\frac{dJ^2}{d\tau} = 2J^2\left[\frac{vk(v)}{L_c\sqrt{F - 2Q^2F'}} - \frac{\dot{a}}{a}\right]$$

$$\frac{dQ^2}{d\tau} = 2Q^2\frac{F' + 2J^2F''}{F' + 2Q^2F''}\left[\frac{vk(v)}{L_c\sqrt{F - 2Q^2F'}} - \frac{\dot{a}}{a}\right]$$

Chirality

$$K = Q^2 - J^2$$

Charge

$$Y = \tfrac{1}{2}(Q^2 + J^2)$$

$$k_{NG}(v) = \frac{2\sqrt{2}}{\pi}\frac{1 - 8v^6}{1 + 8v^6}$$

Energy loss mechanisms

Phenomenological parameter:

- loop chopping efficiency $$\left.\frac{\mathrm{d}E_0}{\mathrm{d}\tau}\right|_{\mathrm{loops}} = -\, c\, v \frac{E_0}{\xi_{\mathrm{c}}}$$

($c \simeq 0.5$ in ordinary "VOS" model)

- current chopping efficiency $$\left.\frac{\mathrm{d}E}{\mathrm{d}\tau}\right|_{\mathrm{loops}} = -\, g\, cv \frac{E}{\xi_{\mathrm{c}}}$$

- charge leakage $$\left.\frac{\mathrm{d}Y}{\mathrm{d}\tau}\right|_{\mathrm{leakage}} = -\, A\, \frac{Y}{\xi_{\mathrm{c}}} = -\, A\, \frac{Y}{L_{\mathrm{c}}\sqrt{F - 2Q^2 F'}} \rightarrow \frac{Y}{L_{\mathrm{c}}\sqrt{1+Y}}$$

Universe expansion: $$a(\tau) = a_{\mathrm{eq}} \left[2\left(\frac{\tau}{\tau_{\mathrm{eq}}}\right) + \left(\frac{\tau}{\tau_{\mathrm{eq}}}\right)^2 \right]$$

Linear CVOS Model

First attempt: Take simplified linear EoS for calculational simplicity

Linear regime $F(K) = 1 - \frac{\kappa_0}{2} K$

$$\boxed{\begin{aligned} \dot{\zeta}\tau &= \frac{v^2+Y}{1+Y}\frac{\dot{a}}{a}\zeta + \frac{gcv(1+Y)+AY}{2(1+Y)^{3/2}} - \zeta \\ \dot{v}\tau &= \frac{1-v^2}{1+Y}\left[\frac{k\,(1-Y)}{\zeta\sqrt{1+Y}} - 2v\frac{\dot{a}}{a}\right] \\ \dot{Y}\tau &= 2Y\left(\frac{vk}{\zeta\sqrt{1+Y}} - \frac{\dot{a}}{a}\right) - \frac{vc(g-1)}{\zeta}\sqrt{1+Y} - \frac{AY}{\zeta\sqrt{1+Y}} \end{aligned}}$$

$$\dot{K} = 2K\left(\frac{vk}{L_c\sqrt{1+Y}} - \frac{\dot{a}}{a}\right) - \frac{2(1-2\rho_\Lambda)AY}{L_c\sqrt{1+Y}} - 2\frac{v}{L_c}c\,(g-1)\,(1-2\rho)\,\sqrt{1+Y}$$

