On the hiding of the cosmological constant

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PPGCosmo Workshop
March 5, 2020

Outline

Introduction: the cosmological constant problem

Reviews

- *S. Weinberg, The cosmological constant problem, Reviews of Modern Physics, Vol. 61, No. 1, January 1989;*
- *S. Carroll, The cosmological constant, Living Rev. Relativity, 4, (2001), 1*;
- *J. Polchinski, The Cosmological Constant and the String Landscape, hep-th/0603249;*
- *J. Martin, Everything you always wanted to know about the cosmological constant problem (but were afraid to ask), C. R. Physique 13 (2012) 566665*;
- *C. P. Burgess, The Cosmological Constant Problem: Why it's hard to get Dark Energy from Micro-physics, arXiv:1309.4133 [hep-th];*
- *A. Padilla, Lectures on the Cosmological Constant Problem, arXiv:1502.05296v1 [hep-th].*

General Relativity

Einstein-Hilbert action plus a cosmological constant term:

$$S = \frac{c^4}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\,(R - 2\Lambda) + S_{\rm matter}\left[g_{\mu\nu}, \Psi\right] \; . \tag{1}$$

Field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu} \; . \tag{2}$$

Λ was introduced by Einstein in order to model a static universe [*A. Einstein, Sitzungs. König. Preuss. Akad. (1917) 142-152*]. It has been rejected afterwards being the expansion of the universe observationally established. It revives today as the most successful model for the accelerated expansion of the universe.

The value of Λ from cosmology

So, there is strong observational evidence of Λ driving the accelerated expansion of the universe. The amount of Λ energy density in the universe is about 70% percent, hence:

$$\rho_\Lambda \approx \Omega_\Lambda \rho_{\rm cr} \approx 10^{-47}\ \mathrm{GeV}^4 \approx 10^{-52}\ \mathrm{m}^{-2}\ . \tag{3}$$

What is the problem with Λ?

- Huge discrepancy with the predictions coming from quantum field theory (old cosmological constant problem);
- Why ρ_Λ has the above tiny value? (new cosmological constant problem).

The first question was tackled already by Zel'dovich and Sakharov [*Y. B. Zeldovich, JETP letters 6 (1967), 316-317*; *A. D. Sakharov, Dokl. Akad. Nauk SSSR (1967) 177, 70-71*].

Vacuum fluctuations

In Minkowski space we have that

$$\langle T_{\mu\nu} \rangle \propto \eta_{\mu\nu} \ , \tag{4}$$

hence by the equivalence principle, in curved space one has:

$$\langle T_{\mu\nu} \rangle = -\rho_{\rm vac}(x) g_{\mu\nu} \ , \tag{5}$$

and because of Bianchi identities $\rho_{\rm vac}$ has to be a constant. So, the field equations become:

$$G_{\mu\nu} + \Lambda_{\rm eff} g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ , \tag{6}$$

with

$$\Lambda_{\rm eff} = \Lambda + 8\pi G_N \rho_{\rm vac} \ , \tag{7}$$

the effective cosmological constant (whose measured value is 10^{-47} GeV^4).

Classical contributions

These come from fields which settle at the minimum of the potential to which they are subject. Consider a simple example:

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi + V(\Phi)\right] . \tag{8}$$

If the field rolls down to a minimum of its potential:

$$\langle T_{\mu\nu}\rangle = -V(\Phi_{\rm min})g_{\mu\nu} . \tag{9}$$

If we can set $V(\Phi_{\rm min}) = 0$ there is no contribution to $\Lambda_{\rm eff}$. Phase transitions are therefore quite problematic because the position of the minimum is shifted and $V(\Phi_{\rm min}) = 0$ can be realised only before or after the phase transition.

