Happy Birthday

BLACK HOLE PHYSICS

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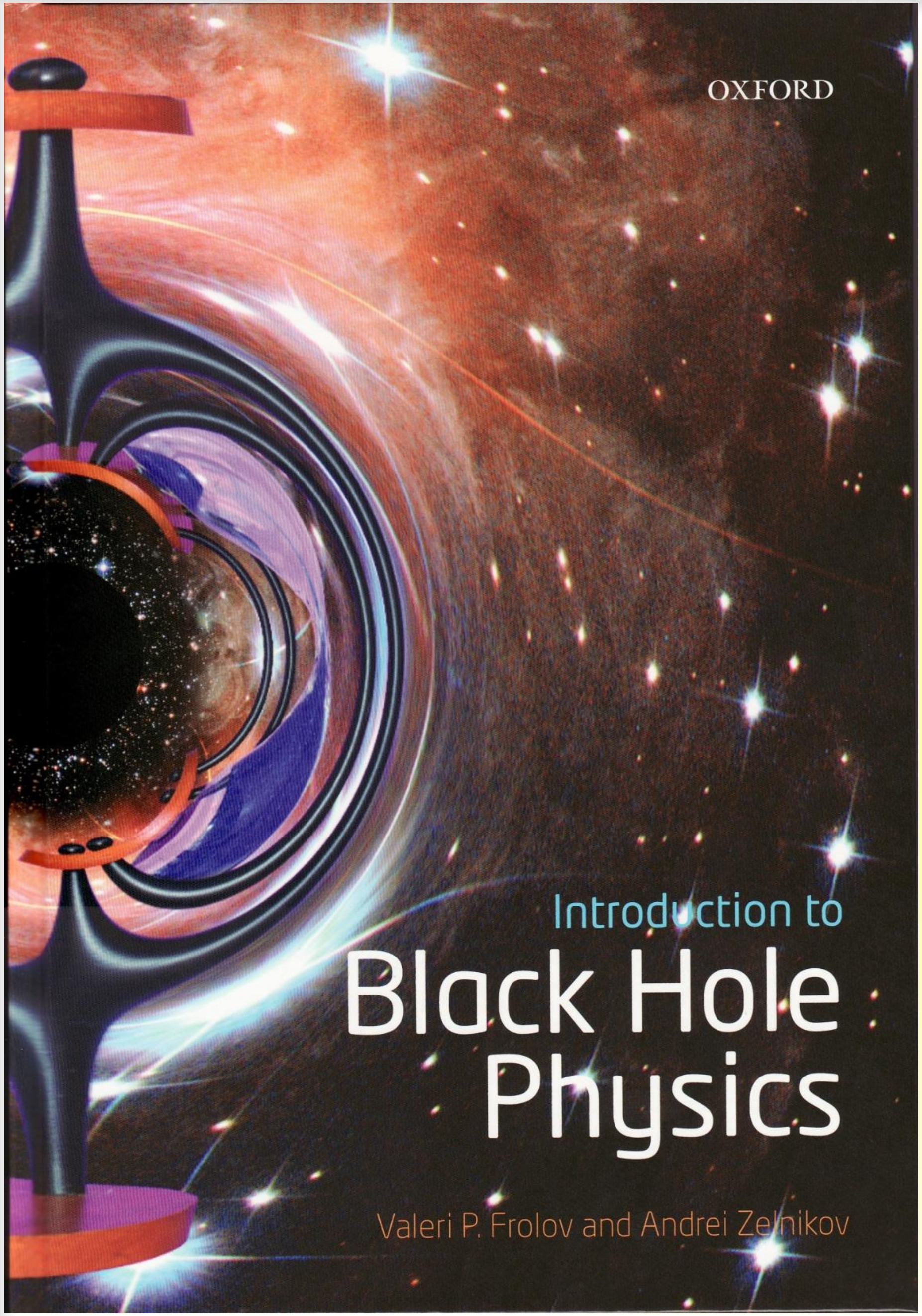
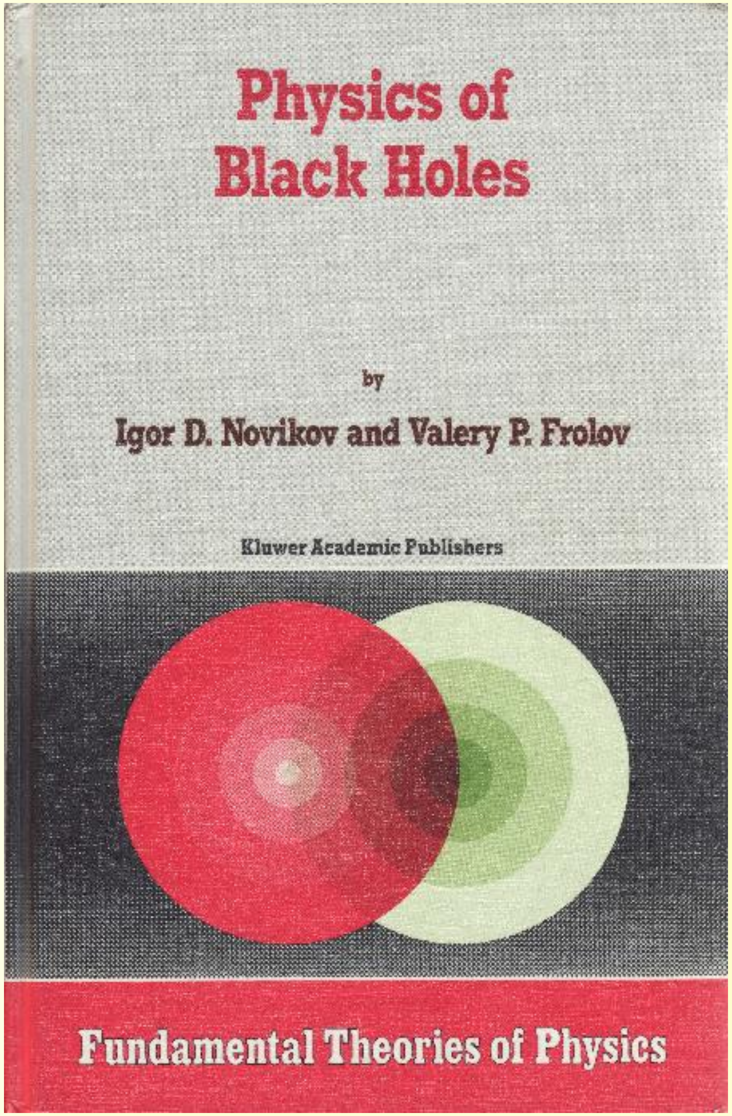
Mini-course of lectures at Verão Quântico 2019

Ubu, Espírito Santo, Brazil, 17 to 22 February 2019,



http://www.amazon.ca/Introduction-Black-Physics-Valeri-Frolov/dp/0199692297



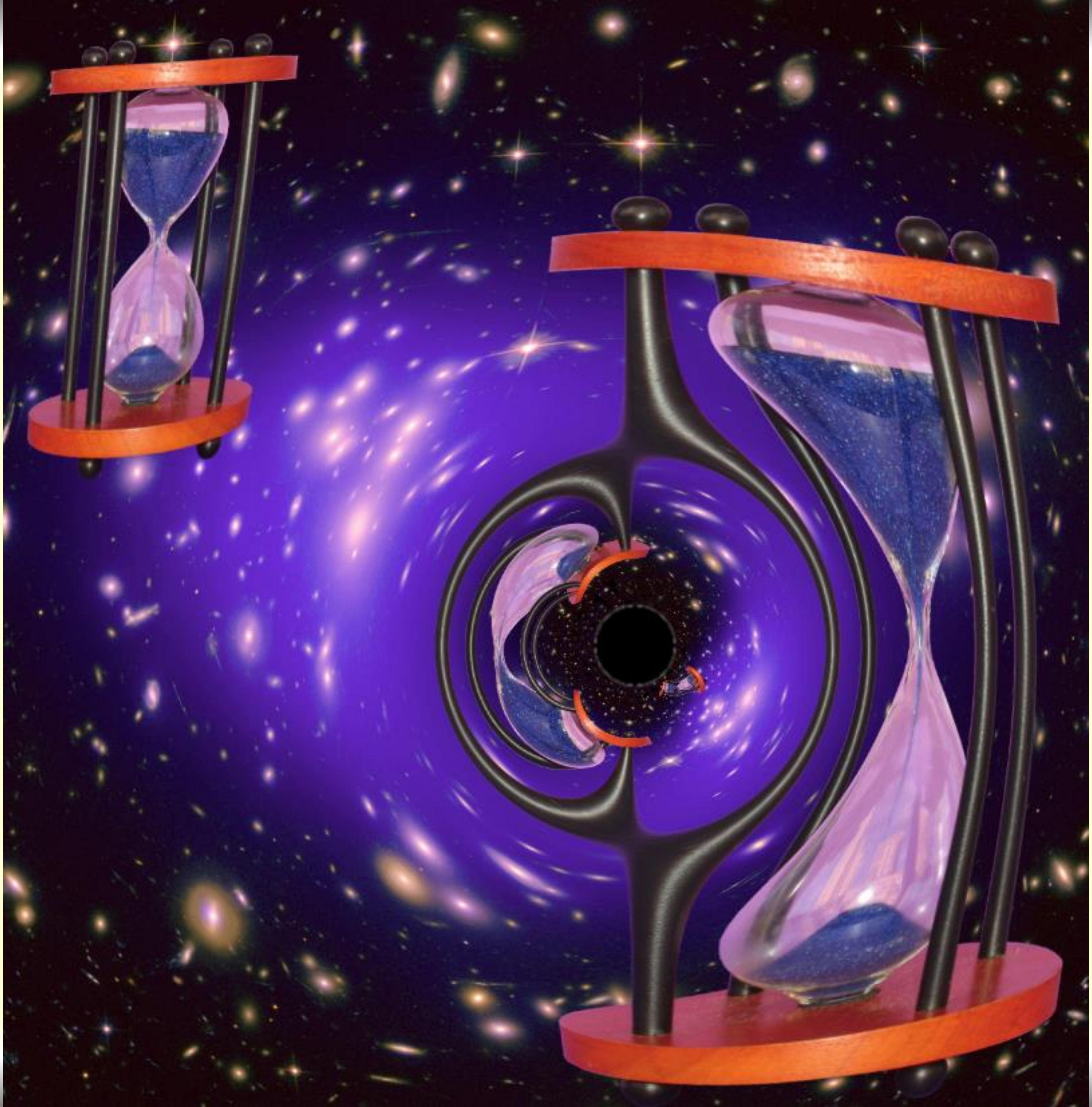
PDF file of the book can be found at http://libgen.org/ (search for “frolov novikov”)

At: http://www.google.ca

BLACK HOLE 913,000,000

NEUTRON STAR 23,800,000

In virtual reality black hole are more `real’ than neutron stars.



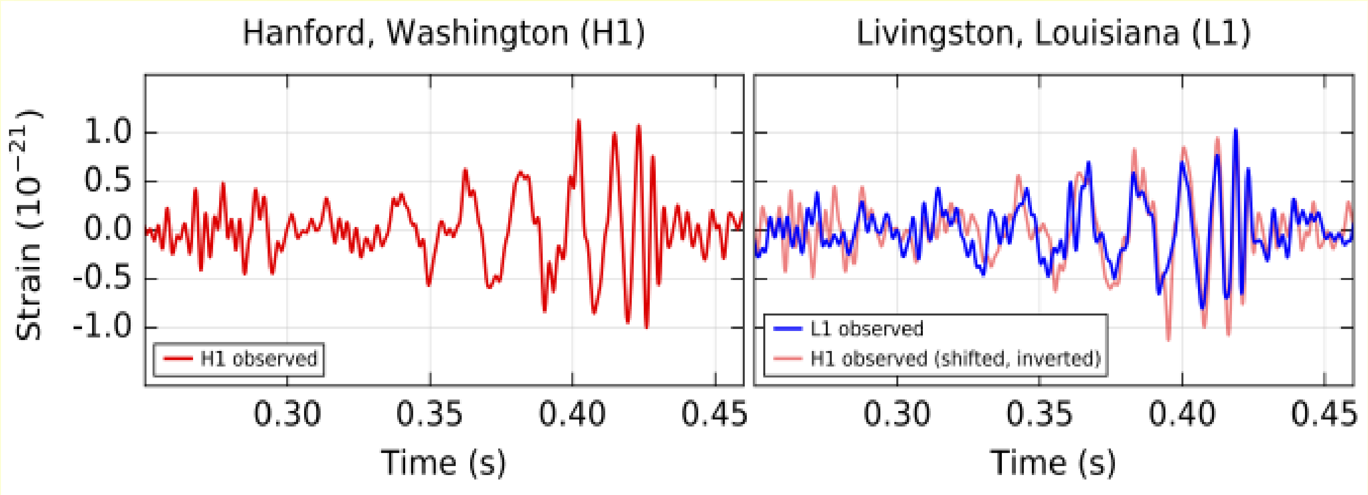
Black hole discovery

Two predictions of the Einstein's General Relativity (1915)

- Gravitational waves
- Black holes

One hundred years later ...