Scaling solution $L_c = \zeta\tau$ with $\dot{\zeta} = 0$

$$\dot{v} = 0$$

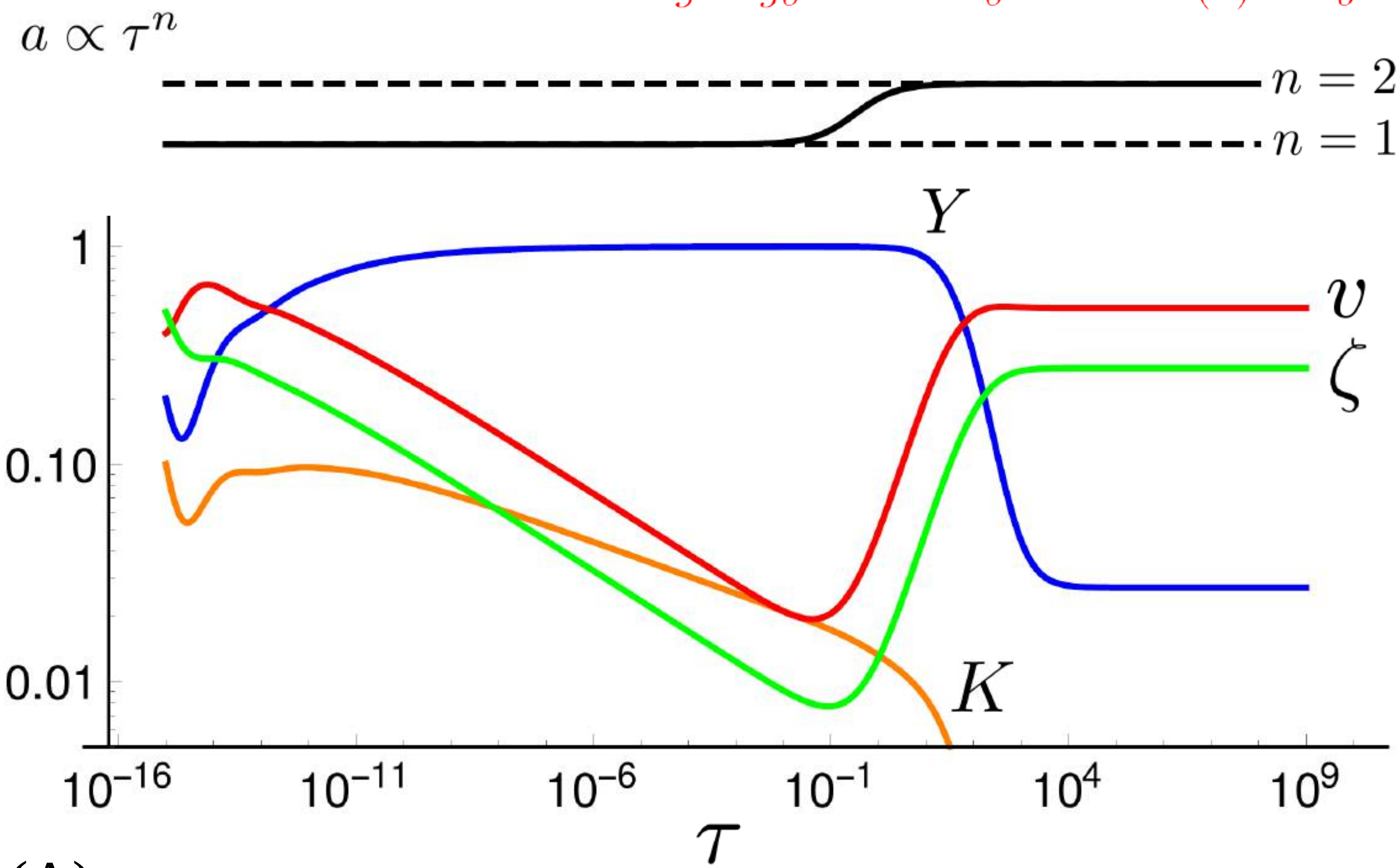
$$\dot{Y} = \dot{K} = 0$$

Charge/current domination (no leakage)

