Null-Plane Dynamics for the Reduced-Order Bopp-Podolsky Model

Ronaldo Thibes
thibes@uesb.edu.br

Universidade Estadual do Sudoeste da Bahia

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Summary

1 - The Bopp-Podolsky Model

$$S_0[A_\mu] = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right.$$

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$$\mathcal{L}_B = -\frac{1}{4} \left[F_{\mu\nu} F^{\mu\nu} - a^2 \partial_\rho F^{\mu\nu} \partial^\rho F_{\mu\nu} \right] \quad \text{Bopp (1940)}$$

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$$S_0 = \frac{1}{2} \int d^4x \left[\mathbf{E}^2 - \mathbf{B}^2 + a^2 (\nabla \cdot \mathbf{E})^2 - a^2 (\dot{\mathbf{E}} - \nabla \times \mathbf{B})^2 \right] , \tag{1}$$

$$(1 + a^2 \Box) \nabla \cdot \mathbf{E} = j^0 , \tag{2}$$

$$(1 + a^2 \Box)(\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}) = \mathbf{j} . \tag{3}$$

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2 $\rightarrow$ A new (physical?) massive mode for the gauge field preserving gauge-invariance.

3 $\rightarrow$ An effective theory which can be used as a building block for other models with larger field contents.

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Consider the interaction between a charged spinless bosonic field ϕ and a Bopp-Podolsky gauge field A_μ given by the Lagrangian density

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$$\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi - m^2|\phi|^2 \,, \tag{5}$$

and

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho} \,, \tag{6}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

Equations of Motion

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho} \tag{8}$$

By demanding stationarity of the total gauge invariant action

$$S = \int d^4x \,[\mathcal{L}_\phi + \mathcal{L}_A + \mathcal{L}_{int}] \tag{9}$$

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By demanding stationarity of the total gauge invariant action

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with respect to arbitrary variations of ϕ and A_μ we obtain

$$(\Box + m^2)\phi = -ieA_\mu\partial^\mu\phi - ie\partial^\mu(\phi A_\mu) + e^2A^2\phi \,, \tag{10}$$

$$(\Box + m^2)\phi^* = ieA_\mu\partial^\mu\phi^* + ie\partial^\mu(\phi^* A_\mu) + e^2A^2\phi^* \tag{11}$$

and

$$(1 + a^2\Box)\partial_\nu F^{\mu\nu} = ie(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + 2e^2A^\mu|\phi|^2 \,. \tag{12}$$

Energy Momentum Tensor

By considering the Noether current associated to space-time translations, the canonical energy-momentum tensor can be directly computed as

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}_{\phi} + \Theta^{\mu\nu}_{A} + \Theta^{\mu\nu}_{int} \tag{13}$$

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$$\Theta^{\mu\nu}_{\phi} = \partial^\mu\phi^*\partial^\nu\phi + \partial^\mu\phi\partial^\nu\phi^* - \eta^{\mu\nu}\partial_\rho\phi^*\partial^\rho\phi + m^2\eta^{\mu\nu}|\phi|^2 \tag{14}$$

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$$\begin{aligned}\Theta^{\mu\nu}_{A} \quad = \quad & F^{\rho\mu}\partial^{\nu}A_{\rho} - \frac{a^{2}}{2}\partial_{\lambda}\partial_{\rho}F^{\mu\rho}\partial^{\nu}A^{\rho} + a^{2}\partial^{\mu}\partial_{\rho}F^{\lambda\rho}\partial^{\nu}A_{\lambda} \\ & + \frac{a^{2}}{2}\partial_{\rho}F^{\lambda\rho}\partial_{\lambda}\partial^{\nu}A^{\mu} + \frac{a^{2}}{2}\partial_{\rho}F^{\mu\rho}\partial_{\lambda}\partial^{\nu}A^{\lambda} - a^{2}\partial_{\rho}F^{\lambda\rho}\partial^{\mu}\partial^{\nu}A_{\lambda} \\ & + \frac{\eta^{\mu\nu}}{4}F^{\rho\lambda}F_{\rho\lambda} - \frac{a^{2}\eta^{\mu\nu}}{2}\partial_{\kappa}F^{\lambda\kappa}\partial^{\rho}F_{\lambda\rho}\end{aligned} \tag{15}$$