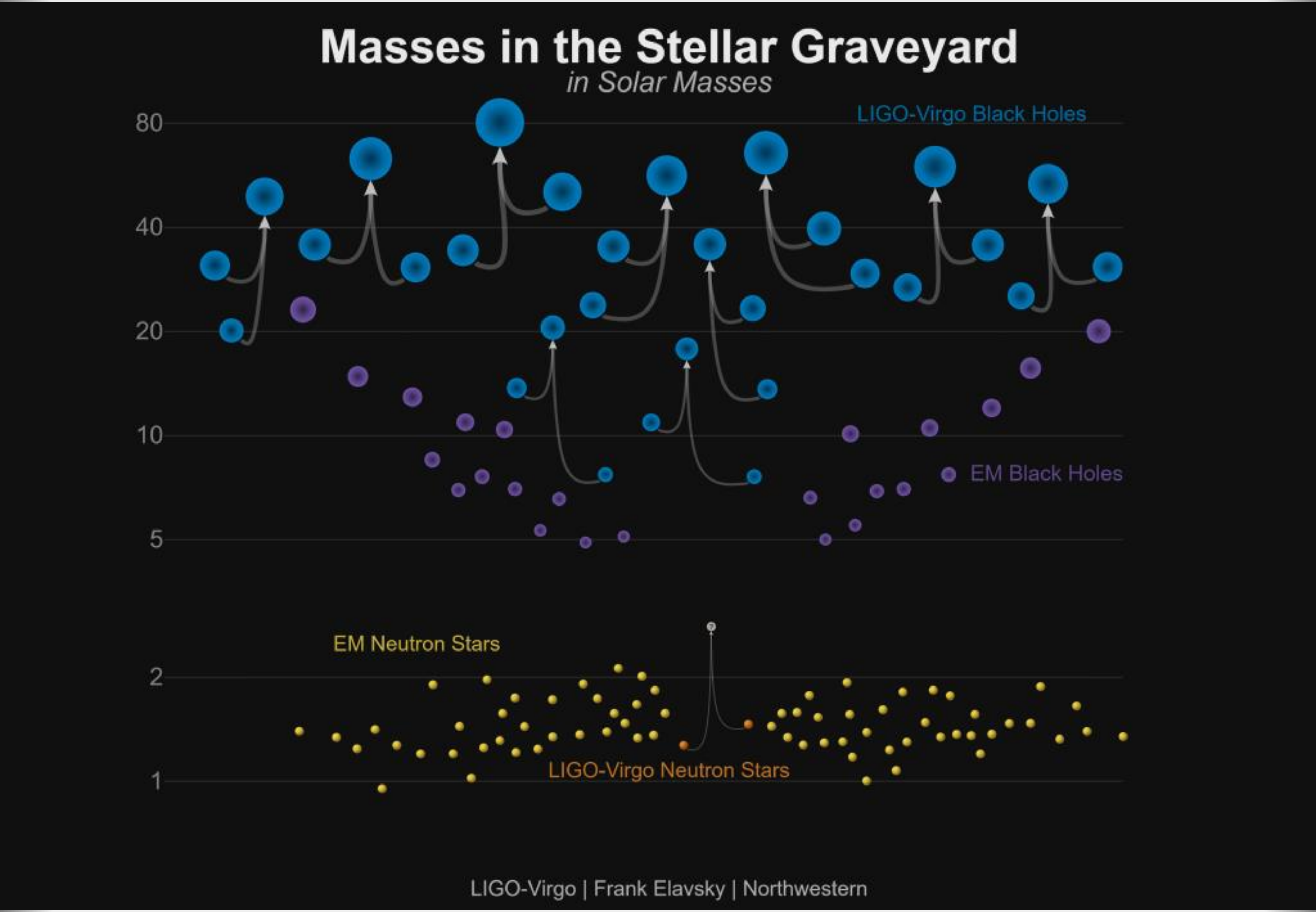
The first observing run (O1) of Advanced LIGO was scheduled to start 9 am GMT (10 am BST), 14 September 2015. Both gravitational-wave detectors were running fine, but there were few a extra things the calibration team wanted to do and not all the automated analysis had been set up, so it was decided to postpone the start of the run until 18 September. No-one told the Universe. At 9:50 am, 14 September there was an event.



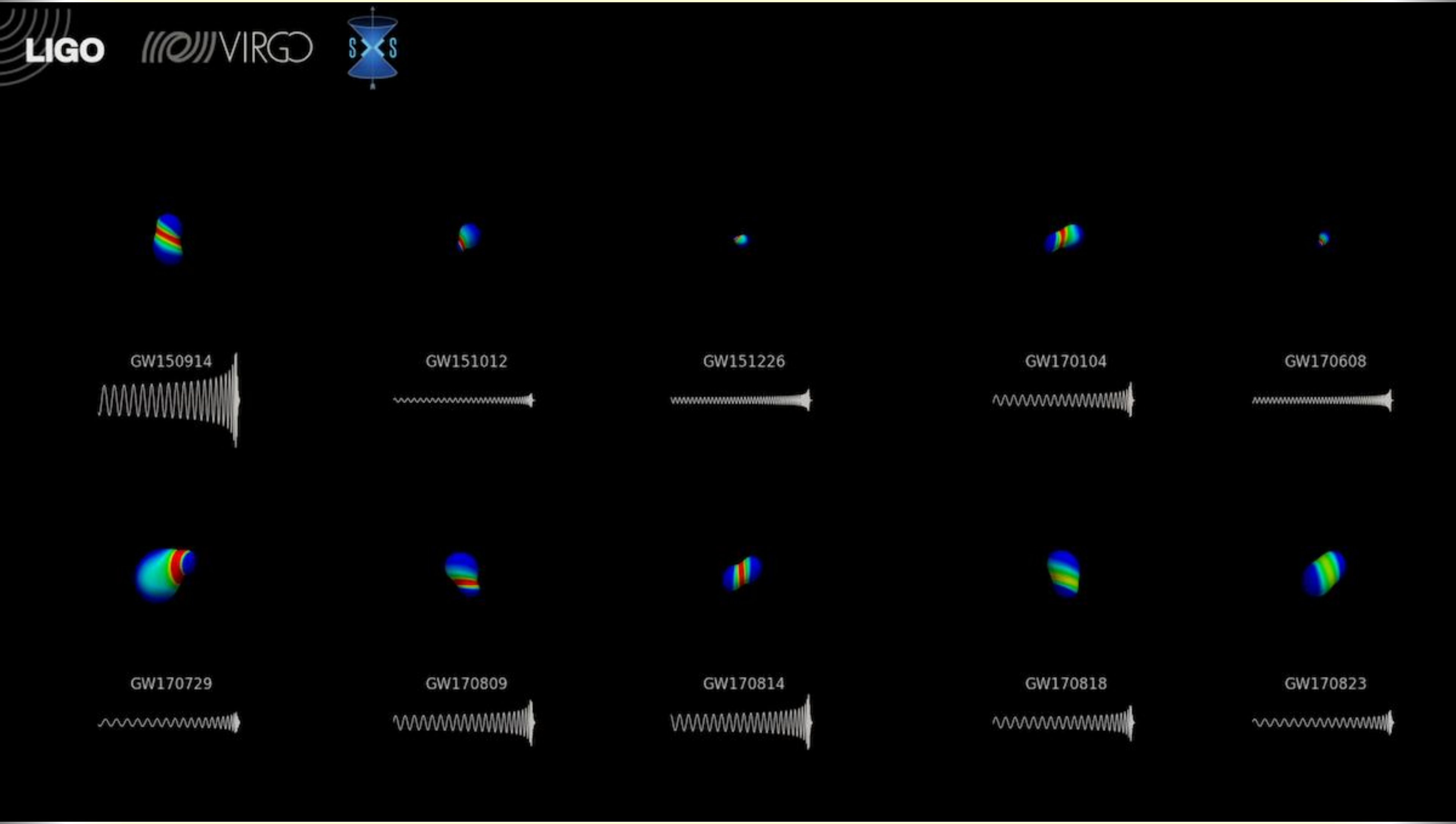
Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
0.30
0.35
0.40
0.45
Time (s)

3 Dec 2018 -- The LIGO Scientific Collaboration and the Virgo Collaboration have released the results of their search for stellar-mass coalescing compact binaries during the first and second observing runs using an advanced gravitational-wave detector network. This includes the confident detection of ten binary black hole mergers and one binary neutron star merger.

4 new events. The "most massive case" GW170729. Its source is made up of black holes with masses 50 $M_{\odot}$ and 34 $M_{\odot}$.



Masses in the Stellar Graveyard
in Solar Masses
LIGO-Virgo Black Holes
EM Black Holes
EM Neutron Stars
LIGO-Virgo Neutron Stars
80
40
20
10
5
2
1
?
LIGO-Virgo | Frank Elavsky | Northwestern


LIGO
VIRGO
GW150914
GW151012
GW151226
GW170104
GW170608
GW170729
GW170809
GW170814
GW170818
GW170823

Facts about GW150914

Masses $35M_\odot$ and $30M_\odot$. Final mass $62M_\odot$, $\sim 3M_\odot$ was emitted in the form of gravitational waves. This is about 5% of the total mass. 3.6×10^{49} watts – 50 times greater than the combined power of all light radiated by all the stars in the observable universe.

It was observed about 8 cycles from 35 Hz to 250 Hz.

Total number of emitted gravitons $\sim 10^{80}$.

Gravitational waves

Metric perturbation $g = \eta + h$, $h \sim \dfrac{G}{rc^4}\ddot{Q}$, $-\dot{E} \sim \dfrac{G}{c^5}(\dddot{Q})^2$,

$$Q \sim MR^2, \quad \ddot{Q} \sim \frac{MR^2}{T^2}, \quad \dddot{Q} \sim \frac{MR^2}{T^3}.$$

Third law of Kepler: $\dfrac{R^3}{T^2} \sim GM$, $T^2 \sim \dfrac{R^3}{GM}$, $\ddot{Q} \sim \dfrac{GM^2}{R}$,

$$h \sim \frac{G^2M^2}{rRc^4} \sim \left(\frac{R_g}{R}\right)^2 \frac{R}{r} = \left(\frac{R_g}{R}\right)\frac{R_g}{r}, \qquad -\dot{E} \sim \frac{c^5}{G}\left(\frac{R_g}{R}\right)^5$$

$\frac{c^5}{G} \approx 3.6 \times 10^{59} erg/s$. If we put $\frac{R_g}{R} \approx 1/4$, then $(\frac{R_g}{R})^5 \approx 10^{-3}$.

$-\dot{E} \approx 3.6 \times 10^{56} erg/s = 3.6 \times 10^{49}$ watt 50 times greater than the combined power of all light radiated by all the stars in the observable universe.

This event happened at the distance $r \sim 400\text{mpc} \approx 1.4 \times 10^9$ light years, $r \sim 1.3 \times 10^{27} cm$. The grav. radius of $60 M_\odot$

$$R_g \sim 60 \times 3km = 1.8 \times 10^7 cm, \quad h \approx 0.25 \frac{1.8 \times 10^7}{1.3 \times 10^{27}} \approx 10^{-21}.$$

Stellar mass BHs

Gravitational collapse will always occur on any star core over 3-5 solar masses, inevitably producing a black hole.

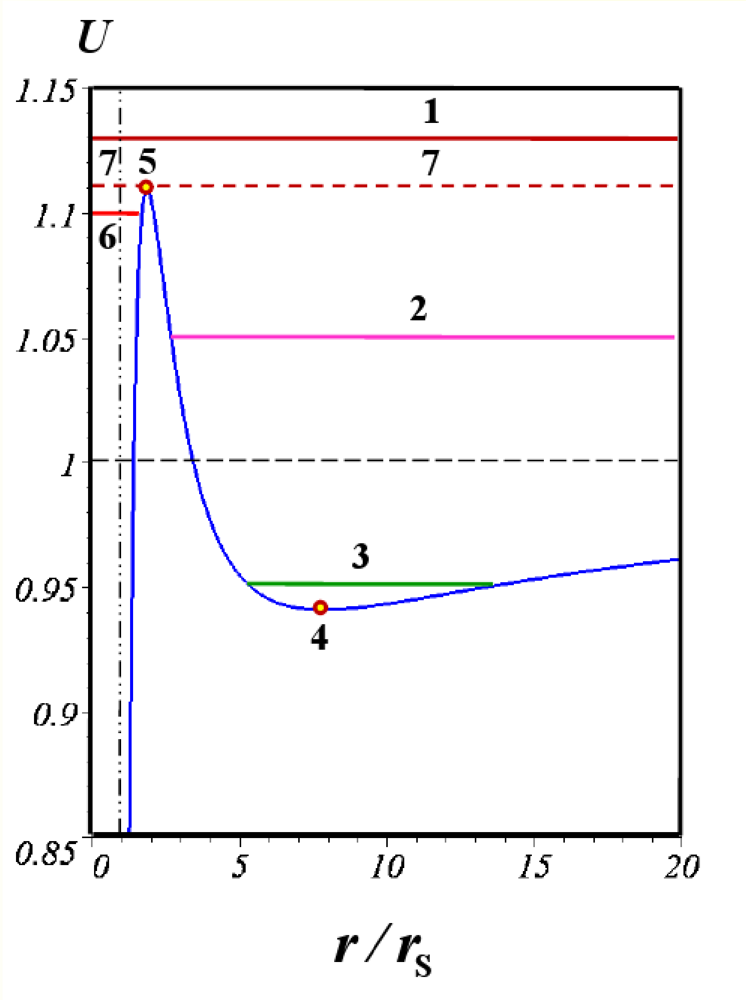
Approximately 1/3 of the star systems in the Milky Way are binary or multiple. The rest 2/3 are single stars. A heavier companion in the stellar binary has faster evolution than the lighter one. If the mass of the heavy star is large enough a system may become a *black hole binary*.

Known candidates for stellar mass black holes were discovered in such systems. The motion of a star companion in the binary can be detected. This gives information about the mass of the black hole candidate.

Estimates give: the number of stellar mass black holes in our Galaxy is about $10^8 - 10^9$. A similar or larger number of neutron stars exist in the Galaxy.

Indirect methods of seach for stellar mass black holes

- X-ray object in a binary system,
- Its mass measuring by means of Newton's law,
- Existence of an accretion disk,
- Velocity of matter at the ISCO: Broading of Fe α lines,
- Total brightness of the accretion disk.



Schwarzschild BH:

$r_{ISCO} = 3r_g = 6M$

$E_{ISCO} / m = \sqrt{8/9}$

BH efficiency: 5.72%

Extremal Kerr BH ($J/M^2=1$):

$r_{ISCO} = M$

$E_{ISCO} / m = \sqrt{1/3}$

BH efficiency: 42.36%

Black Holes classification

- **Stellar-mass black holes with** $M \sim 3-60 M_{\odot}$
- **(Super)massive black holes with** $M \sim 10^5 - 10^9 M_{\odot}$
- **Intermediate-mass black holes with** $M \sim 10^3 M_{\odot}$
- **Primordial black holes with mass up to** $M \sim M_{\odot}$
- **Micro black holes** $M > M_{Planck}$

$$M_{Planck} = \sqrt{\frac{\hbar c}{G}} \approx 10^{-5} g$$

The quantum gravity effects become important for black holes with $M < M_{Planck}$. The black holes of smaller mass do not exist, at least in the standard classical sense.