Dynamical solutions

No leakage

$g = g_o = 0.9 \quad c_o = 0.5 \quad k(v) = k_o = 0.6$



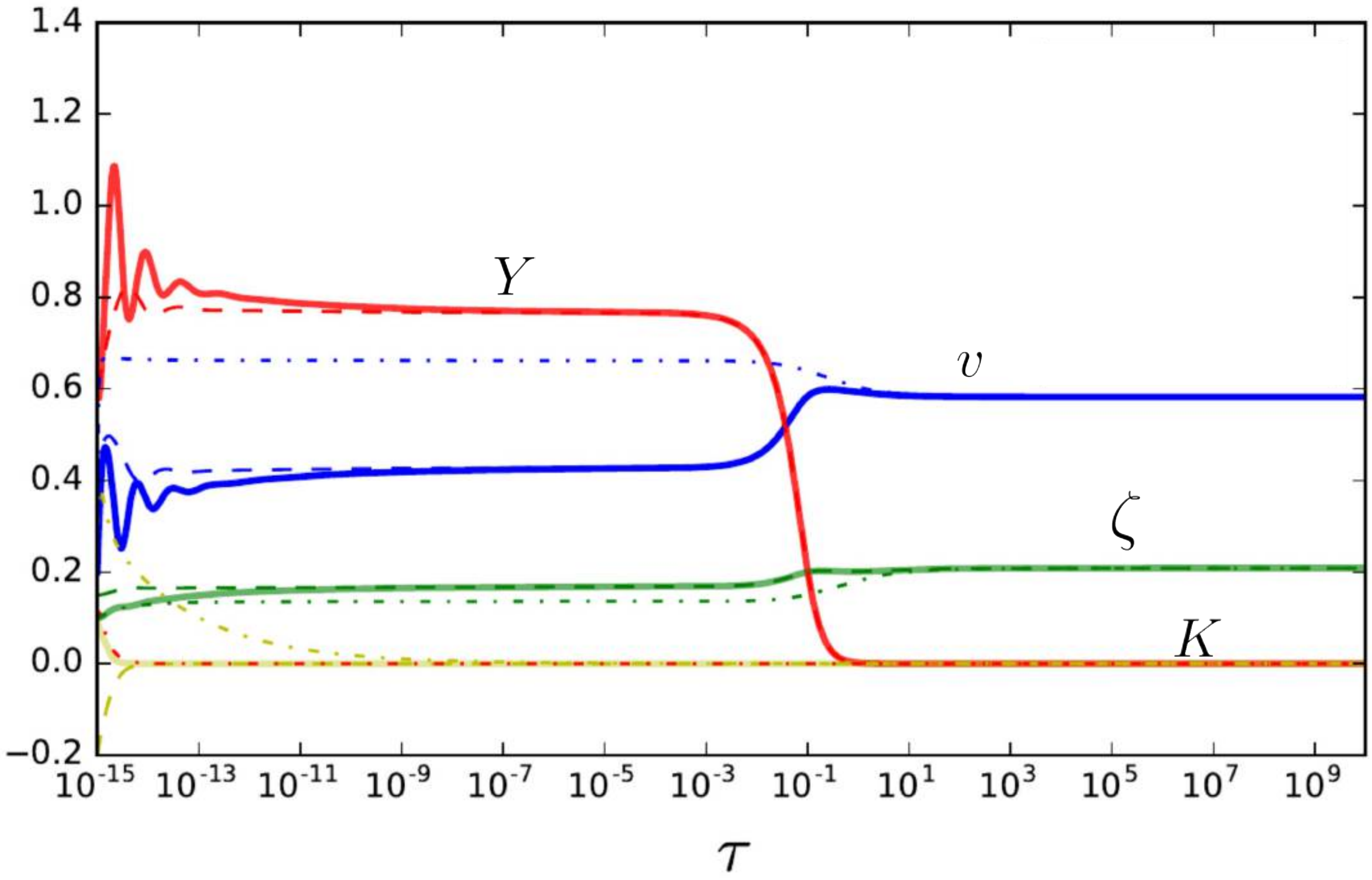
(A)

Charge/current scaling (with leakage)