Electroweak phase transition

Realistic cases are the electroweak phase transition and the QCD transition. In the electroweak case (after the transition) we have the potential ($\lambda \simeq 0.1$ is a coupling):

$$V(H) = -\frac{\lambda v^4}{4} + \frac{1}{2}\lambda v^2 H^2 + \frac{\lambda}{2}\frac{v}{\sqrt{2}}H^3 + \frac{\lambda}{16}H^4 \;, \qquad (10)$$

with $m_H^2 = \lambda v^2$ being the Higgs mass and $v = \langle H \rangle$. Then:

$$\rho_{\rm vac} = -\frac{1}{4}m_H^2 v^2 \;, \qquad v^2 = \frac{\sqrt{2}}{4G_F^2} \;, \qquad (11)$$

lead to:

$$\rho_{\rm vac} = -\frac{\sqrt{2}}{16}\frac{m_H^2}{G_F^2} \approx -1.2 \times 10^8 \text{ GeV}^4 \;. \qquad (12)$$

$G_F \simeq 1.16 \times 10^{-5}$ GeV^{-2} is Fermi's constant and $m_H \approx 125$ GeV.

Electroweak phase transition

Higgs Potential (plots taken from Martin's review)

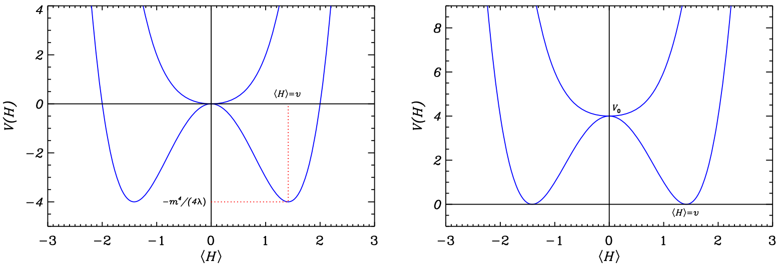

Fig. 2. Effective potential of the Higgs boson before and after the electroweak phase transition. The left panel corresponds to a situation where the vacuum energy vanishes at high temperature. As a consequence $\rho_{\rm vac}$ is negative at temperature smaller than the critical temperature. This is the situation treated in the text where the quantity $-m^4/(4\lambda)$ is explicitly calculated. On the right panel, the off-set parameter V_0 is chosen such that the vacuum energy is zero after the transition. As a consequence, it does not vanish at high temperatures.

Quantum-mechanical contribution to $\Lambda_{\rm eff}$

Simple example of a scalar field

Let us see the calculation of the zero-point energy-momentum tensor for a massive non-interacting scalar field:

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left(a_k e^{ik^\mu x_\mu} + a^\dagger_k e^{-ik^\mu x_\mu}\right) , \tag{13}$$

with

$$\omega_k \equiv k^0 = \sqrt{k^2 + m^2} . \tag{14}$$

The energy-momentum tensor is:

$$\langle T_{\mu\nu}\rangle = \int \frac{d^3k}{(2\pi)^3 2k^0} k_\mu k_\nu . \tag{15}$$

Quantum-mechanical contribution to $\Lambda_{\rm eff}$

Putting a cutoff

Consider a cutoff $M \to \infty$. Then:

$$\langle\rho\rangle = \frac{1}{4\pi^2}\int_0^M dk k^2\sqrt{k^2+m^2} =$$

$$\frac{M^4}{16\pi^2}\left[\sqrt{1+\frac{m^2}{M^2}}\left(1+\frac{m^2}{2M^2}\right) - \frac{m^4}{2M^4}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)\right]$$

$$= \frac{M^4}{16\pi^2}\left(1+\frac{m^2}{M^2}+\dots\right) \,, \quad (16)$$

Quantum-mechanical contribution to $\Lambda_{\rm eff}$

Putting a cutoff

and also for the pressure:

$$\langle p \rangle = \frac{1}{3}\frac{1}{4\pi^2}\int_0^M dk\frac{k^4}{\sqrt{k^2+m^2}} =$$

$$\frac{1}{3}\frac{M^4}{16\pi^2}\left[\sqrt{1+\frac{m^2}{M^2}}\left(1-\frac{3m^2}{2M^2}\right)+\frac{3m^4}{2M^4}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)\right]$$

$$= \frac{1}{3}\frac{M^4}{16\pi^2}\left(1-\frac{m^2}{M^2}+\dots\right) . \quad (17)$$

At the leading order $(m/M \to 0)$ $\langle p \rangle = \langle \rho \rangle/3$, as radiation does, so something is going wrong. Indeed, putting a cutoff spoils Lorentz invariance. Considering just the logarithmic terms instead gives the expected behaviour $\langle p \rangle = -\langle \rho \rangle$.