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$$\begin{aligned}
\Theta^{\mu\nu}_{A} \quad = \quad & F^{\rho\mu}\partial^\nu A_\rho - \frac{a^2}{2}\partial_\lambda\partial_\rho F^{\mu\rho}\partial^\nu A^\rho + a^2\partial^\mu\partial_\rho F^{\lambda\rho}\partial^\nu A_\lambda \\
& + \frac{a^2}{2}\partial_\rho F^{\lambda\rho}\partial_\lambda\partial^\nu A^\mu + \frac{a^2}{2}\partial_\rho F^{\mu\rho}\partial_\lambda\partial^\nu A^\lambda - a^2\partial_\rho F^{\lambda\rho}\partial^\mu\partial^\nu A_\lambda \\
& + \frac{\eta^{\mu\nu}}{4}F^{\rho\lambda}F_{\rho\lambda} - \frac{a^2\eta^{\mu\nu}}{2}\partial_\kappa F^{\lambda\kappa}\partial^\rho F_{\lambda\rho}
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and

$$\Theta^{\mu\nu}_{int} = ie\left[\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\right]A_\alpha\left[\phi\partial_\beta\phi^* - \phi^*\partial_\beta\phi\right] - e^2\eta^{\mu\nu}A^2|\phi|^2 \tag{16}$$

Calculation of the divergence with respect to the first index μ leads to

$$\begin{aligned}
\partial_\mu \Theta^{\mu\nu} &= \left[(\Box + m^2)\phi^* - ie\partial_\mu(A^\mu\phi^*) - e^2A^2\phi^* - ieA_\mu\partial^\mu\phi^*\right]\partial^\nu\phi \\
&\quad + \left[(\Box + m^2)\phi + ie\partial_\mu(A^\mu\phi) - e^2A^2\phi + ieA_\mu\partial^\mu\phi\right]\partial^\nu\phi^* \\
&\quad + \left[-ie(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) - 2e^2A^\mu|\phi|^2 + (1 + a^2\Box)\partial_\rho F^{\mu\rho}\right]\partial^\nu A_\mu
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which shows explicitly the consistent conservation of $\Theta^{\mu\nu}$ modulo the equations of motion (10), (11) and (12). However, $\Theta^{\mu\nu}$ is neither gauge-invariant nor symmetric.

This can be fixed by constructing the improved energy-momentum tensor

$$\mathcal{T}^{\mu\nu} = \Theta_{\phi}^{\mu\nu} + \mathcal{T}_{A}^{\mu\nu} + \mathcal{T}_{int}^{\mu\nu} \tag{17}$$

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and

$$\mathcal{T}^{\mu\nu}_{int} \equiv ieA_\alpha\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)\left[\phi\partial_\beta\phi^* - \phi^*\partial_\beta\phi - ieA_\beta|\phi|^2\right] \, ,$$

which now may be checked to be indeed gauge-invariant, symmetric and divergenceless.

3 - Gauge-Fixing and Propagators

By considering the Landau gauge in the Light-Front $(a, b = +, -, 1, 2)$

$$\mathcal{L}_L = -\frac{1}{2\xi}(\partial_a A^a)^2 + \frac{a^2}{2\xi}(\partial_c \partial_a A^a)(\partial^c \partial_b A^b) \tag{18}$$

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we can invert the corresponding gauge field kinetic term and obtain

$$P_{ab}(k) = \frac{-i}{(1 - a^2 k^2)k^2} \left[\eta_{ab} + (\xi - 1)\frac{k_a k_b}{k^2} \right] . \tag{19}$$

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$$\begin{aligned} Z[j^\mu] \;&=\; N \int DA_\mu DC D\bar{C} DB \exp\Big\{ iS_0 \\ &\quad + i\int d^4x \,\Big[\bar{C}(1 + a^2\Box)\Box C + B(1 + a^2\Box)\partial^\mu A_\mu \\ &\quad - \frac{a^2\xi}{2}\partial_\mu B \partial^\mu B + \frac{\xi B^2}{2} - j^\mu A_\mu\Big]\Big\} . \end{aligned} \tag{20}$$

Concerning the axial gauge in the Light-Front

$$\mathcal{L}_A = -\frac{1}{2\alpha}(n_a A^a)^2 + \frac{a^2}{2\alpha}(n_a \partial_c A^a)(n_b \partial^c A^b)\,, \tag{21}$$

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we have the propagator

$$P_{ab}(k) = \frac{-i}{k^2(1-a^2k^2)}\left[\eta_{ab} + \frac{(\alpha k^2 + n^2)}{(n\cdot k)^2}k_a k_b - \frac{1}{(n\cdot k)}(k_a n_b + k_a n_b)\right]\,.$$

Light-Front Gauges

For the usual light-front gauge we choose a light-like direction n, with $n^2 = 0$, and consider the limit $\alpha \to 0$. In this case

$$P_{ab} = \frac{-i}{k^2(1 - a^2k^2)} \left[\eta_{ab} - \frac{1}{(n \cdot k)}(k_a n_b + k_b n_a)\right] . \tag{22}$$

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In order to obtain the doubly transverse three-term propagator we may use the mixed gauge-fixing

$$\mathcal{L}_3 = -\frac{1}{\beta}(n\cdot A)(\partial\cdot A) + \frac{a^2}{\beta}(n_a\partial_c A^a)(\partial_b\partial^c A^b)\,, \tag{23}$$

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which is the corresponding three-term generalized photon propagator in the light-front gauge for the BP model.