Dynamical solutions

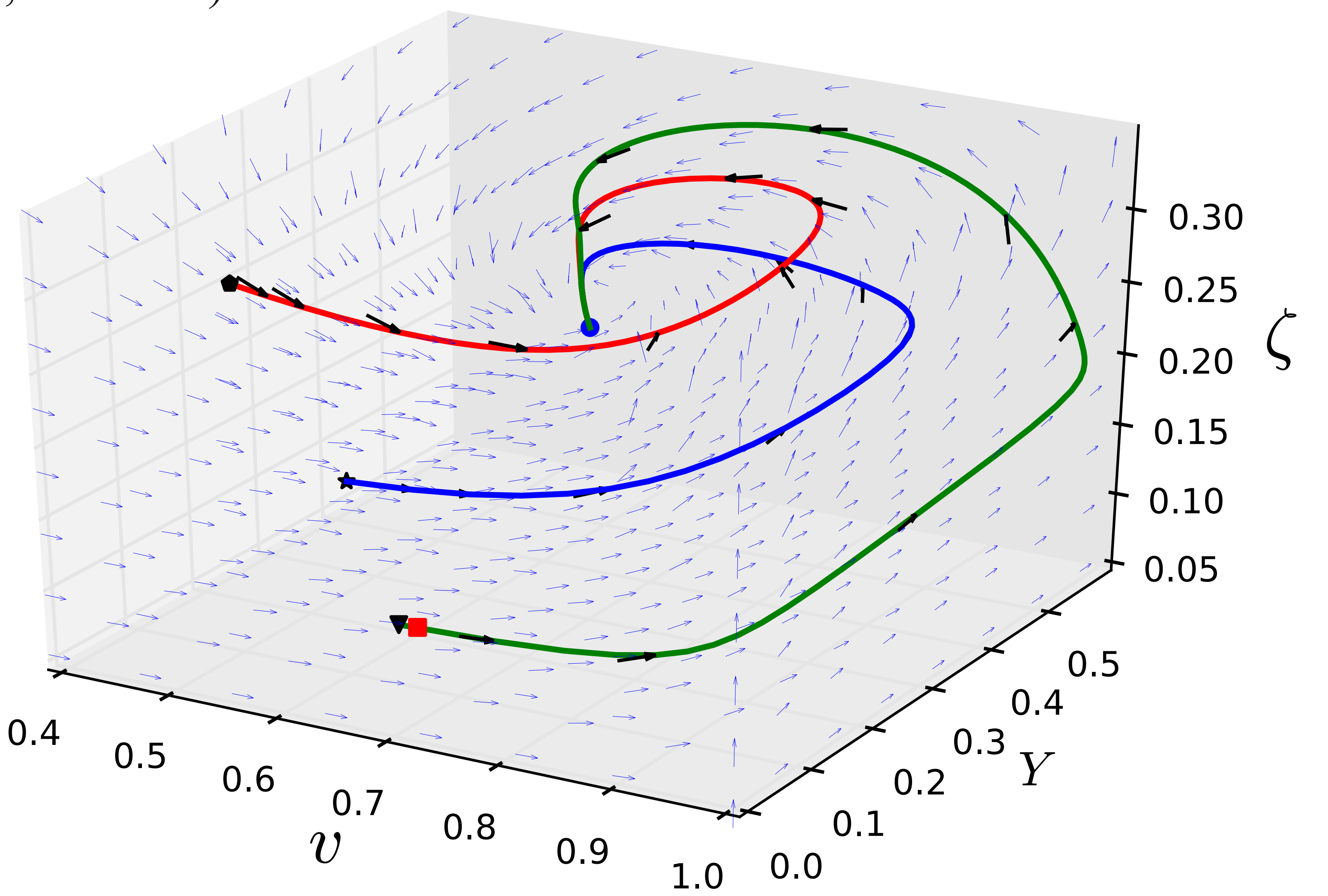
With leakage $g = 1 + 2bY$

$c_o = 0.23 \quad b = 0 \quad A = 0.25$