Quantum-mechanical contributions

Dimensional regularisation

Using dimensional regularisation one gets:

$$\langle\rho\rangle = \frac{\mu^{4-d}}{(2\pi)^{d-1}}\frac{1}{2}\int_0^\infty dk k^{d-2} d^{d-2}\Omega\omega_k = \frac{\mu^4}{2(4\pi)^{d-1}}\frac{\Gamma(-d/2)}{\Gamma(-1/2)}\left(\frac{m}{\mu}\right)^d , \quad (18)$$

with μ an arbitrary scale. Similarly

$$\langle p\rangle = \frac{\mu^4}{4(4\pi)^{d-1}}\frac{\Gamma(-d/2)}{\Gamma(1/2)}\left(\frac{m}{\mu}\right)^d , \quad (19)$$

Now $\langle p\rangle = -\langle\rho\rangle$ as expected.

Quantum-mechanical contributions

Renormalisation

Considering $d = 4 - \epsilon$ one can easily investigate the pole structure of the Gamma function and see that:

$$\langle\rho\rangle = -\frac{m^4}{64\pi^2}\left[\frac{2}{\epsilon} + \frac{3}{2} - \gamma - \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] + \ldots \ . \tag{20}$$

By subtracting the divergent term we finally have:

$$\langle\rho\rangle = \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) \ . \tag{21}$$

In general one can show the same result for any free field, provided a minus sign for the fermionic ones. Hence:

$$\langle\rho\rangle = \frac{1}{64\pi^2}\sum_n (-1)^{2S_n} g_n m_n^4 \ln\left(\frac{m_n^2}{\mu^2}\right) \ . \tag{22}$$

Quantum-mechanical contributions

Pauli Sum Rules

Pauli already observed in 1951 (ETH lectures) that even using a UV cutoff the correct result is obtained if the following conditions are met:

$$\sum_n (-1)^{2S_n} g_n = 0 \, , \sum_n (-1)^{2S_n} g_n m_n^2 = 0 \, , \sum_n (-1)^{2S_n} g_n m_n^4 = 0 \, . \tag{23}$$

Visser shows how these conditions provide a bridge between the finiteness of the zero-point energy and Lorentz invariance. He also speculates on the consequences of taking these relations to be valid non-perturbatively, leading to the necessity of physics beyond the standard model (*M. Visser, Phys. Lett. B* **791** *(2019) 43 [arXiv:1808.04583 [hep-th]]*).

The value of the cosmological constant and the problem

Summing up all the contributions considered so far, we have:

$$\rho_{\rm vac} = \frac{1}{64\pi^2}\sum_n (-1)^{2S_n} g_n m_n^4 \ln\left(\frac{m_n^2}{\mu^2}\right) + \rho_\Lambda + \rho_{\rm vac}^{\rm EW} + \rho_{\rm vac}^{\rm QCD} + \ldots \; . \tag{24}$$

We don't know μ, so we could simply fine tune it and ρ_B in order to give the observed result. The problem is that such a fine-tuning is needed at each loop order, since the coupling constants are not all smalls in the Standard Model. In other words, perturbation theory doesn't work here. Just taking the electroweak scale:

$$\rho_{\rm vac}^{\rm EW} \approx -1.2 \times 10^8 \ {\rm GeV}^4 \; . \tag{25}$$

This has the wrong sign and it is in modulus way larger than the observed value.

Hiding the cosmological constant

Hiding the cosmological constant

In [S. Carlip, PRL 123, 131302 (2019)] the author describes a mechanism by which the effect of a huge cosmological constant might average out on macroscopic scales.