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As can be directly checked, the three-term propagator satisfies

$$k^a P_{ab} = 0 \tag{25}$$

and

$$n^a P_{ab} = 0 \tag{26}$$

being in this sense doubly transverse.

4 - The Reduced-Order BP Model

In order to decouple the two massive and massless photon modes we may work with the reduced order Lagrangian

$$\mathcal{L}_{AB} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{a^2}{2}B_\mu B^\mu + a^2\partial_\mu B_\nu F^{\mu\nu}\,, \tag{27}$$

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$$S_{red} = \int d^4x \, \{\mathcal{L}_\phi + \mathcal{L}_{AB} + \mathcal{L}_{int}\} \tag{28}$$

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$$(\Box + m^2)\phi = -ieA_\mu\partial^\mu\phi - ie\partial^\mu(\phi A_\mu) + e^2A^2\phi\,, \tag{29}$$

$$(\partial^\mu\partial^\nu - \Box\eta^{\mu\nu})A_\nu = B^\mu\,, \tag{30}$$

$$(\partial^\mu\partial^\nu - \Box\eta^{\mu\nu})(A_\nu - a^2B_\nu) = ie\,[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi] + 2e^2A^\mu|\phi|^2\,. \tag{31}$$

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with

$$\begin{aligned} \Theta^{\mu\nu}_{B} &= F^{\rho\mu}\partial^\nu A_\rho + a^2(\partial^\mu B^\lambda - \partial^\lambda B^\mu)\partial^\nu A_\lambda + a^2 F^{\mu\lambda}\partial^\nu B_\lambda \\ &\quad + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} + a^2\eta^{\mu\nu}\left(\frac{1}{2}B_\lambda B^\lambda - \partial_\lambda B_\rho F^{\lambda\rho}\right). \end{aligned} \tag{33}$$

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By applying the Belinfante procedure, the canonical energy-momentum tensor can be upgraded to the symmetric gauge-invariant version

$$
\begin{aligned}
\mathcal{T}^{\mu\nu}_{(r)} &= \Theta^{\mu\nu}_{\phi} + \mathcal{T}^{\mu\nu}_{int} + F^{\mu}{}_{\rho}F^{\rho\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\rho\lambda}F_{\rho\lambda} \\
&\quad + a^2\eta^{\mu\nu}\left(\frac{1}{2}B^{\lambda}B_{\lambda} - \partial_{\lambda}B_{\rho}F^{\lambda\rho}\right) \\
&\quad + a^2F^{\mu}{}_{\lambda}(\partial^{\nu}B^{\lambda} - \partial^{\lambda}B^{\nu}) + a^2F^{\nu}{}_{\lambda}(\partial^{\mu}B^{\lambda} - \partial^{\lambda}B^{\mu}) - a^2B^{\mu}B^{\nu}
\end{aligned}
$$

The reduced-order model for BP electromagnetic theory can also be written as

$$S = \int_\Omega d\omega \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{a^2}{2} B_\mu B^\mu + \frac{a^2}{2} G_{\mu\nu} F^{\mu\nu} \right) , \tag{34}$$

which clearly, contrary to BP's original theory, presents only first-order derivatives. Here, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$.

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The field equations are given by

$$\partial_\mu F^{\mu\nu} - a^2 \partial_\mu G^{\mu\nu} = 0, \tag{35a}$$

$$a^2 \left(\partial_\mu F^{\mu\nu} + B^\nu \right) = 0, \tag{35b}$$

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from which follows, for $a^2 \neq 0$,

$$B^\mu = a^2 \partial_\nu G^{\mu\nu}. \tag{36}$$

Eq. (36) then implies $\partial_\mu B^\mu = 0$, and

$$\left(1 + a^2 \Box \right) B^\mu = 0\,. \tag{37}$$

Moreover, eq. (35b) results in

$$\left(1 + a^2\Box\right)\partial_\nu F^{\nu\mu} = \left(1 + a^2\Box\right)\left(\delta^\mu_\nu\Box - \partial^\mu\partial_\nu\right)A^\nu = 0, \tag{38}$$

reproducing the field equations of BP's theory. The canonical energy-momentum tensor density can be obtained from (34) as

$$H^\mu{}_\nu = \left(a^2 G^{\mu\gamma} - F^{\mu\gamma}\right)\partial_\nu A_\gamma + a^2 F^{\mu\gamma}\partial_\nu B_\gamma - \delta^\mu_\nu \mathcal{L}\,, \tag{39}$$

with $\mathcal{L}$ denoting the integrand of (34), Moreover, eq. (35b) results in