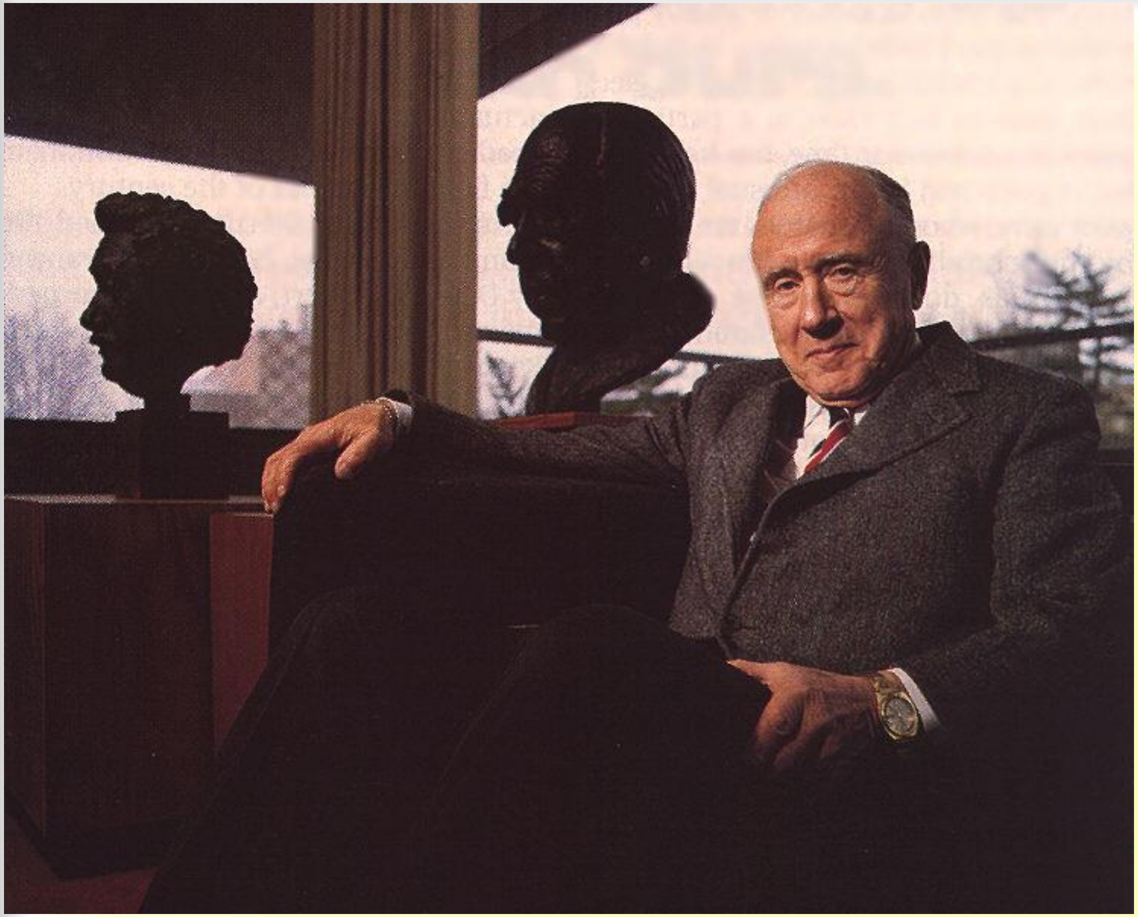
Supermassive black holes

- BH in the center of our Galaxy, Milky Way. Sagittarius A* is the location of a supermassive black hole of mass $\sim 4\times10^6 M_\odot$.
- SMBHs in the center of other galaxies.

A black hole is a compact massive object, the gravitational field of which is so strong that nothing (even the light) can escape from it.

The boundary of the black hole, called the event horizon, is a surface at which the escape velocity is equal to the velocity of light.

$$\tfrac{1}{2}mv^2 = \frac{GMm}{R} \quad \Rightarrow \quad R_g = \frac{2GM}{c^2}$$



John
Archibald
Wheeler
(1911--
2008)

The name "black hole" was invented in December 1967 by John Archibald Wheeler. Before Wheeler, these objects were often referred to as "black stars“ or "frozen stars".

4D black holes and their properties

Event horizon (BH boundary) is almost everywhere null surface

Black hole surface topology is S^2.

Its surface area never decreases (EC)

Soon after their formation black holes become stationary (the `balding phase' $T \sim r_g / c$) .

Stationary black holes are either static (Schwarzschild) or axially symmetric (Kerr).

Uniqueness theorem: Stationary isolated BHs are uniquely specified by their mass and angular momentum and are described by the Kerr metric.

The Kerr metric besides the evident ST symmetries has also hidden symmetry.

Geodesic equation of motion are completely integrable. Massless field equations allow the separation of variables.

Black holes are classically stable.

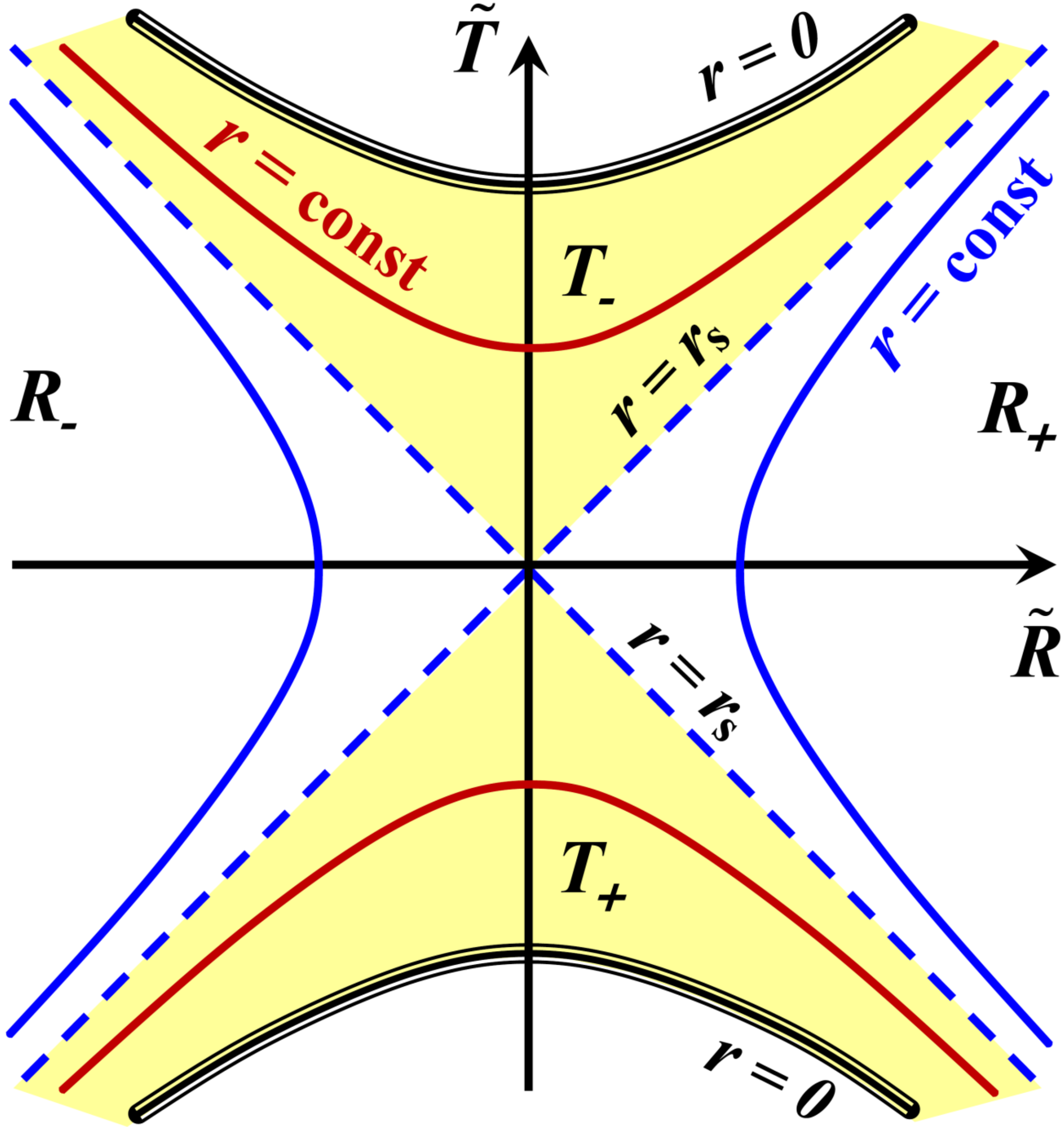
The radius $=2M$ is known as the gravitational radius or the Schwarzschild radius. In physical units, after restoring G and c constants, it has the value

$$r_S = \frac{2GM}{c^2}$$

Schwarzschild metric

$$ds^2 = -\left(1-\frac{r_S}{r}\right)dt^2 + \left(1-\frac{r_S}{r}\right)^{-1} dr^2 + r^2\, d\omega^2$$

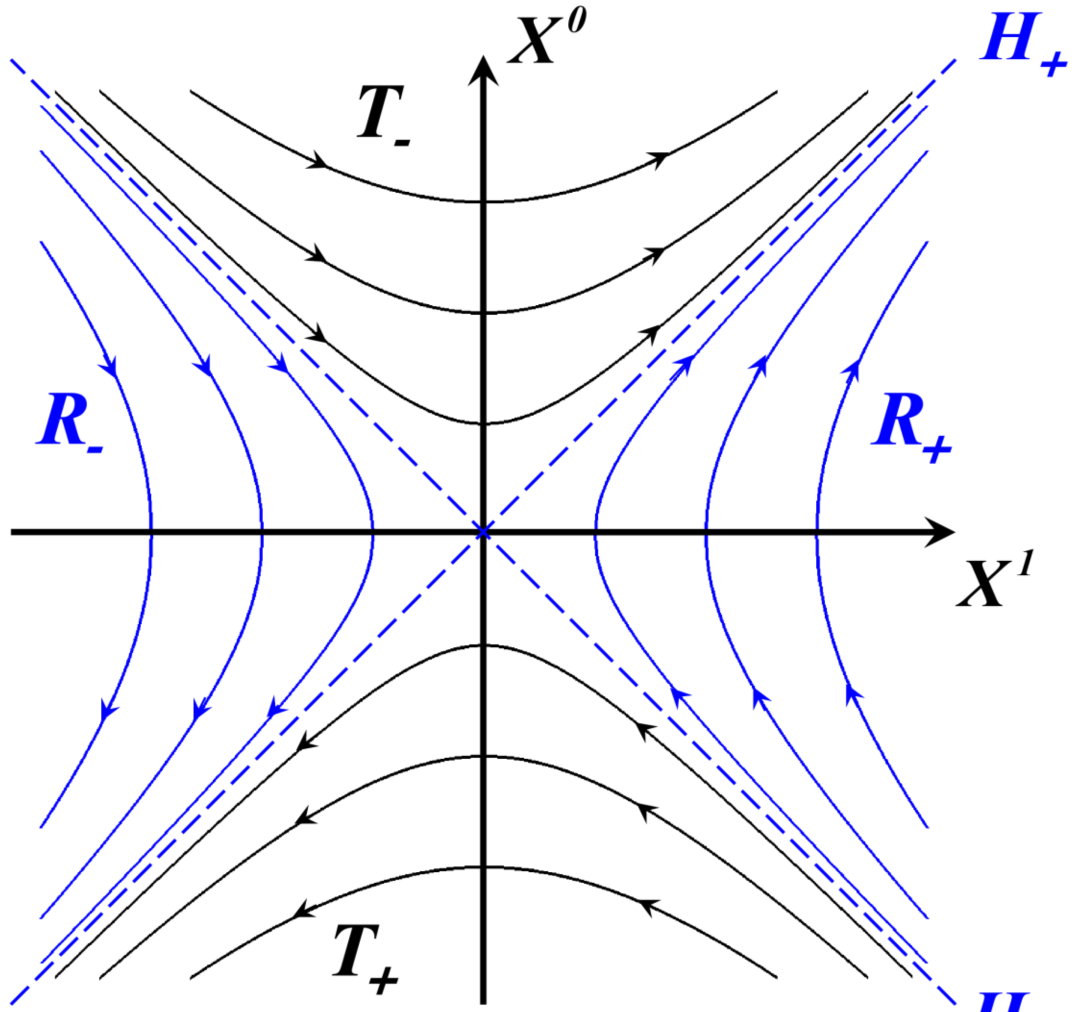
describes the gravitational field in vacuum, outside a spherical distribution of matter. This matter may be either static or have radial motion preserving the spherical symmetry. According to Birkhoff's theorem, the external metric does not depend on such motion. In the absence of matter, the metric describes an exterior of a spherically symmetric static black hole. In this case r_S is the radius of its event horizon.



$\tilde{T}$
$r = 0$
$r = \text{const}$
T_-
$r = \text{const}$
$r = r_s$
R_-
R_+
$\tilde{R}$
$r = r_s$
T_+
$r = 0$

Rindler space

$$ds^2 \approx -\kappa^2 \rho^2 dt^2 + d\rho^2 + r_S^2 d\omega^2$$



Einstein-Rosen bridge

The equation *U = -V* determines a three-dimensional spacelike slice of the Kruskal spacetime. This slice passes through the bifurcation surface of the horizons. It has two branches. This subspace is called the Einstein-Rosen bridge. Its internal geometry

$$dl^2 = \frac{dr^2}{1-2M/r} + r^2\, d\omega^2 = \Omega^4\, dl_0^2\,,$$

$$r = \rho(1+M/2\rho)^2\,.$$

$$\Omega = 1 + \frac{M}{2\rho},$$

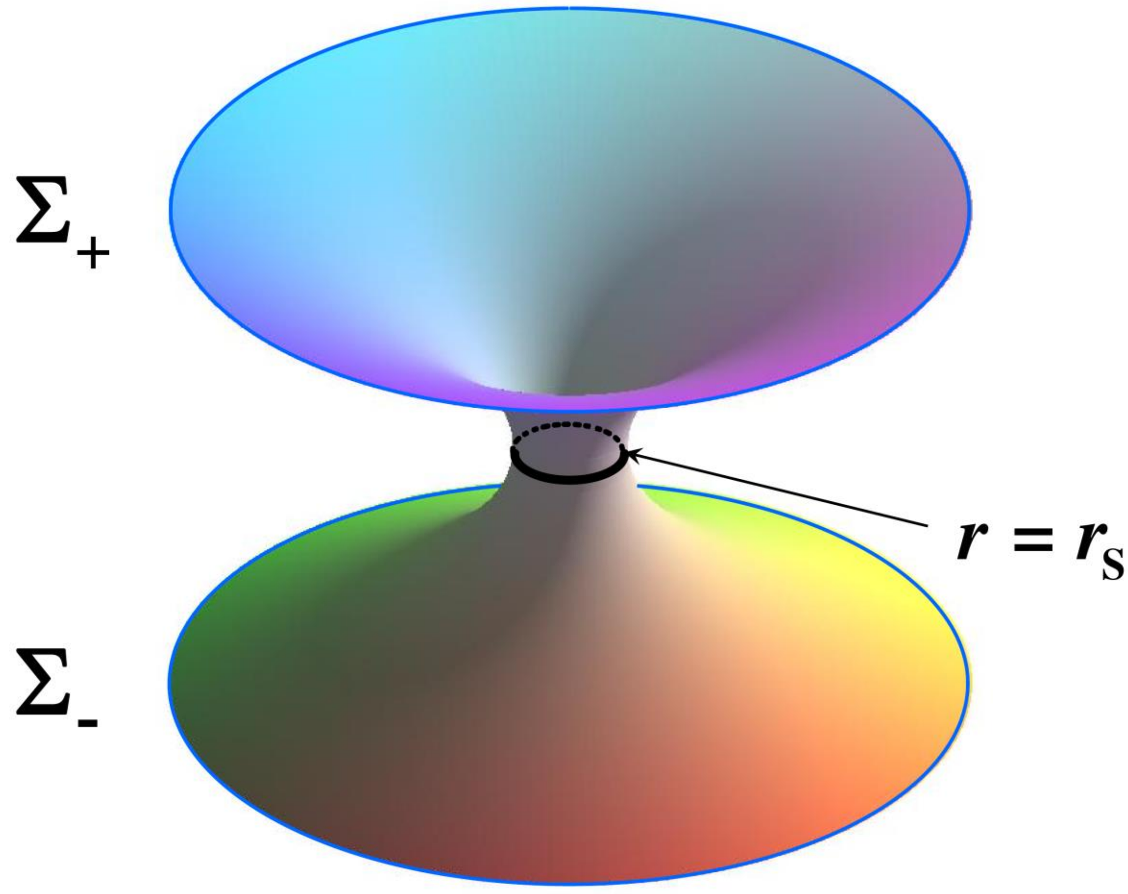
$$dl_0^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