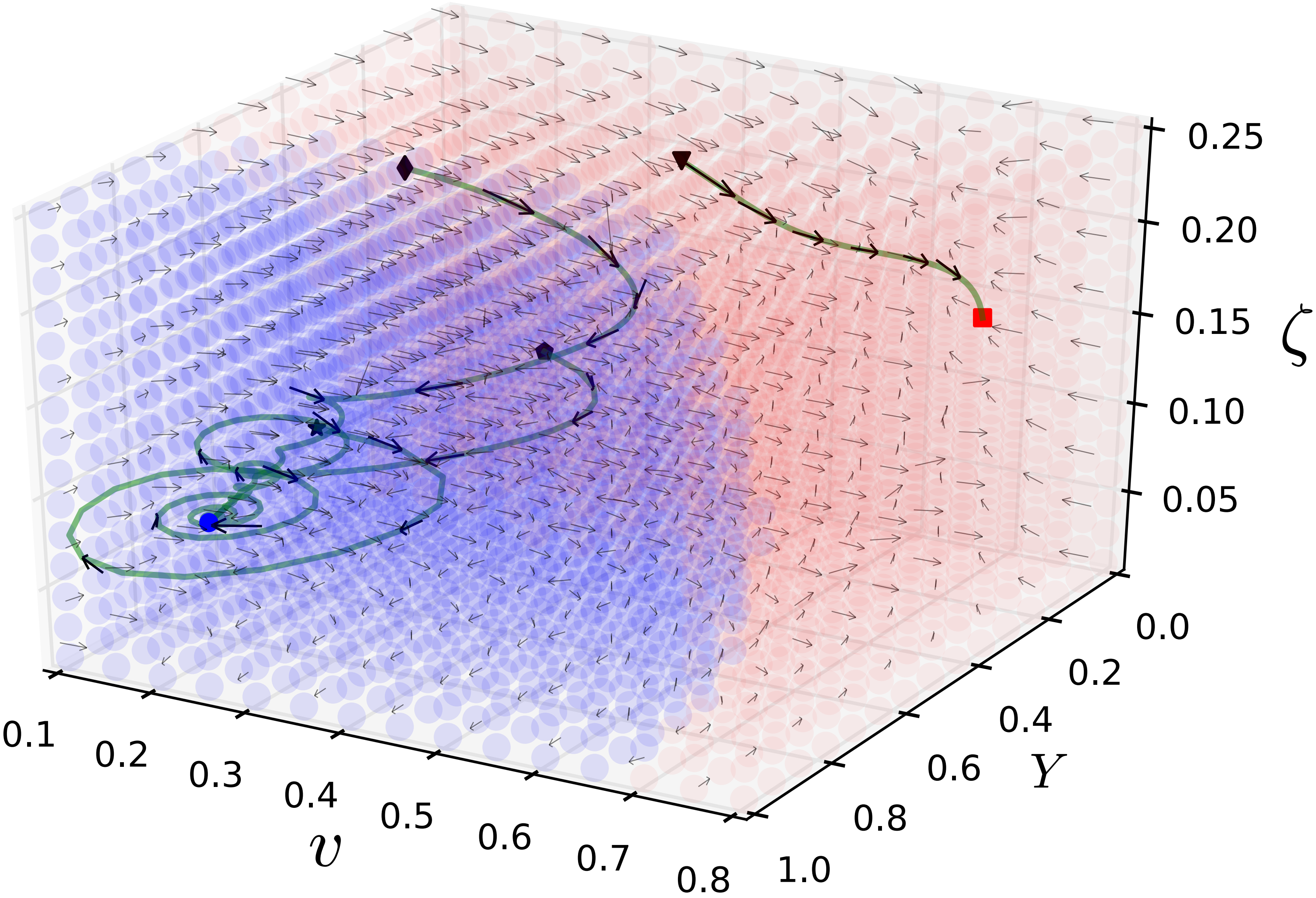
Attractor phase diagram

Attractor ($n = 1$, radiation)