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with $\mathcal{L}$ denoting the integrand of (34), and can be recast into the corresponding gauge-invariant symmetric form

$$\begin{aligned} T^{\mu\nu} \;&=\; F^\mu{}_\gamma F^{\gamma\nu}+\frac{1}{4}\eta^{\mu\nu}F^{\gamma\lambda}F_{\gamma\lambda}+\frac{a^2}{2}\eta^{\mu\nu}\left(B^\gamma B_\gamma-G_{\gamma\lambda}F^{\gamma\lambda}\right)\\ &\quad+a^2F^\mu{}_\gamma G^{\nu\gamma}+a^2F^\nu{}_\gamma G^{\mu\gamma}-a^2B^\mu B^\nu\,. \end{aligned} \tag{40}$$

The massive and massless modes for the Podolsky gauge field decouple in a precise way such that B_μ acquires the full Podolsky mass while A_μ becomes massless. Gauge invariance is naturally preserved.

Light-front dynamics is characterized by the choice of

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$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}.$$

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The components of Minkowski's metric are now given by

$$(\hat{\eta}_{\mu\nu}) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{41}$$

with $\mu, \nu = +, -, 1, 2$, enabling light-front indexes raising and lowering.

The derivative operators can be written as

$$\partial_\mu \equiv (\partial_+, \partial_-, \partial_i)\,, \quad \partial^\mu \equiv (\partial_-, \partial_+, -\partial_i)\,, \qquad i = 1, 2, \tag{42}$$

while the d'Alembertian operator takes the form

$$\Box \equiv \partial_\mu \partial^\mu = 2\partial_- \partial_+ - \nabla^2, \quad \nabla^2 \equiv \partial_i \partial_i. \tag{43}$$

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For convenience, we introduce the notation

$$\bar{A} \equiv \partial_+ A \tag{44}$$

for a corresponding given observable A.

The Hamiltonian function H can be found from the canonical energy-momentum tensor as its double projection over $(u^\mu) \equiv (1, 0, 0, 0)$:

$$\begin{aligned} H &= \int_\Sigma d\sigma \, u_\mu H^\mu_{\ \nu} u^\nu = \int_\Sigma d\sigma H^+_{\ +} \\ &= \int_\Sigma d\sigma \left[\left(F^{\mu +} - a^2 G^{\mu +} \right) \bar{A}_\mu + a^2 F^{+\mu} \bar{B}_\mu - \mathcal{L} \right] , \end{aligned} \tag{45}$$

where Σ is a characteristic surface of constant $\tau \equiv x^+$ with volume element $d\sigma = dx^- dx^1 dx^2$.

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&= \int_\Sigma d\sigma \left[\left(F^{\mu+} - a^2 G^{\mu+} \right) \bar{A}_\mu + a^2 F^{+\mu} \bar{B}_\mu - \mathcal{L} \right], \qquad (45)
\end{aligned}
$$

where Σ is a characteristic surface of constant $\tau \equiv x^+$ with volume element $d\sigma = dx^- dx^1 dx^2$.Therefore, the canonical variables of the theory are the fields A_μ and B_μ conjugated respectively to the momenta

$$p^\mu \equiv F^{\mu+} - a^2 G^{\mu+}, \qquad (46a)$$

$$\pi^\mu \equiv a^2 F^{+\mu}. \qquad (46b)$$

5 - Constraint Structure

The definitions (46) result in the primary constraints

$$\phi_1 \equiv p^+ \approx 0, \tag{47a}$$

$$\phi_2^i \equiv p^i - F_{-i} + a^2 G_{-i} \approx 0, \tag{47b}$$

$$\phi_3 \equiv \pi^+ \approx 0, \tag{47c}$$

$$\phi_4^i \equiv \pi^i + a^2 F_{-i} \approx 0, \tag{47d}$$

and the relations

$$\bar{A}_- = \partial_- A_+ - \frac{1}{a^2}\pi^-, \tag{48a}$$

$$\bar{B}_- = \partial_- B_+ - \frac{1}{a^2}\left(p^- + \frac{1}{a^2}\pi^-\right). \tag{48b}$$