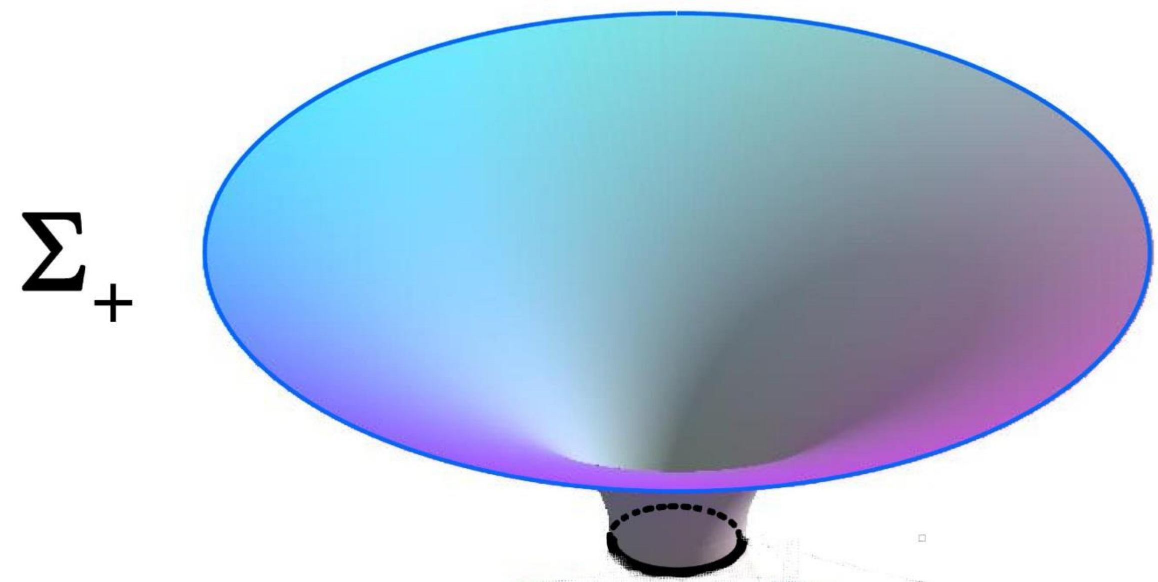
The geometry of a two-dimensional section $\theta = \pi/2$ of the metric can be embedded in a at three-dimensional space as a revolution surface

$$dL^2 = dz^2 + dr^2 + r^2\, d\phi^2 = dr^2(1 + z'^2) + r^2\, d\phi^2\,,$$

where $z = \pm 2\sqrt{2M(r - 2M)}$

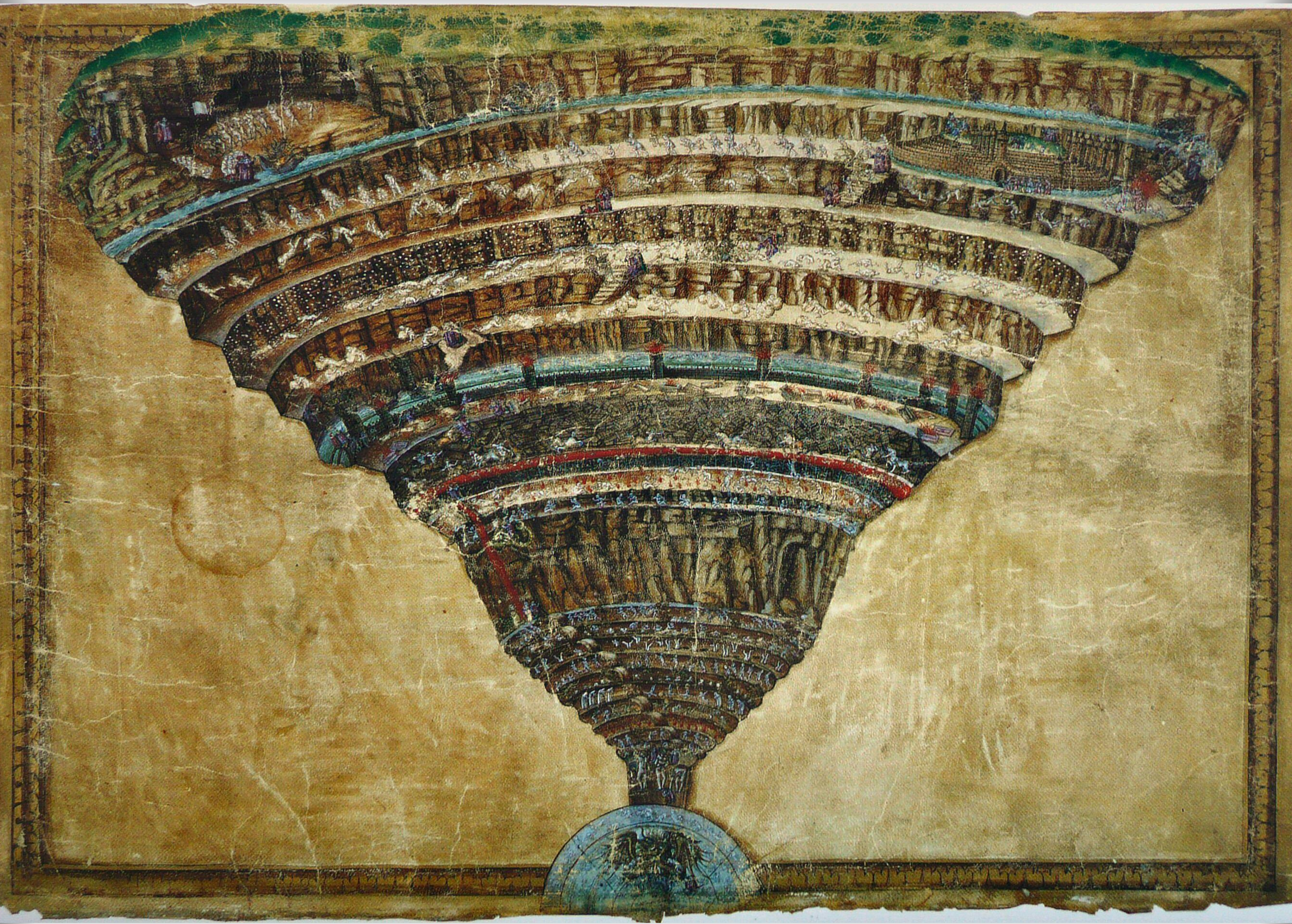
Einstein-Rosen bridge





Sandro Botticelli (1480) “La Carte de l'Enfer”
(“The Abyss of Hell’). An illustration to Dante’s “The Divine Comedy”

“If the hell exists, the best place for it is inside a black hole.”



Euclidean black hole

In the `physical' spacetime the signature of the metric is (- ; + ; + ; +). The Schwarzschild metric is static and it allows an analytical continuation to the Euclidean one (+ ; + ; + ; +) . This continuation can be obtained by making the Wick's rotation $t = it_E$. The corresponding space, called a Euclidean black hole, has interesting mathematical properties and has important physical applications.

The two-dimensional part of the Euclidean metric $$d\gamma_E^2 = g\, dt_E^2 + \frac{dr^2}{g}$$

Near the Euclidean horizon is has the form $$d\gamma_E^2 \approx \kappa^2 \rho^2\, dt_E^2 + d\rho^2$$

In a general case this metric has a conical singularity. This singularity vanishes if t_E is a periodic coordinate with the period

$$t_E \in \left(0, \frac{2\pi}{\kappa} \right)$$

Euclidean Schwarzschild metric in the vicinity of the horizon

Consider the (t-r)-sector of the Schwarzschild metric

$$ds^2 = \underbrace{g\,dt^2 + \frac{dr^2}{g}}_{d\gamma^2} + r^2\,d\omega^2\,,$$

$$g = 1 - \frac{2M}{r}\,.$$

In the vicinity of the horizon: $r = r_S(1+y), \quad y \ll 1.$

the proper length distance from the horizon $\rho = \int_{r_S}^{r} \frac{dr}{\sqrt{g}} \simeq 2r_S\sqrt{y}\,,$

$$d\gamma^2 = \kappa^2 \rho^2 dt^2 + d\rho^2$$

$\kappa = 1/(2r_S) = 1/(4M)$ **is the surface gravity of the black hole.**

$$ds^2 \approx \kappa^2 \rho^2 dt^2 + d\rho^2 + r_S^2 d\omega^2$$

Regularity at ρ=0 requires t_E is periodic with a period $2\pi/\kappa$.

This condition determines Hawking themperature

$$\Theta_H = \frac{\hbar c^3}{8\pi G k_B M}.$$

$$\Theta_H = \frac{\hbar c^3}{8\pi G k_B M}.$$

Why $\hbar$ enters the expression for the temperature?

We keep $\hbar$ in the relations, but put $c = G = 1$

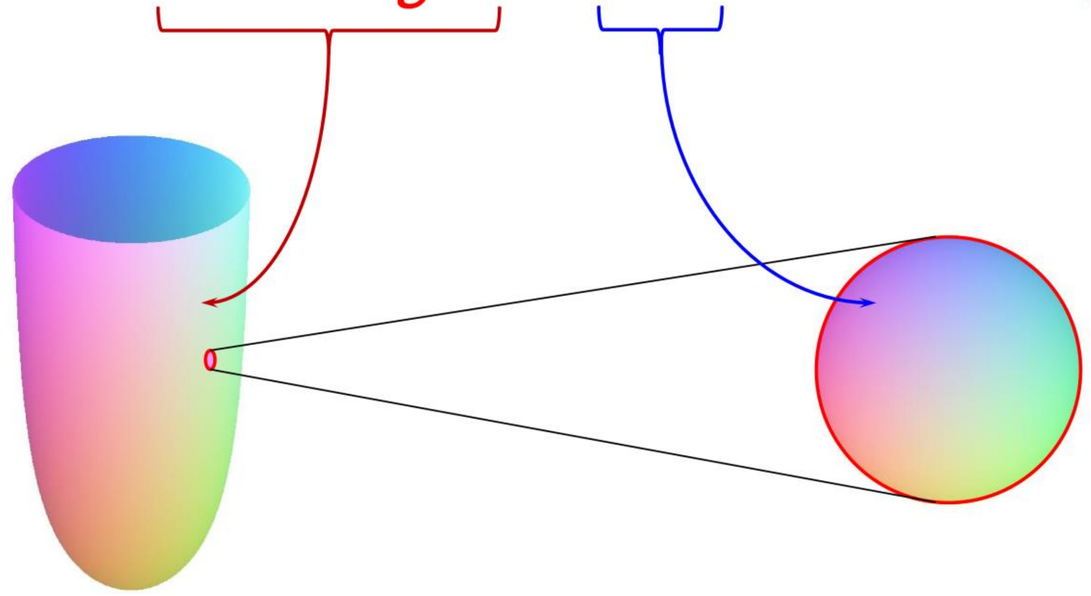
$$\Theta \sim E \sim \hbar\omega \sim \frac{\hbar}{\lambda} \sim \frac{\hbar\kappa}{2\pi} \sim \frac{\hbar}{8\pi M}$$

We used here that λ is equal to the period of the Euclidean time.

Gibbons-Hawking instanton

This regular four dimensional Euclidean space is called the Euclidean black hole or the Gibbons-Hawking instanton. The inverse period in Euclidean time is equal

$$ds^2 = \underbrace{g\,d\tau^2 + \frac{dr^2}{g}}\, + \underbrace{r^2\,d\omega^2},\qquad g = 1 - \frac{r_S}{r}$$



$$\Theta_H = \frac{\kappa}{2\pi}$$

is called the Hawking temperature of the black hole.

Rotating black holes

4D Kerr-NUT-(A)dS
Derivation in 3 Simple Steps

Step 1: Write flat ST metric in ellipsoidal coordinates

$$dS^2 = -dt^2 + dX^2 + dY^2 + dZ^2$$

$$X = \sqrt{r^2 + a^2}\,\sin\theta\cos\phi, \quad Y = \sqrt{r^2 + a^2}\,\sin\theta\sin\phi,$$

$$Z = r\cos\theta, \quad \tfrac{X^2+Y^2}{r^2+a^2} + \tfrac{Z^2}{r^2} = 1$$

$$dS^2 = -dt^2 + (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) +$$

$$+ (r^2 + a^2)\sin^2\theta\, d\phi^2$$

Step 2: Rewrite this metric in `algebraic' form

$$y = a\cos\theta, \tau = t - a\phi, \quad \psi = a^{-1}\phi$$

$$ds^2 = -\frac{\Delta_r}{r^2+y^2}(d\tau + y^2 d\psi)^2 + \frac{\Delta_y}{r^2+y^2}(d\tau - r^2 d\psi)^2$$

$$+(r^2+y^2)[\frac{dr^2}{\Delta_r} + \frac{dy^2}{\Delta_y}], \quad \Delta_r = r^2 + a^2, \quad \Delta_y = a^2 - y^2$$

(i) Coefficients are rational functions;
(ii) `Almost symmetric' form

Step 3: Use this form of the metric to solve Einstein equations

Assume that $\Delta_r = \Delta_r(r)$ and $\Delta_y = \Delta_y(y)$ and solve the equation $R_{\mu\nu} = -3\lambda g_{\mu\nu}$ (λ=-Λ/3)

$$ds^2 = -\frac{\Delta_r}{r^2+y^2}(d\tau + y^2 d\psi)^2 + \frac{Y(y)}{r^2+y^2}(d\tau - r^2 d\psi)^2$$

$$+(r^2+y^2)[\frac{dr^2}{\Delta_r}+\frac{dy^2}{\Delta_y}]$$

[Carter 1968]

'Trace equation' gives:

$$\partial_r^2 \Delta_r + \partial_y^2 \Delta_y = 12\lambda(r^2 + y^2),$$

$$\partial_r^2 \Delta_r - 12\lambda r^2 = c, \quad \partial_y^2 \Delta_y - 12\lambda y^2 = -c,$$

$$\Delta_r = \lambda r^4 + \frac{c}{2} r^2 + \alpha r + \beta,$$

$$\Delta_y = \lambda y^4 - \frac{c}{2} y^2 + \gamma r + \delta$$

The other equations fix some constants and we have

$$\Delta_r = (r^2 + a^2)(1 + \lambda r^2) - 2Mr,$$

$$\Delta_y = (a^2 - y^2)(1 - \lambda y^2) + 2Ny$$

a – rotation parameter, M – mass,

N – `NUT' parameter, $\lambda$ – `cosmological term'

If we put $\lambda = N = 0$ and change coordinates

$$y = a\cos\theta, \tau = t - a\phi, \quad \psi = a^{-1}\phi,$$

we obtain Kerr solution in standard Boyer-Lindquist coordinates (t, r, θ, ϕ).