He relies on Wheeler's notion of a spacetime foam, where a possibly huge cosmological constant is generated at the Planck scale.

The framework is the initial value formulation of General Relativity.

The initial value formulation of General Relativity.

Adopting a 3+1 foliation of spacetime, with Σ a Cauchy 3-hypersurface, where to establish our initial conditions, the metric can be written as:

$$ds^2 = -(N^2 - N_i N^j)dt^2 + 2N_i dx^i dt + g_{ij} dx^i dx^j \;, \qquad (26)$$

and the usual Einstein equations in vacuum and in presence of a cosmological constant are equivalent to two constraints:

$$R - K^i{}_j K^j{}_i + K^2 - 2\Lambda = 0 \;, \qquad (27)$$

$$D_i(K^i{}_j - \delta^i_j K) = 0 \;, \qquad (28)$$

and two evolution equations:

The initial value formulation of General Relativity.

$$\frac{1}{N}\partial_t g_{ij} = 2K_{ij} + \frac{1}{N}\left(D_j N_i + D_i N_j\right) , \tag{29}$$

$$\frac{1}{N}\partial_t K^i{}_j = -R^i{}_j - KK^i{}_j + \delta^i_j\Lambda$$
$$+\frac{D^i D_j N}{N} + \frac{1}{N}\left(K^i{}_k D_j N^k - K^k{}_j D_k N^i + N^k D_k K^i{}_j\right) , \tag{30}$$

where K_{ij} is the extrinsic curvature.

Two important properties

Since:

1. The equations are time reversal invariant, i.e. if (g, K) is allowed initial data for a manifold Σ, so is $(g, -K)$
2. Two manifolds Σ_1 and Σ_2 with initial data (g_1, K_1) and (g_2, K_2) can be glued to form a manifold $\Sigma_1 \# \Sigma_2$ (connected sum) for which the initial data are unchanged outside arbitrarily small neighborhoods of the points where the gluing is performed. [P. T. Chrusciel, J. Isenberg, and D. Pollack, Phys. Rev. Lett. 93, 081101 (2004), Commun. Math. Phys. 257, 29 (2005)]

then it is natural that, averaging over a sufficiently large volume:

$$\langle K^i{}_j \rangle \approx 0 \,. \tag{31}$$

Generality

The result is quite general because any orientable 3-manifold is isomorphic to a connected sum of prime manifolds and this sum is unique. [J. Milnor, Am. J. Math. 84, 1 (1962); D. Giulini, Int. J. Theor. Phys. 33, 913 (1994)]

Time evolution

Is the above property preserved with time?

Define the average [T. Buchert, Gen. Relativ. Gravit. 32, 105 (2000)]:

$$\langle X\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} X\sqrt{g}d^3x\;, \qquad V_{\mathcal{D}} = \int_{\mathcal{D}}\sqrt{g}d^3x\;. \tag{32}$$

Use $\langle K\rangle = 0$ as initial condition. On the same hypersurface:

$$\langle K\rangle^{\bullet} = \frac{1}{V}\int N(-R + 3\Lambda)\sqrt{g}d^3x\;. \tag{33}$$

Since $N > 0$, there are infinite possible choices of it such that $\langle K\rangle^{\bullet} = 0$, as long as $-R + 3\Lambda$ has not a definite sign. A similar argument goes on for higher derivatives of $\langle K\rangle$, which envolve higher derivatives of N.