We find the canonical Hamiltonian substituting (47) and (48) in (45) as

$$
\begin{aligned}
H_c &= \int_\Sigma d\sigma \left[-\frac{1}{2}\frac{1}{a^4}\left(\pi^-\right)^2 - \frac{1}{a^2} p^- \pi^- + \left(p^- \partial_- + p^i \partial_i\right) A_+ \right. \\
&\quad + \left(\pi^- \partial_- + \pi^i \partial_i + a^2 B_-\right) B_+ + \frac{1}{4} F_{ij} F^{ij} \\
&\quad \left. - \frac{a^2}{2} G_{ij} F^{ij} + \frac{a^2}{2} B_i B^i \right] , \qquad (49)
\end{aligned}
$$

We find the canonical Hamiltonian substituting (47) and (48) in (45) as

$$\begin{aligned} H_c &= \int_\Sigma d\sigma \left[-\frac{1}{2}\frac{1}{a^4}\left(\pi^-\right)^2 - \frac{1}{a^2}p^-\pi^- + \left(p^-\partial_- + p^i\partial_i\right)A_+ \right. \\ &\quad + \left(\pi^-\partial_- + \pi^i\partial_i + a^2 B_-\right)B_+ + \frac{1}{4}F_{ij}F^{ij} \\ &\quad \left. - \frac{a^2}{2}G_{ij}F^{ij} + \frac{a^2}{2}B_iB^i \right] , \end{aligned} \tag{49}$$

from which the primary Hamiltonian can be written as

$$H_P \equiv H_c + \int_\Sigma d\sigma\, u^a\left(x\right)\phi_a\left(x\right) , \tag{50}$$

with a repeated index sum convention in $a = 1, 2^{(i)}, 3, 4^{(i)}$ to account for the six primary constraints (47) and u^a denoting the corresponding Lagrange multiplier functions.

The fundamental PBs of the theory are given by

$$\{A_\mu(x), p^\nu(y)\} = \{B_\mu(x), \pi^\nu(y)\} = \delta_\mu^\nu \delta^3(x-y), \tag{51}$$

where $\delta^3(x-y) \equiv \delta(x^- - y^-)\,\delta\left(x^1 - y^1\right)\delta\left(x^2 - y^2\right)$ is the three-dimensional Dirac delta in null-plane coordinates.

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Consistency for the constraints ϕ_1 and ϕ_3 results in the two secondary constraints

$$\chi_1 \equiv \partial_- p^- + \partial_i p^i \approx 0, \tag{52a}$$

$$\chi_2 \equiv \partial_- \pi^- + \partial_i \pi^i - a^2 B_- \approx 0. \tag{52b}$$

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$$\left\{A_\mu\left(x\right), p^\nu\left(y\right)\right\} = \left\{B_\mu\left(x\right), \pi^\nu\left(y\right)\right\} = \delta^\nu_\mu \delta^3\left(x-y\right), \tag{51}$$

where $\delta^3\left(x-y\right) \equiv \delta\left(x^- - y^-\right)\delta\left(x^1 - y^1\right)\delta\left(x^2 - y^2\right)$ is the three-dimensional Dirac delta in null-plane coordinates.

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For the remaining primary constraints within Eqs (47), we have the relations

$$\partial_+ \phi_2^i \approx \partial_i p^- + \partial_k\left(F^{ki} - a^2 G^{ki}\right) - 2\partial_- u_i^2 + 2a^2 \partial_- u_i^4 \approx 0, \tag{53a}$$

$$\partial_+ \phi_4^i \approx \partial_i \pi^- - a^2\left(\partial_k F^{ki} + B^i\right) + 2a^2 \partial_- u_i^2 \approx 0, \tag{53b}$$

which do not lead to new constraints, resulting in consistency conditions

In turn, the set of secondary constraints (52) should also be Dirac-Bergmann consistent. For χ_1, the consistency condition is identically satisfied, but for χ_2 it results in the tertiary constraint

$$\chi_3 \equiv p^- + \frac{1}{a^2}\pi^- - 2a^2\partial_- B_+ - a^2\partial_i B^i \approx 0. \tag{54}$$

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whose stability requires $\partial_+\chi_3 \approx 0$, leading to

$$\partial_+\chi_3 \approx -\left(1 - a^2\nabla^2\right) B_+ - 2a^2\partial_- u^3 \approx 0\,. \tag{55}$$

In this last case, once more a condition over a Lagrange multiplier appears and no more constraints are found from the consistency conditions.