Kerr metric

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mra\sin^2\theta}{\Sigma}dt\,d\phi + \frac{A\sin^2\theta}{\Sigma}d\phi^2$$

$$+\frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2,$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2,$$

$$A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta.$$

The event horizon is located at $r = r_+ ,\qquad r_\pm = M \pm \sqrt{M^2 - a^2}\,.$

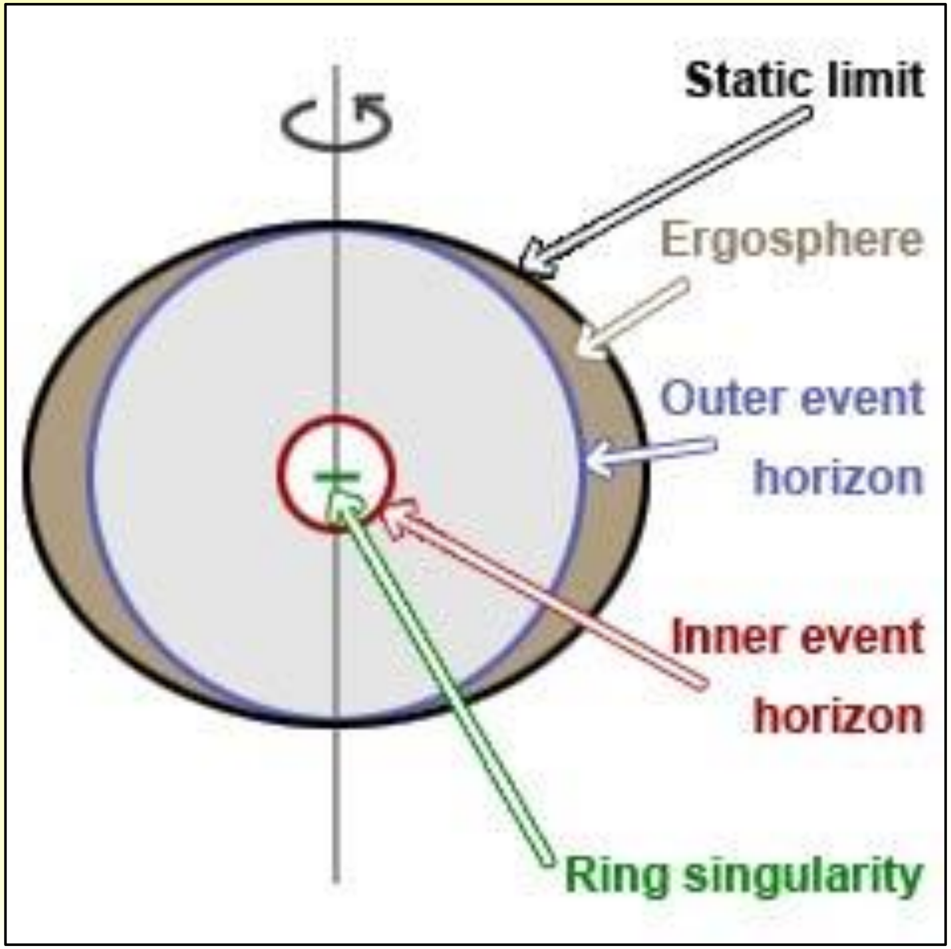
Which are the roots of the equation $\Delta(r) = 0$

The event horizon of the Kerr spacetime is a null 3-dimensional surface. Its spatial slices have the geometry of a 2-dimensional distorted sphere.

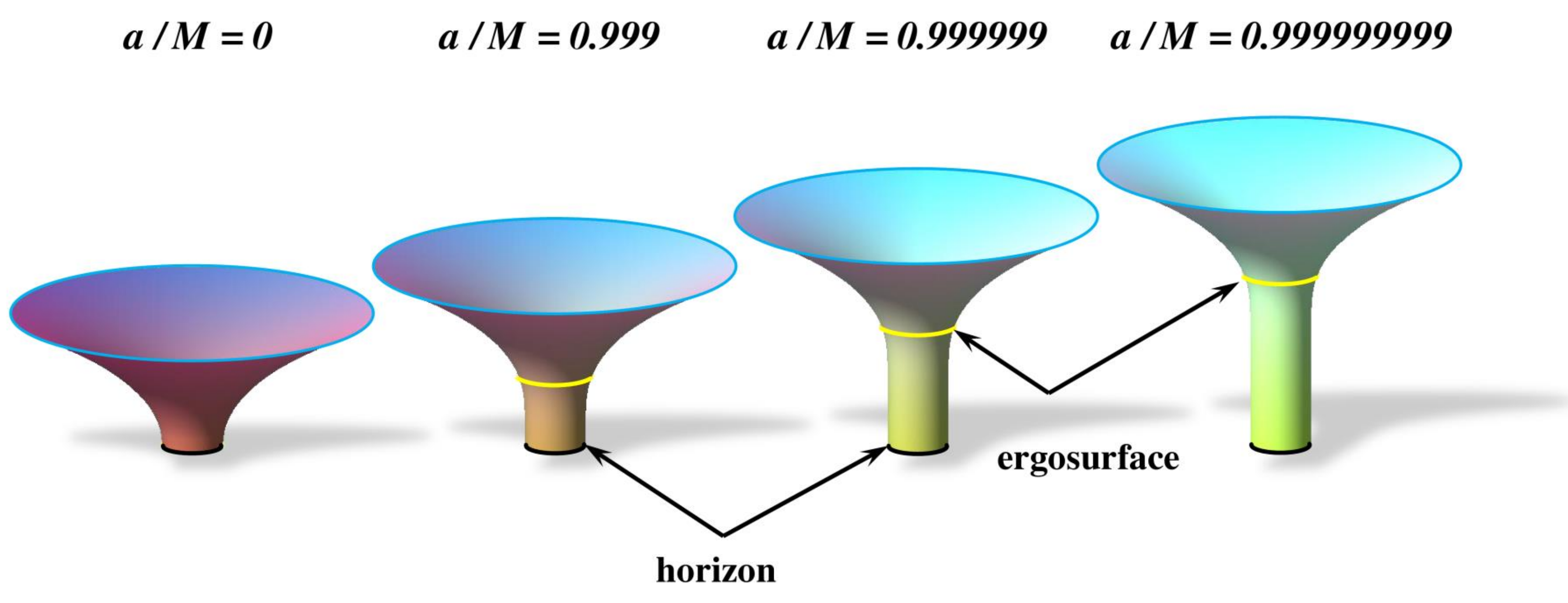
The rotating black holes exist only for

$$a \le M$$

For $a > M$ the Kerr solution does not have a horizon and it describes a naked singularity. It is generally believed that such a singularity does not arise in real physical processes.



Einstein-Rosen bridges for Kerr spacetime



Metric $g_{\mu\nu}$ is a trivial example of the Killing tensor.

Kerr metric has 2 Killing vectors ξ_{t}^{μ} and ξ_{ϕ}^{μ}.

Carter (1968) found that it also has a Killing tensor.

This gives 4 integral of motion and make particle and light equations completely integrable.

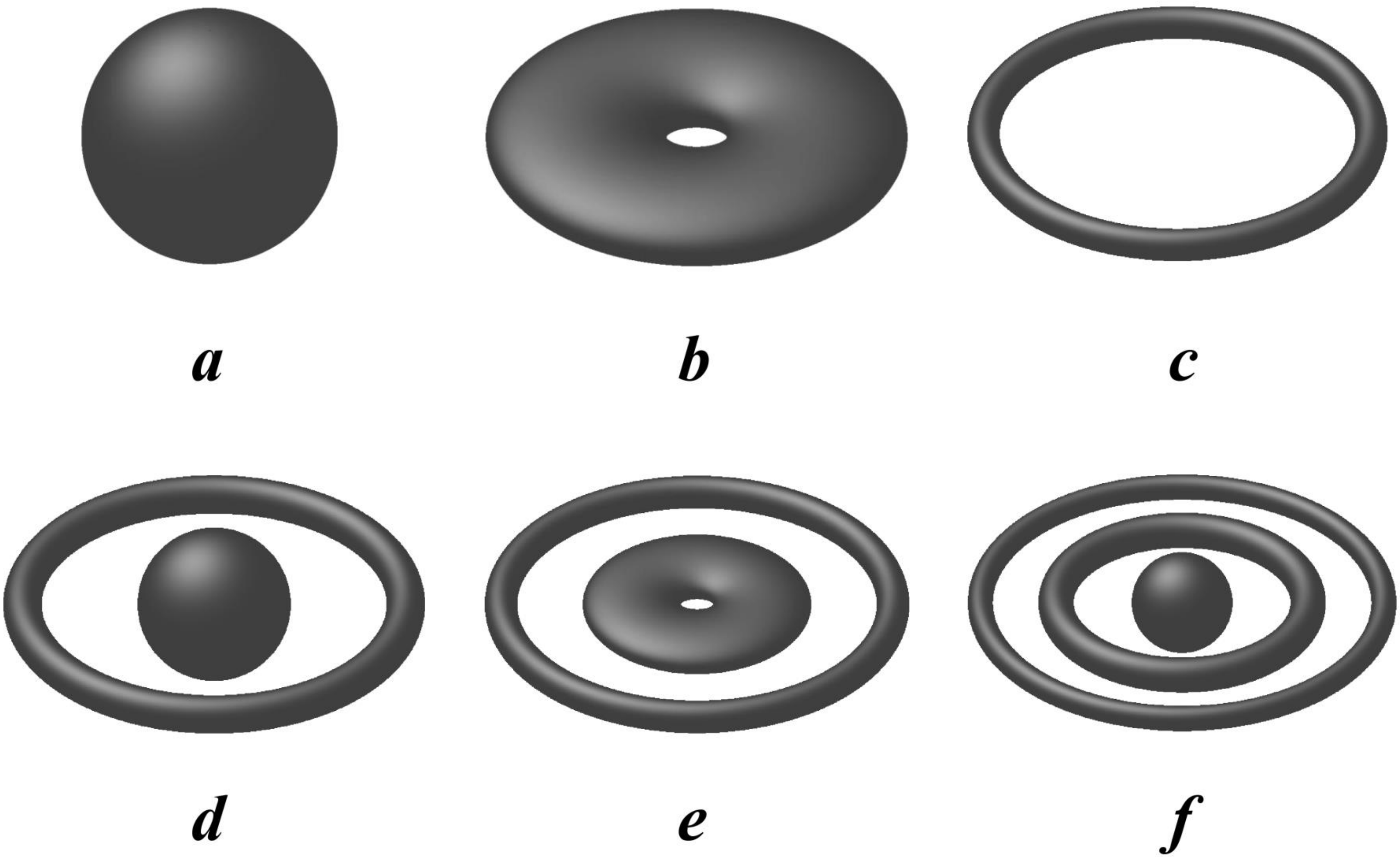
Higher dimensional BHs

Angular momentum in Higher Dimensions

$$(D = 2n + \varepsilon)$$

$$\mathbf{J} = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_{n-1+\varepsilon} \end{pmatrix}, \quad J_i = \begin{pmatrix} 0 & j_i \\ -j_i & 0 \end{pmatrix}$$

5D vac. stationary black holes



(1) No Uniquness Theorem: For given M

and J more than one BH solution

(2) Complete integrability and separation

of variables are generic properties

of HD analogues of Kerr BH

$(hidden\ symmetry\ generators\ V.F\ \&\ D.K.\,07)$

(3) Stability of HD BHs ?

Main Results

In the most general $(Kerr-NUT-(A)dS)$ *higher-dimensional black hole spacetime:*

(1) *Geodesic motion is completely integrable.*

(2) *Hamilton-Jacobi and Klein-Gordon equations allow the complete separation of variables*

Motivations

Separation of variables allows one to reduce a physical problem to a simpler one in which physical quantities depend on less number of variables. In case of complete separability original partial differential equations reduce to a set of ordinary differential equations

Separation of variables in the Kerr metric
is used for study:
(1) Black hole stability
(2) Particle and field propagation
(3) Quasinormal modes
(4) Hawking radiation

Main lesson: Properties of higher dimensional rotating BHs and the 4D Kerr metric are very similar.