Time evolution

Evolution equation for $\langle K\rangle$

I wanted to better characterise this argument, so I thought of obtaining a closed differential equation for $\langle K\rangle$.

$$\langle K\rangle^{\bullet} = -\frac{\dot{V}}{V}\langle K\rangle + \frac{1}{V}\int[\dot{K} + K\dot{g}/(2g)]\sqrt{g}d^3x = -\frac{\dot{V}}{V}\langle K\rangle$$
$$+\frac{1}{V}\int[N(-R+3\Lambda) + D^kD_kN + N^kD_kK + KD_kN^k]\sqrt{g}d^3x \; . \quad (34)$$

The derivative of the volume is:

$$\frac{\dot{V}}{V} = \frac{1}{V}\int[\dot{g}/(2g)]\sqrt{g}d^3x = \frac{1}{V}\int(NK + D_kN^k)\sqrt{g}d^3x \; . \quad (35)$$

Time evolution

Evolution equation for $\langle K \rangle$

We then have:

$$\langle K \rangle^\bullet = -[\langle NK \rangle + \mathcal{B}_1]\langle K \rangle - \langle NR \rangle + 3\langle N \rangle \Lambda + \mathcal{B}_2 + \mathcal{K} \;, \qquad (36)$$

where $\mathcal{B}_{1,2}$ are the boundary terms depending only on the shift and lapse functions, respectively:

$$\mathcal{B}_1 := \frac{1}{V}\int D_k N^k \sqrt{g} d^3x \;, \quad \mathcal{B}_2 := \frac{1}{V}\int D^k D_k N \sqrt{g} d^3x \;, \qquad (37)$$

and $\mathcal{K}$ is a boundary term involving K itself:

$$\mathcal{K} := \frac{1}{V}\int D_k(KN^k)\sqrt{g} d^3x \;. \qquad (38)$$

Time evolution

Evolution equation for $\langle K \rangle$

The evolution of $\langle K \rangle$ is then described by:

$$\langle K \rangle^\bullet = -\langle NK \rangle \langle K \rangle - \langle NR \rangle + 3\langle N \rangle \Lambda + \mathcal{B}_2 \; . \tag{39}$$

We now get rid of $\langle NK \rangle$ and find a closed equation for $\langle K \rangle$. This can be done using the following lemma. For a generic quantity Ψ, one has:

$$\langle \Psi \rangle^\bullet = \langle \dot{\Psi} \rangle + \langle NK\Psi \rangle - \langle NK \rangle \langle \Psi \rangle \; . \tag{40}$$

Therefore, choosing $\Psi = 1/N$, we obtain:

$$\langle 1/N \rangle^\bullet = -\langle \dot{N}/N^2 \rangle + \langle K \rangle - \langle NK \rangle \langle 1/N \rangle \; , \tag{41}$$

from which we can obtain:

$$\langle NK \rangle = \frac{\langle K \rangle}{\langle 1/N \rangle} - \frac{\langle 1/N \rangle^\bullet + \langle \dot{N}/N^2 \rangle}{\langle 1/N \rangle} \; . \tag{42}$$

Time evolution

Evolution equation for $\langle K \rangle$

Therefore, we can find a closed equation for $\langle K \rangle \equiv X$ which takes the form of a Riccati equation:

$$\dot{X} = -f(t)X^2 + g(t)X + h(t) \;, \tag{43}$$

where:

$$f(t) := \frac{1}{\langle 1/N \rangle} \;, \quad g(t) := \frac{\langle 1/N \rangle^{\bullet} + \langle \dot{N}/N^2 \rangle}{\langle 1/N \rangle} \;, \tag{44}$$

$$h(t) = -\langle NR \rangle + 3\langle N \rangle \Lambda + \mathcal{B}_2 \;. \tag{45}$$

Time evolution

Evolution equation for $\langle K \rangle$

It should be possible to employ our freedom in the choice of N and of the integration volume in order to set $h(t) = 0$. But then:

$$\dot{X} = f(t)X^2 + g(t)X \;, \quad \ddot{X} = (\dot{f})X^2 + 2fX\dot{X} + (\dot{g})X + g\dot{X} \;, \quad \ldots \tag{46}$$

implies that if $X = 0$ at some initial time, then all its derivatives are also zero at that time (provided f and g are regular) and thus $X = 0$ is the solution of the equation. In practice, we have explicitly realised Carlip's proposal.

An issue to check here is whether this solution $X = 0$ is stable and if f and g are regular indeed.