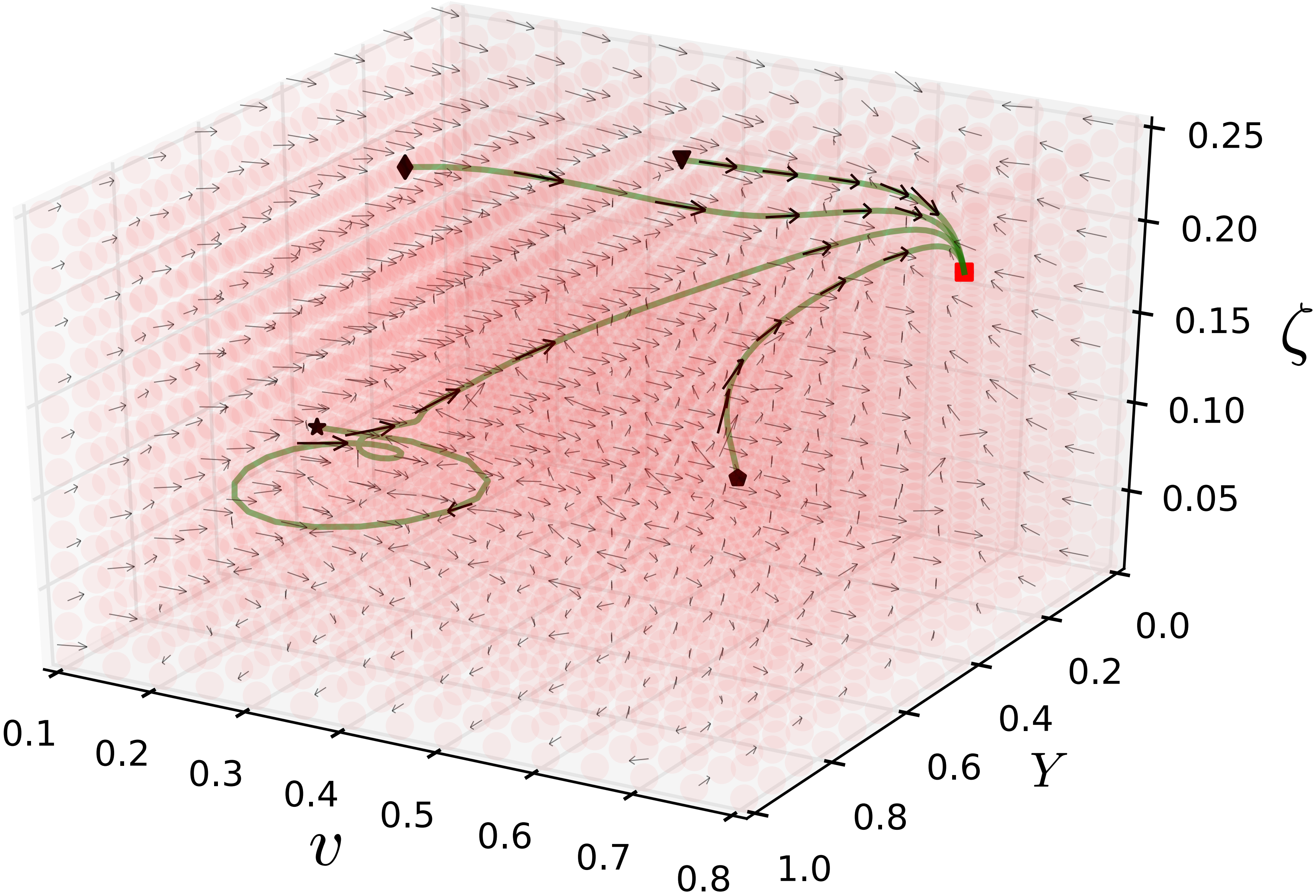
Dependence on initial conditions

Radiation era (charged scaling possible)