HDBHs give a new wide class of completely integrable systems

"General Kerr-NUT-AdS metrics in all dimensions“, Chen, Lü and Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$n=\left[D/2\right],\ \ D=2n+\varepsilon$$

$$R_{\mu\nu}=(D-1)\lambda g_{\mu\nu}$$

$$\lambda,\, M-mass,\, a_k-(n-1+\varepsilon)\ rotation\ \ parameters,$$
$$M_{\alpha}-(n-1-\varepsilon)\ `NUT'\ parameters$$

$$\boxed{Total\ \#\ of\ \ parameters\ is\ D-\varepsilon}$$

Generator of Symmetries

PRINCIPLE CONFORMAL KY TENSOR

2-form $h_{[\mu\nu]}$ with the following properties:

(i) Non-degenerate (maximal matrix rank)

(ii) Closed $dh = 0$

(iii) Conformal KY tensor

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \qquad \xi_a = \tfrac{1}{D-1}\nabla^b h_{ba}$$

ξ_a is a primary Killing vector

All Kerr-NUT-AdS metrics possess a PRINCIPLE CONFORMAL KY TENSOR

(V.F.&Kubiznak '07)

Uniqueness Theorem

A solution of Einstein equations with the cosmological constant which possess a PRINCIPLE CONFORMAL KY TENSOR is a Kerr-NUT-AdS metric

(Houri,Oota&Yasui ’07 ’09;
Krtous, V.F. .&Kubiznak ’08;)

4D Kerr metric example

Principal CCKY tensor $h = db$

$$b = \frac{1}{2}[(y^2 - r^2)d\tau - r^2 y^2 d\psi]$$

$$H^{\nu}{}_{\mu} = h^{\nu\lambda} h_{\mu\lambda}; \quad \det(H^{\nu}{}_{\mu} - \kappa\delta^{\nu}{}_{\mu}) = 0,$$

$$\kappa_{+} = y^2, \quad \kappa_{-} = -r^2$$

Lesson: Essential coordinates (r, y),

Killing coordinates (τ, ψ)

Killing-Yano Tower

$CCKY: \quad h \;\Rightarrow\; h^{\wedge(j)} = \underbrace{h \wedge h \wedge \ldots \wedge h}_{j\ times}$

KY tensors: $k_{(j)} = {*h^{\wedge(j)}}$

Killing tensors: $K^{(j)} = k_{(j)} \bullet k_{(j)}$

Primary Killing vector: $\xi_a = \frac{1}{D-1} \nabla^b h_{ba}$

Secondary Killing vectors: $\xi_j = K_j \bullet \xi$

Total number of conserved quantities:

$$(n+\varepsilon)+(n-1)+1=2n+\varepsilon=D$$

KV KT g

The integrals of motion are functionally independent and in involution. The geodesic equations in the Kerr-NUT -AdS ST are completely integrable.

Separation of variables in HD Black Holes

In $D = 2n + \varepsilon$ dimensional ST the Principal CKY tensor (as operator) has n 2D eigenspaces with eigenvalues x^a. They can be used as coordinates.

D canonical coordinates: n essential coordinates x^a and $n + \varepsilon$ Killing coordinates ψ_j.

Complete separability takes place in these canonical coordinates.

Hamilton-Jacobi $(\nabla S)^2 = m^2$

$\Updownarrow$ WKB $\Phi \sim \exp(iS)$

Klein-Gordon $(\Box - m^2)\Phi = 0$

$\Updownarrow$ "Dirac eqn=$\sqrt{\text{KG eqn}}$ "

Dirac equation $(\gamma^\mu \nabla_\mu + m)\Psi = 0$

Recent results

- Complete separability of Maxwell equations in 4 and higher dimensional spacetime in the off-shell generalized Kerr-NUT-(A)dS metrics [Lunin (2017, FKK (2017);
- Complete separability of massive field equations in 4 and higher dimensional spacetime in the off-shell generalized Kerr-NUT-(A)dS metrics [FKK (2017)];

"Black holes, hidden symmetries, and complete integrability", VF,Pavel Krtous and David Kubiznak. Living Rev.Rel. 20 (2017) no.1, 6 (195 pp.); arXiv:1705.05482.

Quantum black holes

Denote by q `charge' of the particle of mass m and by Γ the field strength. Probability=

$P \sim \exp(-l/\lambda_m)\,\delta(q\Gamma l - 2mc^2)$,

$\lambda_m = \hbar / mc$ is the Compton length and l is a separation of particles when they become `real'.

$$P \sim \exp\left(-\frac{2m^2c^3}{q\Gamma\hbar}\right)$$

Static electric field: $q = e, \Gamma = E \Rightarrow$

$P \sim \exp\left(-\frac{\pi m^2 c^3}{eE\hbar}\right)$ (Schwinger, 1951)

In gravity: $q = m, \Gamma = \kappa = \dfrac{c^4}{4GM}$,

$$P \sim \exp\left(-\frac{2mc^3}{\kappa\hbar}\right) = \exp\left(-\frac{mc^2}{\Theta_H}\right),$$

$$\Theta_{\mathrm{H}} = \frac{\hbar c^3}{8\pi GkM}$$

Order of magnitude estimation

$$[\dot{E}] \sim \frac{[m][L]^2}{[T]^3}, \quad [\hbar] \sim \frac{[m][L]^2}{[T]},$$

$\dot{E} \sim \dfrac{\hbar c^2}{r_S^2}$. This relation is valid in any number of dimensions.

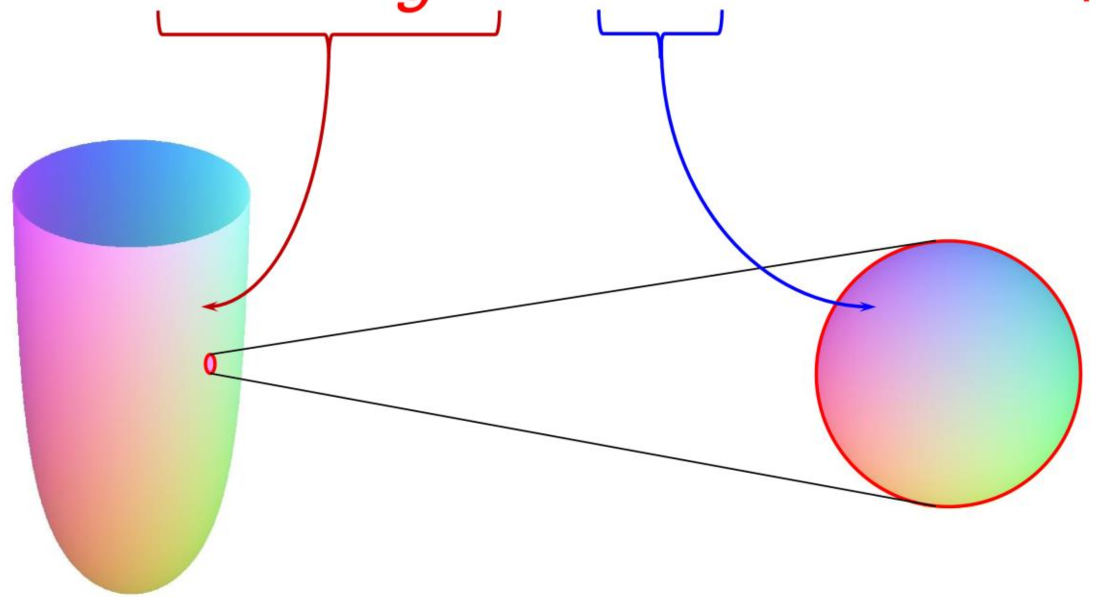
`Ground state'- no-boundary wave function

Barvinsky, V.F., Zelnikov, “Wavefunction of a Black Hole and the Dynamical Origin of Entropy”, Phys.Rev. D51, 1741 (1995)

Gibbons-Hawking instanton

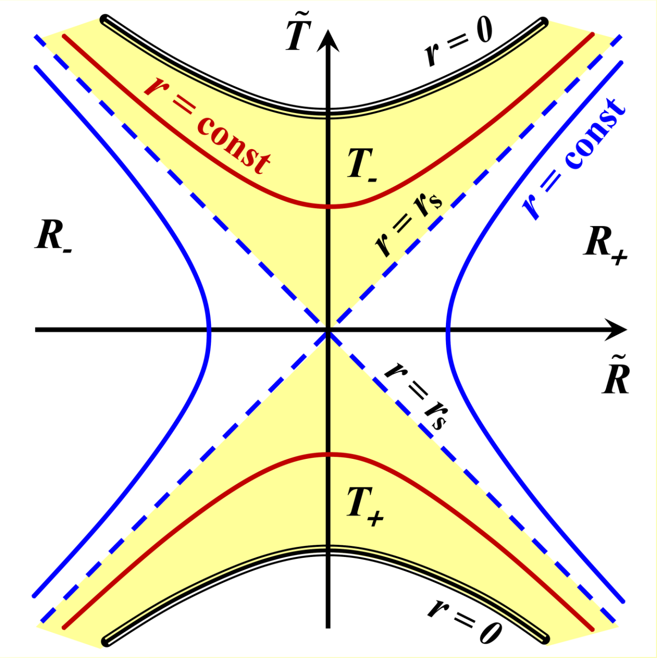
This regular four dimensional Euclidean space is called the Euclidean black hole or the Gibbons-Hawking instanton. The inverse period in Euclidean time is equal

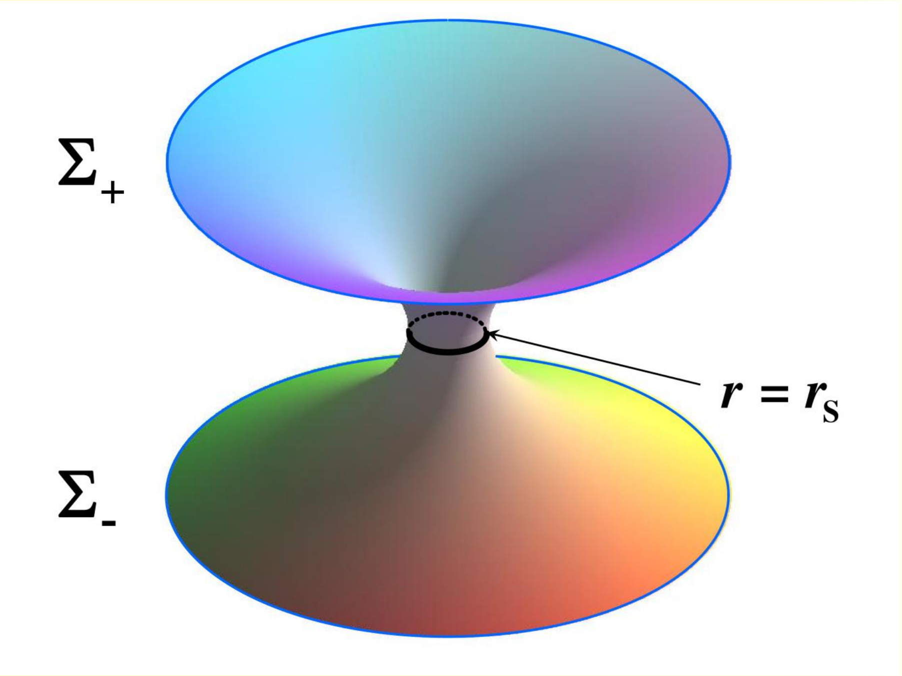
$$ds^2 = g\,d\tau^2 + \frac{dr^2}{g} + r^2\,d\omega^2, \qquad g = 1 - \frac{r_s}{r}$$



$$\Theta_H = \frac{\kappa}{2\pi}$$

is called the Hawking temperature of the black hole.





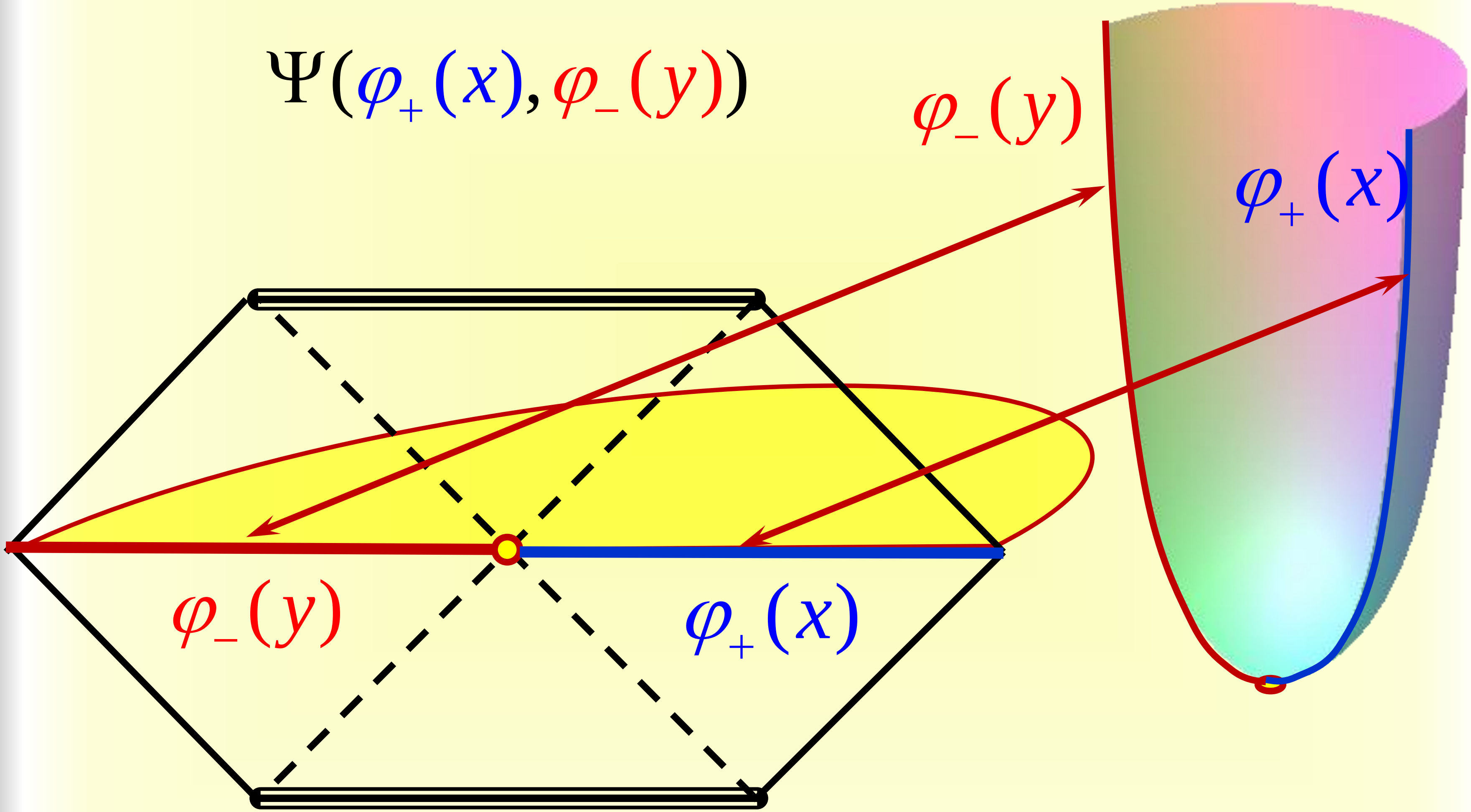
$|\Psi\rangle$ is a wavefunction of (free) quantum field $\hat{\Phi}$.

$$\varphi(\vec{x}) = \Phi(t=0,\vec{x}) \Rightarrow \hat{\Phi}(t=0,\vec{x})\,|\varphi(\vec{x})\rangle = \varphi(\vec{x})\,|\varphi(\vec{x})\rangle$$

$$\langle\varphi(\vec{x})|\Psi\rangle = \Psi(\varphi(\vec{x}))$$

$$\varphi(\vec{x}) = \varphi_+(x) + \varphi_-(y)$$

$$\Psi(\varphi_+(x), \varphi_-(y))$$



$$\Psi(\varphi_+(x),\varphi_-(y)) = Z^{-1}\int_{BC}[D\Phi(\tau,\vec{x})]\exp(-S_E(\Phi))$$

$$BC:\ \ \Phi\,|_{\Sigma_-} = \Phi(0,y) = \varphi_-(y),\qquad \Phi\,|_{\Sigma_+} = \Phi(0,x) = \varphi_+(x)$$

Statement: So defined ground state wavefunction coinsides with Hartle-Hawking vacuum in the eternal black hole.