Summarizing: reduced-order BP model whole constraints set

$$\phi_1 \equiv p^+ \approx 0, \tag{56a}$$

$$\phi_2^i \equiv p^i - F_{-i} + a^2 G_{-i} \approx 0, \tag{56b}$$

$$\phi_3 \equiv \pi^+ \approx 0, \tag{56c}$$

$$\phi_4^i \equiv \pi^i + a^2 F_{-i} \approx 0, \tag{56d}$$

$$\chi_1 = \partial_- p^- + \partial_i p^i \approx 0, \tag{57a}$$

$$\chi_2 = \partial_- \pi^- + \partial_i \pi^i - a^2 B_- \approx 0. \tag{57b}$$

$$\chi_3 = p^- + \frac{1}{a^2}\pi^- - 2a^2\partial_- B_+ - a^2\partial_i B^i \tag{58}$$

Since A_+ does not show up in the constraint relations and the pair (A_-, A_i) appears only through the combination F_{-i}, we identify

$$\phi_1 = p^+ \approx 0, \tag{59a}$$

$$\chi_1 = \partial_- p^- + \partial_i p^i \approx 0, \tag{59b}$$

as a set of first-class constraints.

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as a set of first-class constraints. The remaining constraints can be split into two independent second-class sets. Indeed, we have

$$\phi_2^i = p^i - F_{-i} + a^2 G_{-i} \approx 0, \tag{60a}$$

$$\phi_4^i = \pi^i + a^2 F_{-i} \approx 0, \tag{60b}$$

as a first subset of second-class constraints,

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as a first subset of second-class constraints, while

$$\phi_3 = \pi^+ \approx 0, \tag{61a}$$

$$\chi_2 = \partial_- \pi^- + \partial_i \pi^i - a^2 B_- \approx 0, \tag{61b}$$

$$\chi_3 = p^- + \frac{1}{a^2}\pi^- - 2a^2 \partial_- B_+ - a^2 \partial_i B^i \approx 0, \tag{61c}$$

compose a second subset of second-class constraints.

This can be seen from their PB relations, whose non-null results can be found to be

$$\left\{\phi_2^i\left(x\right),\phi_2^j\left(y\right)\right\}\approx 2\eta^{ij}\partial_-^x\delta^3\left(x-y\right),\tag{62a}$$

$$\left\{\phi_2^i\left(x\right),\phi_4^j\left(y\right)\right\}\approx -2a^2\eta^{ij}\partial_-^x\delta^3\left(x-y\right),\tag{62b}$$

$$\left\{\chi_2\left(x\right),\chi_2\left(y\right)\right\}\approx 2a^2\partial_-^x\delta^3\left(x-y\right),\tag{62c}$$

$$\left\{\chi_3\left(x\right),\phi_3\left(y\right)\right\}\approx -2a^2\partial_-^x\delta^3\left(x-y\right),\tag{62d}$$

$$\left\{\chi_3\left(x\right),\chi_2\left(y\right)\right\}\approx\left(1-a^2\nabla_x^2\right)\delta^3\left(x-y\right)\ ,\tag{62e}$$

confirming that (60) and (61) represent two independent second-class subsets.

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$$\left\{\phi_2^i(x), \phi_2^j(y)\right\} \approx 2\eta^{ij}\partial_-^x\delta^3(x-y), \tag{62a}$$

$$\left\{\phi_2^i(x), \phi_4^j(y)\right\} \approx -2a^2\eta^{ij}\partial_-^x\delta^3(x-y), \tag{62b}$$

$$\{\chi_2(x), \chi_2(y)\} \approx 2a^2\partial_-^x\delta^3(x-y), \tag{62c}$$

$$\{\chi_3(x), \phi_3(y)\} \approx -2a^2\partial_-^x\delta^3(x-y), \tag{62d}$$

$$\{\chi_3(x), \chi_2(y)\} \approx \left(1-a^2\nabla_x^2\right)\delta^3(x-y)\ , \tag{62e}$$

confirming that (60) and (61) represent two independent second-class subsets.

From this result, me may build the total Hamiltonian

$$H_T \equiv H_c + \int_\Sigma d\sigma u^a(x)\,\phi_a(x) + \int_\Sigma d\sigma \lambda^I(x)\,\chi_I(x), \quad I = 1, 2, 3, \tag{63}$$

which adds to the primary Hamiltonian a linear combination of the secondary constraints.

Actually, a redefinition of constraints can ease calculations and enhance clarity. Conveniently, we redefine and rename the constraint relations as

$$\Phi_1 \equiv p^+ \approx 0, \tag{64a}$$
$$\Phi_2 \equiv \partial_- p^- + \partial_i p^i, \tag{64b}$$
$$\Theta_1^i \equiv p^i - F_{-i} + a^2 G_{-i} \approx 0, \tag{64c}$$
$$\Theta_2^i \equiv \pi^i + a^2 F_{-i} \approx 0, \tag{64d}$$
$$\Gamma_1 \equiv \pi^+ \approx 0, \tag{64e}$$
$$\begin{aligned}\Gamma_2 \equiv\ & p^- + \frac{1}{2a^2}\left(1 + a^2\nabla^2\right)\pi^- - 2a^2\partial_- B_+ - a^2\partial_i B^i \\ & -\frac{1}{2a^2}\left[\partial_-\right]^{-1}\left(1 - a^2\nabla^2\right)\left(\partial_i\pi^i - a^2 B_-\right) \approx 0,\end{aligned} \tag{64f}$$
$$\chi_2 \equiv \partial_-\pi^- + \partial_i\pi^i - a^2 B_- \approx 0\,, \tag{64g}$$

with (64a)-(64b) first-class and (64c)-(64g) second-class.