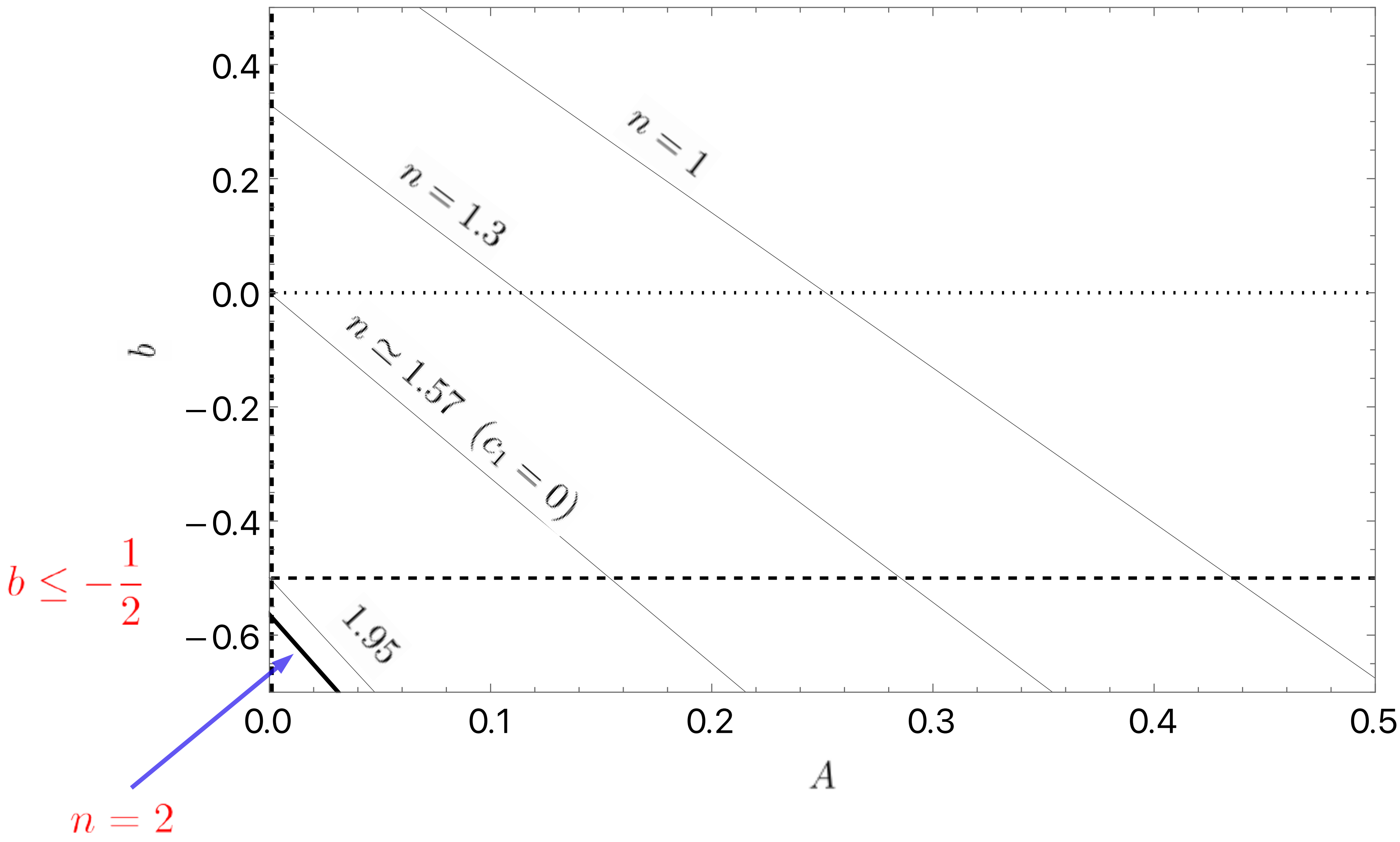
Dependence on initial conditions

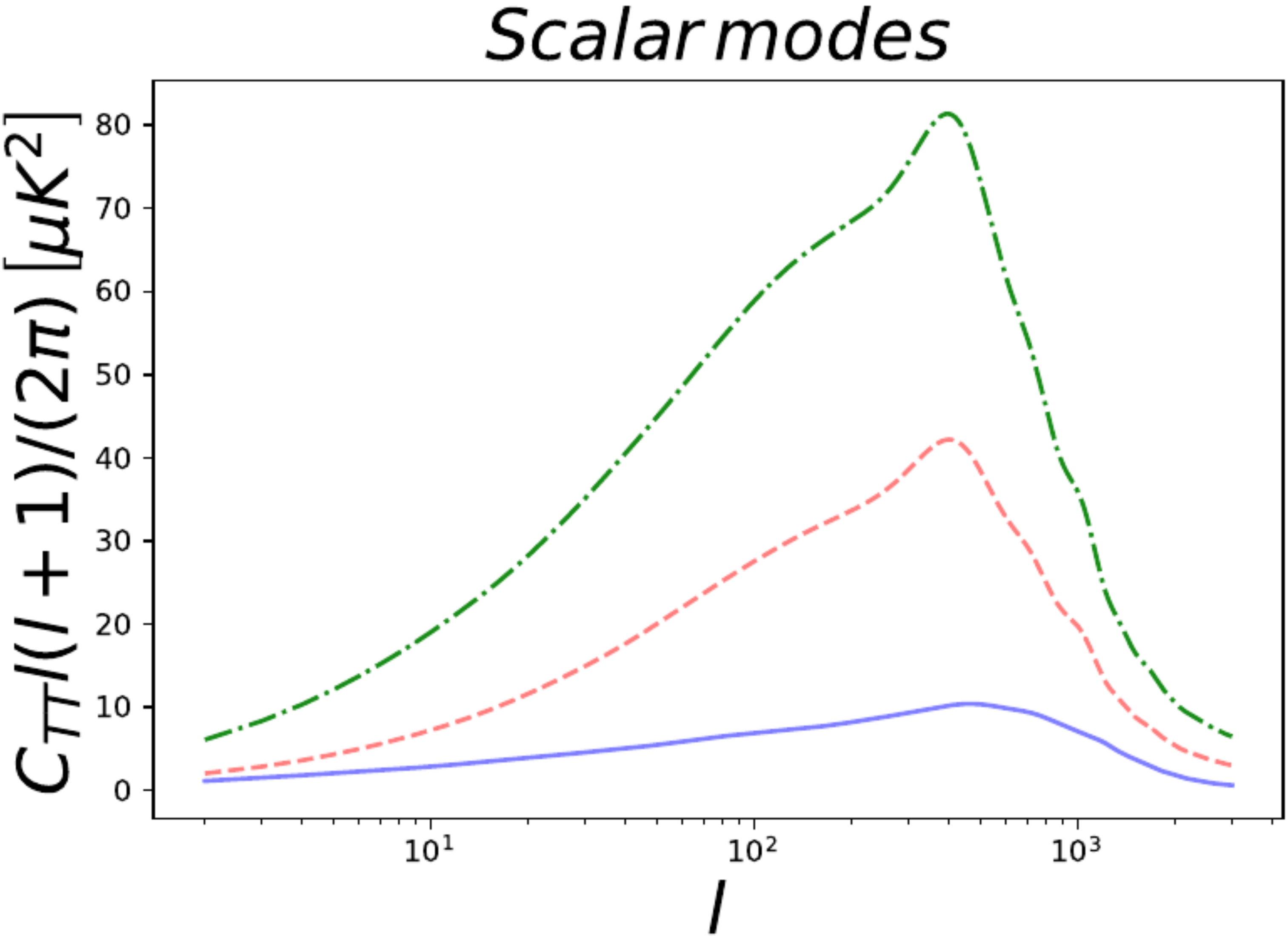
Towards Matter era (only Nambu-Goto networks)



Scaling and cosmological epochs

Constraints





Scalar modes
$C_{TT} l(l+1)/(2\pi)$ [μK^2]
0
10
20
30
40
50
60
70
80
10^1
10^2
10^3
l