["Wavefunction of a Black Hole and the Dynamical Origin of Entropy"

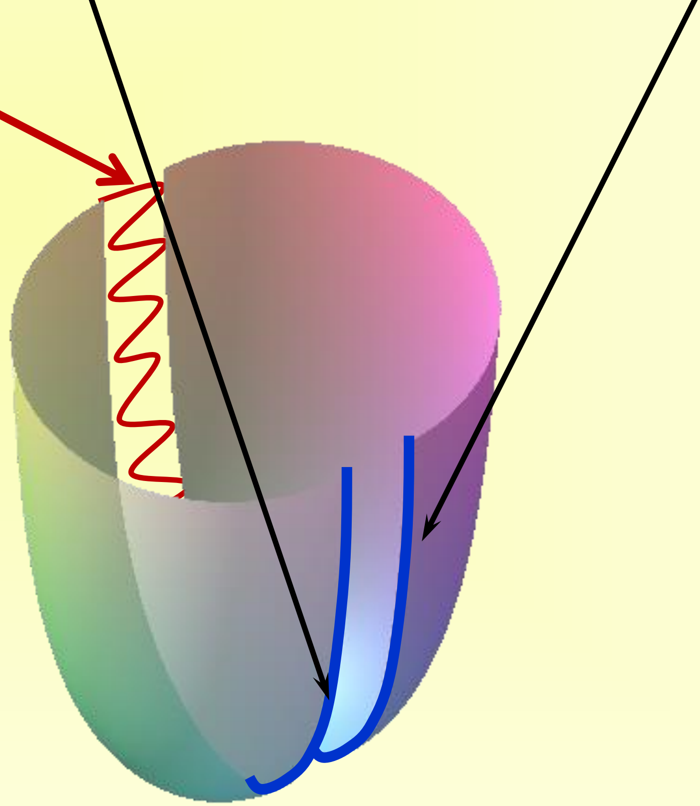
Barvinsky,V.F., Zelnikov, Phys.Rev. D51 $\left(1995\right)$ 1741-1763.]

Density matrix

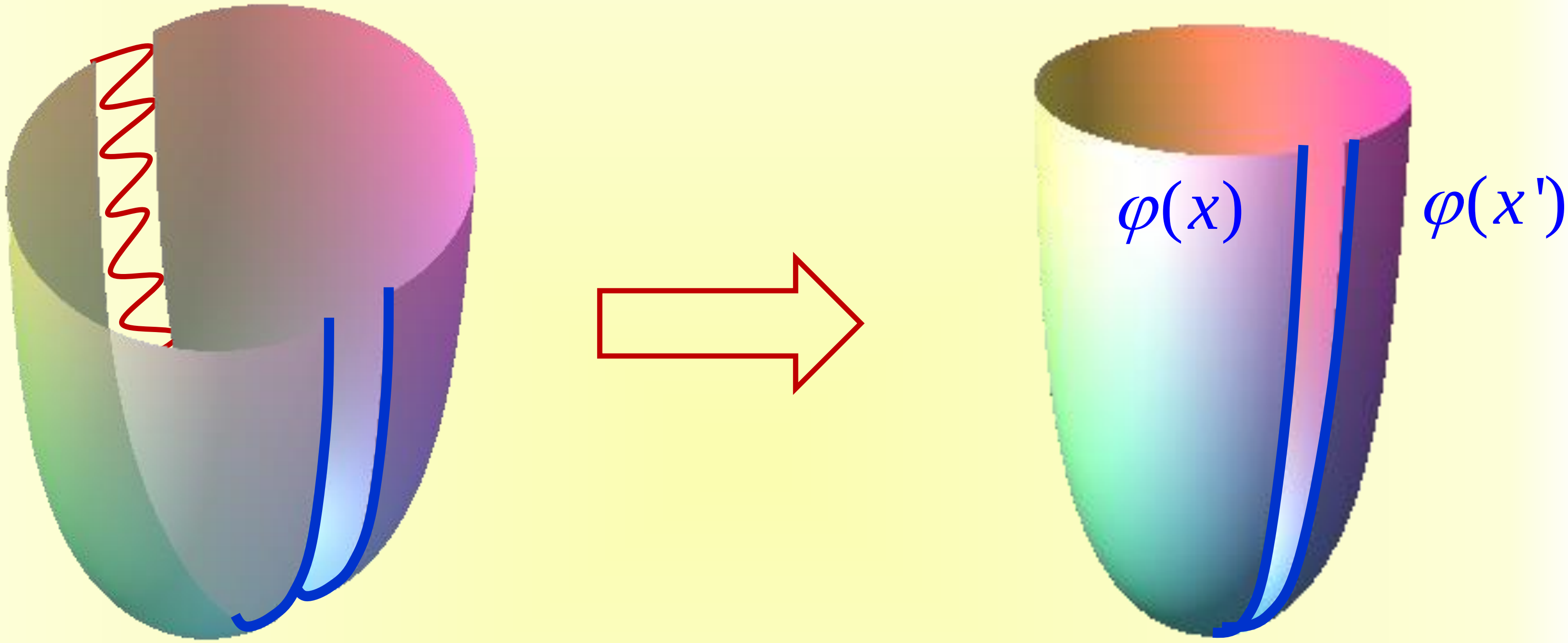
$$\hat{\rho}_+ = Z_+^{-1} Tr_-\left(\left|\Psi\right\rangle\left\langle\Psi\right|\right)$$

Density matrix for quantum fields near BH

$$\rho(\varphi_+(x),\varphi_+(x')) \sim \int [D\varphi_-(y)]\,\Psi(\varphi_+(x),\varphi_-(y))\,\Psi(\varphi_-(y),\varphi_+(x'))$$



Density matrix



$$\rho(\varphi_+(x), \varphi_+(x')) = Z^{-1} \int_{\varphi_+(x)}^{\varphi_+(x')} [D\Phi] \exp(-S_E[\Phi])$$

$$Z = \oint [D\Phi] \exp(-S_E[\Phi])$$

$$\rho_+(\varphi_+(x),\varphi_+(x')) = \langle \varphi_+(x) | \hat{\rho}_+ | \varphi_+(x') \rangle$$

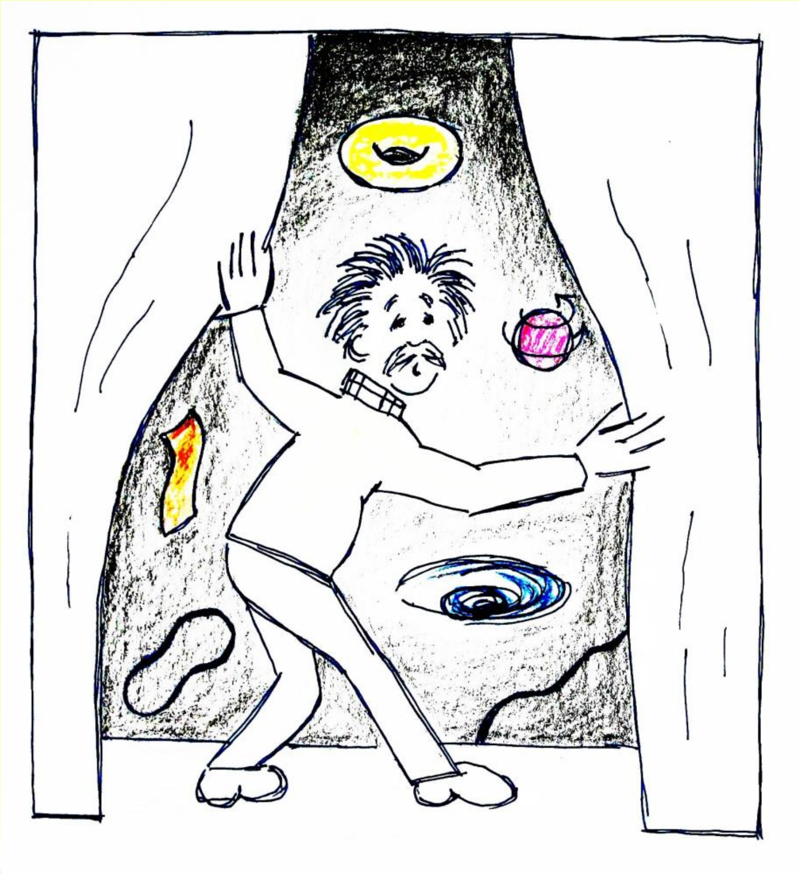
$$\hat{\rho}_+ = Z_+^{-1} \exp(-\beta \hat{H}_+),$$

$\hat{H}_+$ is a Hamiltonian of the field Φ in R_+,

$\beta = \kappa / 2\pi$ is the inverse Hawking temperature

(period of the Hawking-Gibbons instanton)

Problems and Puzzles of Black Holes



Black Hole Entropy Problem

$$dM = TdS \quad \rightarrow \quad S = \frac{A}{4\ell_{Pl}^2}$$

Entropy scales

Stellar mass black hole $(10M_{\odot}) \Rightarrow S \sim 10^{79}$

Milky Way black hole $(M \sim 4\times10^{6} M_{\odot}) \Rightarrow S \sim 1.7\times10^{90}$

Supermassive black hole $(M \sim 10^{9} M_{\odot}) \Rightarrow S \sim 10^{95}$

All stars in the observable Universe $\Rightarrow S \sim 10^{79}$

Relic gravitons $\Rightarrow S \sim 10^{86}$

CMB photons $\Rightarrow S \sim 10^{88}$

Relic neutrinos $\Rightarrow S \sim 10^{88}$

A single Black Hole in the Milky Way has larger entropy than all visible matter in the observable Universe

Microscopic Origin of Black Hole Entropy

What are microscopical degrees of freedom

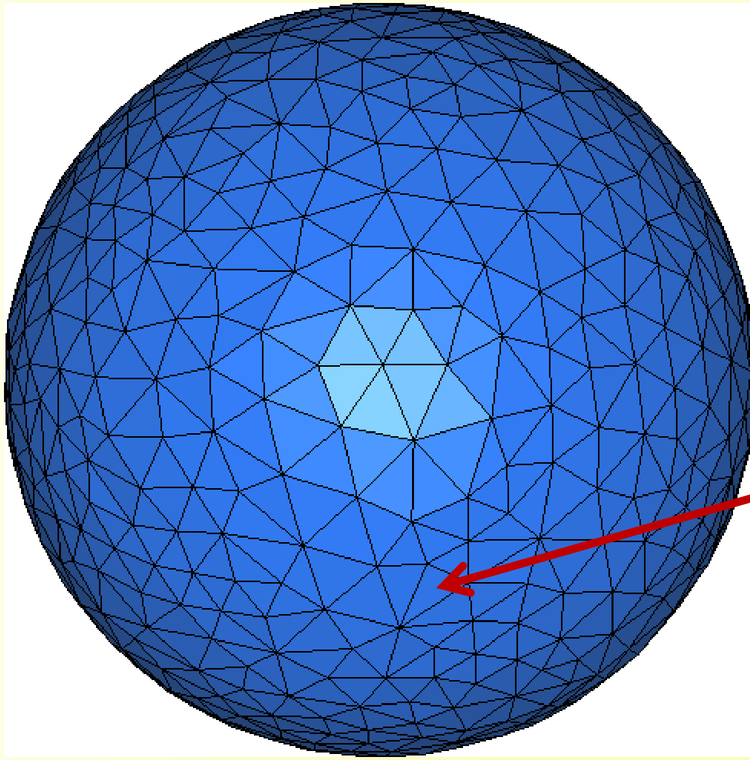
responcible for BH entropy $S = \dfrac{A}{4\, l_{Pl}^2}$?

(1) Each $(2l_{Pl})^2$ element of BH surface has 1 bit of entropy $\Rightarrow$ Quantum gravity is required

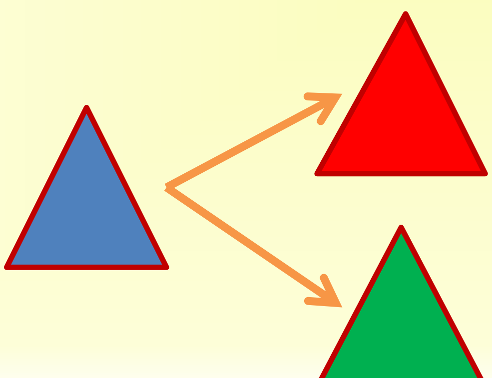
(2) Universality problem: BH is determined by low energy physics parameters

(3) BH is a ground (vacuum) state of classical gravitational field $\Rightarrow$ gravity should be an emergent phenomenon

$$S_{BH} = \frac{A}{a^2}, \quad a = 2l_{Pl}$$



a



There exists $\mathbb{Z}=2^N$ 2-colored triangulations of the sphere for N-mono colored one. Entropy S is:

$$S = \ln \mathbb{Z} = N \ln 2$$

Entanglement Entropy

If the quantum state of a pair of particles is in a definite superposition, and that superposition cannot be factored out into the product of two states (one for each particle), then that pair is entangled.

“Quantum source of entropy for black holes”, Bombelli, Koul, Lee, and Sorkin [Phys.Rev. D34, 373 (1986)]

Entropy of a vacuum domain with a sharp boundary $S \sim \dfrac{A}{l_{cut-off}^2}$, A is the surface area of the boundary.

“Entropy and Area”, by M. Srednicki, Phys. Rev. Lett. 71, 666 (1993).

“Dynamical origin of the entropy of a black hole” by Frolov and Novikov, Phys.Rev. D48, 4545 (1993)

Quantum fluctuations of the horizon as the origin of the ultraviolet cut-off.