6 - Dirac Brackets

Since we are working with a highly constrained system, the canonical quantization cannot be achieved by standard Poisson brackets. In fact, we need Dirac brackets.

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For the second-class sector, the Dirac matrix can be written as

$$M(x,y) \equiv \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}, \tag{65}$$

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$$M(x,y) \equiv \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}, \tag{65}$$

with

$$A \equiv \left\{ \Theta^i_a(x), \Theta^j_b(y) \right\} = 2\eta^{ij} \begin{pmatrix} 1 & -a^2 \\ -a^2 & 0 \end{pmatrix} \partial^x_- \delta^3(x-y), \tag{66a}$$

$$B \equiv \{\Gamma_a(x), \Gamma_b(y)\} = \begin{pmatrix} 0 & -2a^2\partial^x_- \\ -2a^2\partial^x_- & D^2_x \left[2a^2\partial^x_-\right]^{-1} \end{pmatrix} \delta^3(x-y), \tag{66b}$$

$$C \equiv \{\chi_2(x), \chi_2(y)\} = 2a^2\partial^x_- \delta^3(x-y)\ , \tag{66c}$$

where $D_x \equiv 1 - a^2\nabla^2_x$.

The block diagonal form of (65) clearly shows benefits from the redefinition (64) as its inverse can be easily obtained by inverting the internal block matrices A, B and C. Explicitly, we have

$$A^{-1} = -\eta_{ij} \begin{pmatrix} 0 & 1 \\ 1 & 1/a^2 \end{pmatrix} \left[2a^2 \partial_-^x\right]^{-1} \delta^3 (x-y) \,, \tag{67a}$$

$$B^{-1} = -\begin{pmatrix} D_x^2 \left[2a^2 \partial_-^x\right]^{-2} & 1 \\ 1 & 0 \end{pmatrix} \left[2a^2 \partial_-^x\right]^{-1} \delta^3 (x-y) \tag{67b}$$

$$C^{-1} = \left[2a^2 \partial_-^x\right]^{-1} \delta^3 (x-y) \,, \tag{67c}$$

From the inverses (67), we may readily compute Dirac brackets (DB) defined for two given generic phase space functions F and G as

$$
\begin{aligned}
\{F, G\}_D \equiv & \{F, G\} - \int d^3z d^3w \left\{F, \Theta^i_a(z)\right\} \left(A^{-1}\right)^{ab}_{ij}(z, w) \left\{\Theta^j_b(w), G\right\} \\
& - \int d^3z d^3w \left\{F, \Gamma_a(z)\right\} \left(B^{-1}\right)^{ab}(z, w) \left\{\Gamma_b(w), G\right\} \\
& - \int d^3z d^3w \left\{F, \chi_2(z)\right\} \left(C^{-1}\right)(z, w) \left\{\chi_2(w), G\right\},
\end{aligned}
$$

which leads to the following fundamental DB relations among the phase space variables:

$$
\begin{aligned}
\{A_\mu(x), A_\nu(y)\}_D &= 0, \\
\{A_\mu(x), B_\nu(y)\}_D &= -\left(\delta_\mu^i \delta_\nu^j \eta_{ij} + \delta_\mu^- \delta_\nu^+\right)\left[2a^2 \partial_-^x\right]^{-1} \delta^3(x-y), \\
\{A_\mu(x), p^\nu(y)\}_D &= \delta_\mu^\nu \delta^3(x-y) - a^2 \delta_\mu^j \left(\delta_j^\nu \partial_-^x - \delta_-^\nu \partial_j^x\right)\left[2a^2 \partial_-^x\right]^{-1} \delta^3(x-y), \\
\{A_\mu(x), \pi^\nu(y)\}_D &= 0,
\end{aligned}
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\{A_\mu(x), p^\nu(y)\}_D &= \delta^\nu_\mu \delta^3(x-y) - a^2\delta^j_\mu\left(\delta^\nu_j \partial^x_- - \delta^\nu_- \partial^x_j\right)\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{A_\mu(x), \pi^\nu(y)\}_D &= 0, \\
\{B_\mu(x), B_\nu(y)\}_D &= -\left[\delta^-_\mu \delta^-_\nu \partial^x_- \partial^x_- + \frac{1}{2a^2}\delta^{(-}_\mu \delta^{+)}_\nu\left(1 + a^2\nabla^2_x\right)\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y) \\
&\quad + \left[\delta^{(-}_\mu \delta^{i)}_\nu \partial^x_- \partial^x_i + \eta_{\mu i}\left(\delta^i_\nu + \delta^j_\nu \partial^i_x \partial^x_j\right)\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y) \\
&\quad - \left[\delta^+_\mu \delta^+_\nu D^2_x\left[2a^2\partial^x_-\right]^{-2} - \delta^{(+}_\mu \delta^{i)}_\nu D_x \partial^x_i\left[2a^2\partial^x_-\right]^{-1}\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{B_\mu(x), p^\nu(y)\}_D &= a^2\eta_{\mu j}\left[\delta^\nu_j \partial^x_- - \delta^\nu_- \partial^x_j\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{B_\mu(x), \pi^\nu(y)\}_D &= \left(\delta^\nu_\mu - \delta^+_\mu \delta^\nu_+ - \frac{1}{2}\delta^-_\mu \delta^\nu_-\right)\delta^3(x-y) \\
&\quad - a^2\left[\delta^{(+}_\mu \eta^{i)\nu}\partial^x_i - \delta^+_\mu \delta^\nu_- D_x\left[2a^2\partial^x_-\right]^{-1}\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y),
\end{aligned}$$

which leads to the following fundamental DB relations among the phase space variables:

$$
\begin{aligned}
\{A_\mu(x), A_\nu(y)\}_D &= 0, \\
\{A_\mu(x), B_\nu(y)\}_D &= -\left(\delta^i_\mu \delta^j_\nu \eta_{ij} + \delta^-_\mu \delta^+_\nu\right)\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{A_\mu(x), p^\nu(y)\}_D &= \delta^\nu_\mu \delta^3(x-y) - a^2\delta^j_\mu\left(\delta^\nu_j \partial^x_- - \delta^\nu_- \partial^x_j\right)\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{A_\mu(x), \pi^\nu(y)\}_D &= 0, \\
\{B_\mu(x), B_\nu(y)\}_D &= -\left[\delta^-_\mu \delta^-_\nu \partial^x_- \partial^x_- + \frac{1}{2a^2}\delta^{(-}_\mu \delta^{+)}_\nu\left(1 + a^2\nabla^2_x\right)\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y) \\
&\quad + \left[\delta^{(-}_\mu \delta^{i)}_\nu \partial^x_- \partial^x_i + \eta_{\mu i}\left(\delta^i_\nu + \delta^j_\nu \partial^i_x \partial^x_j\right)\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y) \\
&\quad - \left[\delta^+_\mu \delta^+_\nu D^2_x\left[2a^2\partial^x_-\right]^{-2} - \delta^{(+}_\mu \delta^{i)}_\nu D_x \partial^x_i\left[2a^2\partial^x_-\right]^{-1}\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{B_\mu(x), p^\nu(y)\}_D &= a^2\eta_{\mu j}\left[\delta^\nu_j \partial^x_- - \delta^\nu_- \partial^x_j\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{B_\mu(x), \pi^\nu(y)\}_D &= \left(\delta^\nu_\mu - \delta^+_\mu \delta^\nu_+ - \frac{1}{2}\delta^-_\mu \delta^\nu_-\right)\delta^3(x-y) \\
&\quad - a^2\left[\delta^{(+}_\mu \eta^{i)\nu}\partial^x_i - \delta^+_\mu \delta^\nu_- D_x\left[2a^2\partial^x_-\right]^{-1}\right]\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{p^\mu(x), p^\nu(y)\}_D &= -a^2\left(\eta^{\mu j}\delta^\nu_j \partial^x_- \partial^x_- + \delta^\mu_{(-}\delta^\nu_{j)}\partial^x_j \partial^x_- - \delta^\mu_- \delta^\nu_- \nabla^2_x\right)\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{p^\mu(x), \pi^\nu(y)\}_D &= a^4\left(\eta^{\mu j}\delta^\nu_j \partial^x_- \partial^x_- + \delta^\mu_{(-}\delta^\nu_{j)}\partial^x_j \partial^x_- - \delta^\mu_- \delta^\nu_- \nabla^2_x\right)\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y), \\
\{\pi^\mu(x), \pi^\nu(y)\}_D &= a^4\delta^\mu_- \delta^\nu_-\left[2a^2\partial^x_-\right]^{-1}\delta^3(x-y).
\end{aligned}
$$

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- We have discussed the scalar BP model in the light-front.
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- We have performed the canonical quantization of the light-front form evolution of the reduced-order model, obtaining the corresponding Dirac brackets.

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