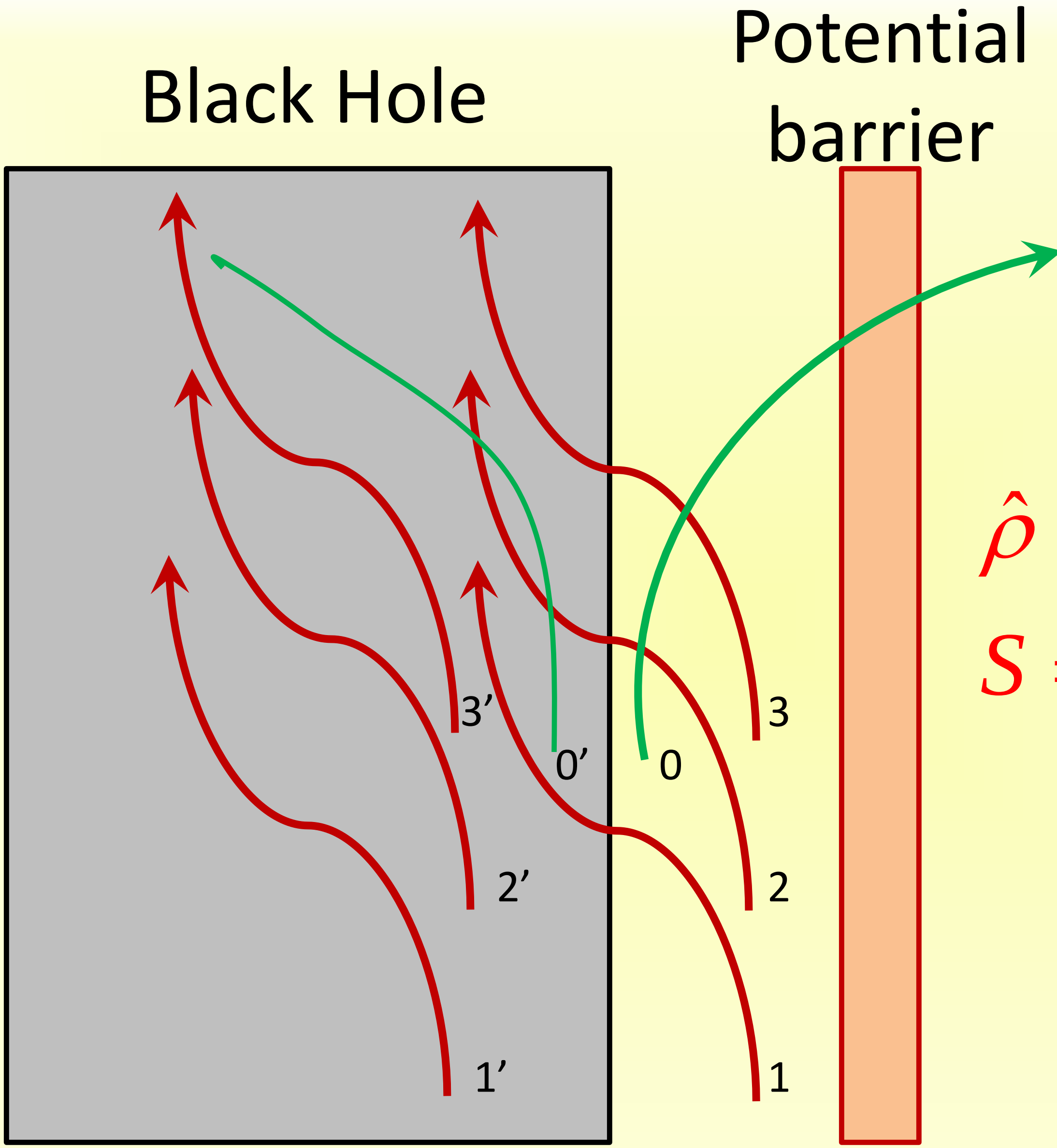
$$S_+ = -Tr(\hat{\rho}_+ \ln \hat{\rho}_+)$$

$S_+ \Leftrightarrow$ (volume contribution)+(surface contribution)

(volume contribution) $\Leftrightarrow$ entropy of thermal radiation outside the black hole

(surface contribution) $\Leftrightarrow$ entanglement entropy of Rindler particles near black hole horizon

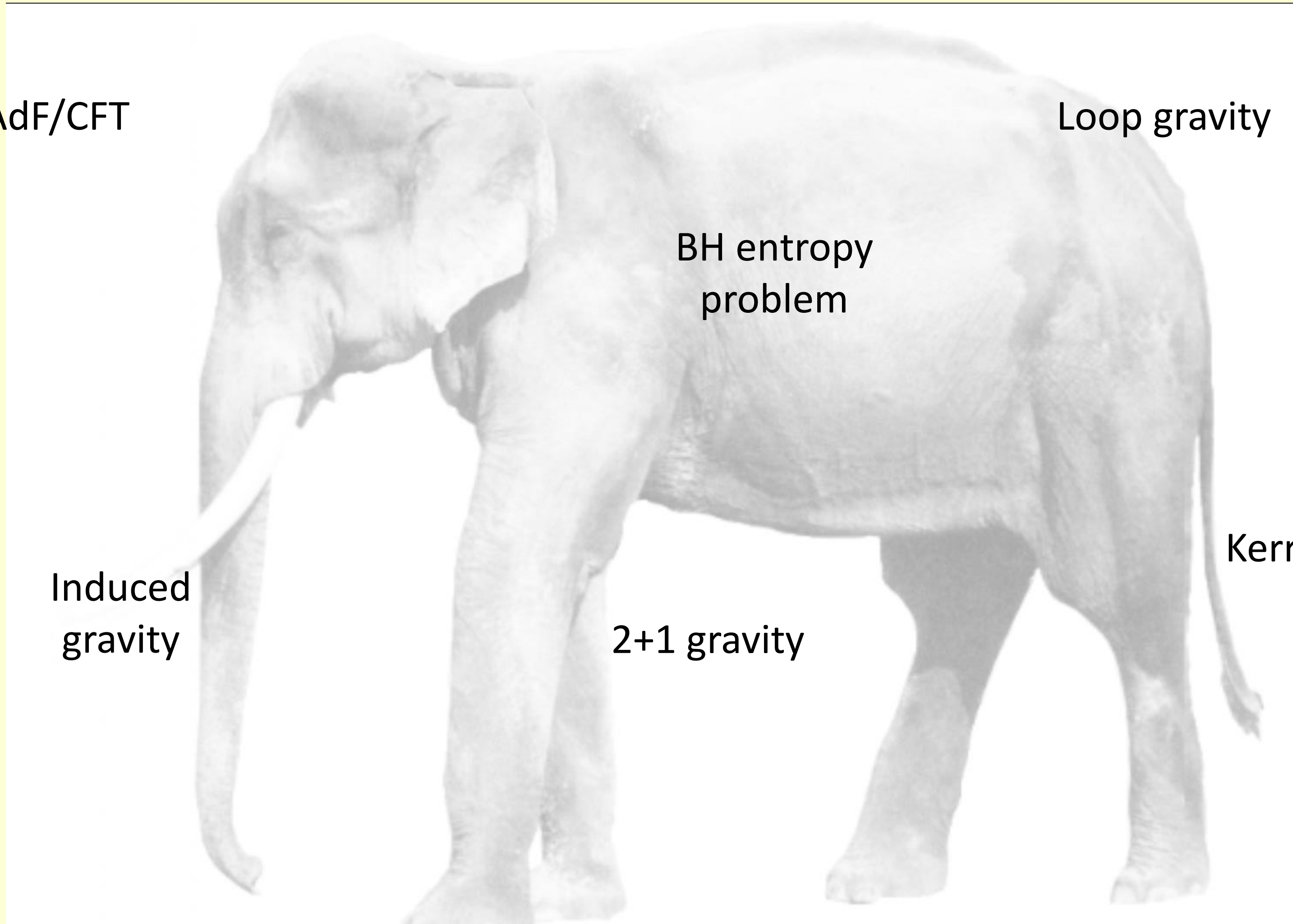
The latter is the black hole entropy: $S \sim \dfrac{A}{\ell_{Pl}^2}$



$$\hat{\rho} = Tr^{(')}\hat{\rho}_{vacuum};$$

$$S = -Tr(\hat{\rho}\ln\hat{\rho})$$

String theory



Information Loss Puzzle

$$\begin{pmatrix}\text{Pure}\\ \text{state}\end{pmatrix} \overset{\text{Collapse}}{\Rightarrow} \begin{pmatrix}\text{Black}\\ \text{hole}\end{pmatrix} \overset{\text{Evaporation}}{\Rightarrow} \begin{pmatrix}\text{Thermal}\\ \text{radiation}\end{pmatrix}$$

If an initial pure state evolves into a mixed state, described by the density matrix, the unitarity of quantum evolution is lost. The unitarity is a basic principle of the quantum mechanics. Does it mean that

Gravity + Quantum Mechanics= Inconsistent theory ?

Main ways out:

1. Quantum mechanics is still valid on a causal evolution of the initial Cauchy surface, but after evaporation an external observer has access only to a part of the system;
2. Information gradually leaks out during the evaporation process;
3. Information is collected in a small mass remnant;
4. Entanglement is immediately broken between the in-falling particle and the outgoing particle. A falling observer will see a `firewall’, when he/she crosses the horizon.

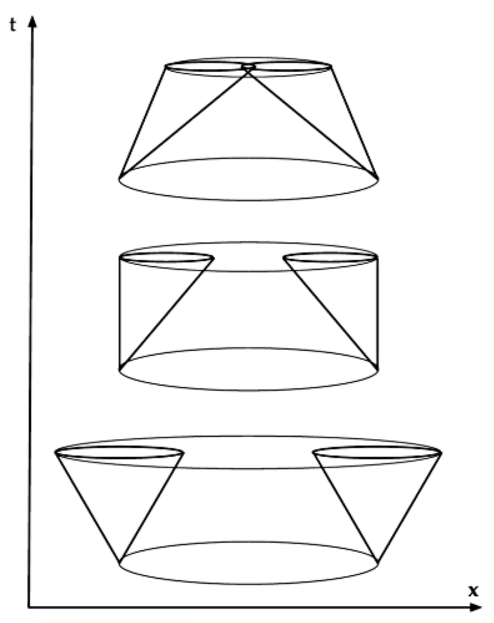
5. No event horizon but only apparent horizon. Information returns `back' after apparent horizon disappears.

V.F. and G.Vilkovisky "Spherically symmetric collapse in quantum gravity" , Trieste preprint (1979), Phys.Lett. B106, 307 (1981)

S.Hawking, January 22, 2014

Event horizon vs.apparent horizon

`Quasi-local definition' of BH: Apparent horizon

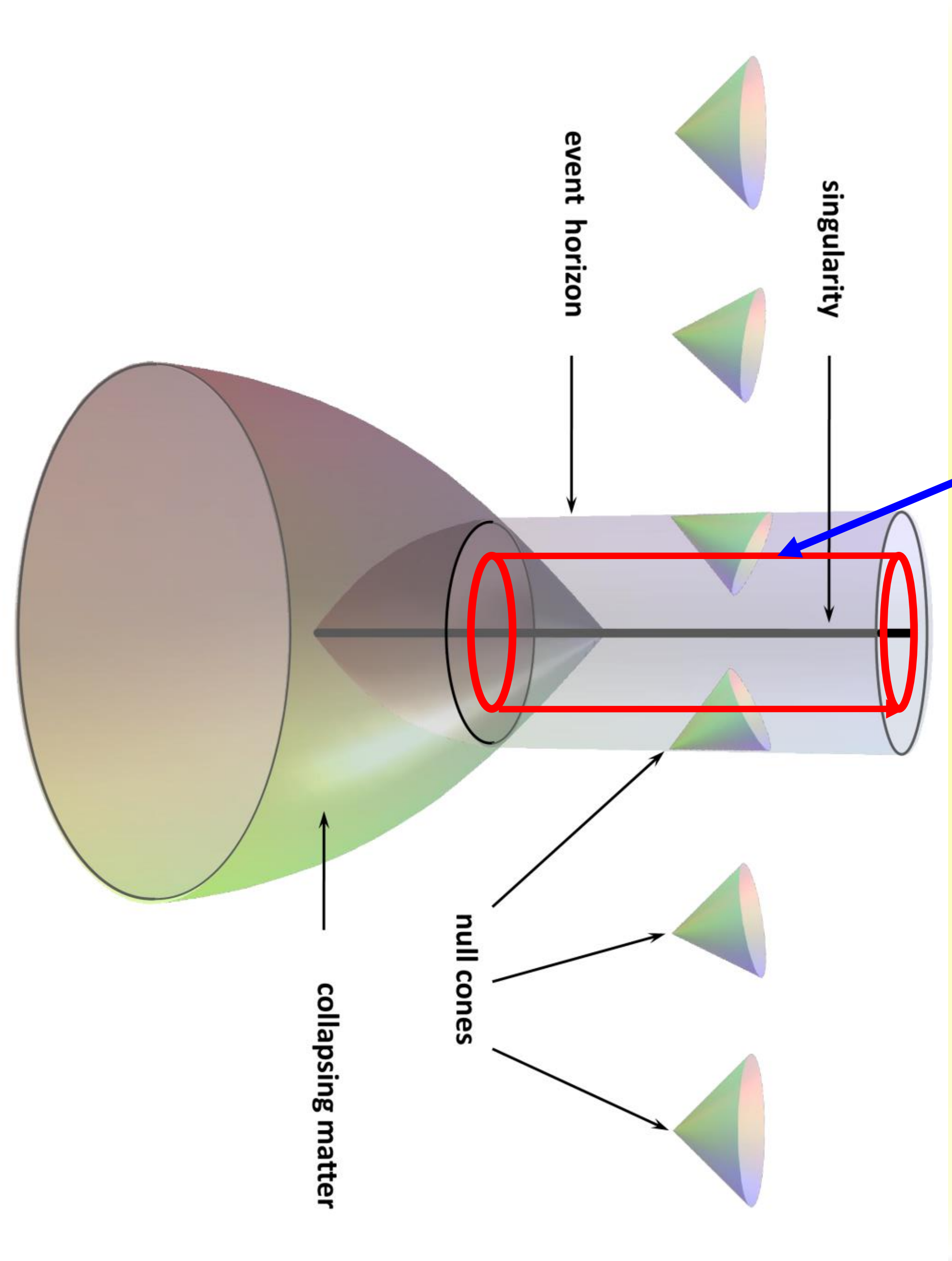


A compact smooth surface *B* is called a trapped surface if both, in- and out-going null surfaces, orthogonal to *B*, are non-expanding .

A trapped region is a region inside *B*.

A boundary of all trapped regions is called an apparent horizon.

Black hole interior



r plays the role of time inside BH

Slice $r = \text{const}$ has topology $S^2 \times R$

Spatial volume $\sim r_g^2 \times (t/c) >>> r_g^3$

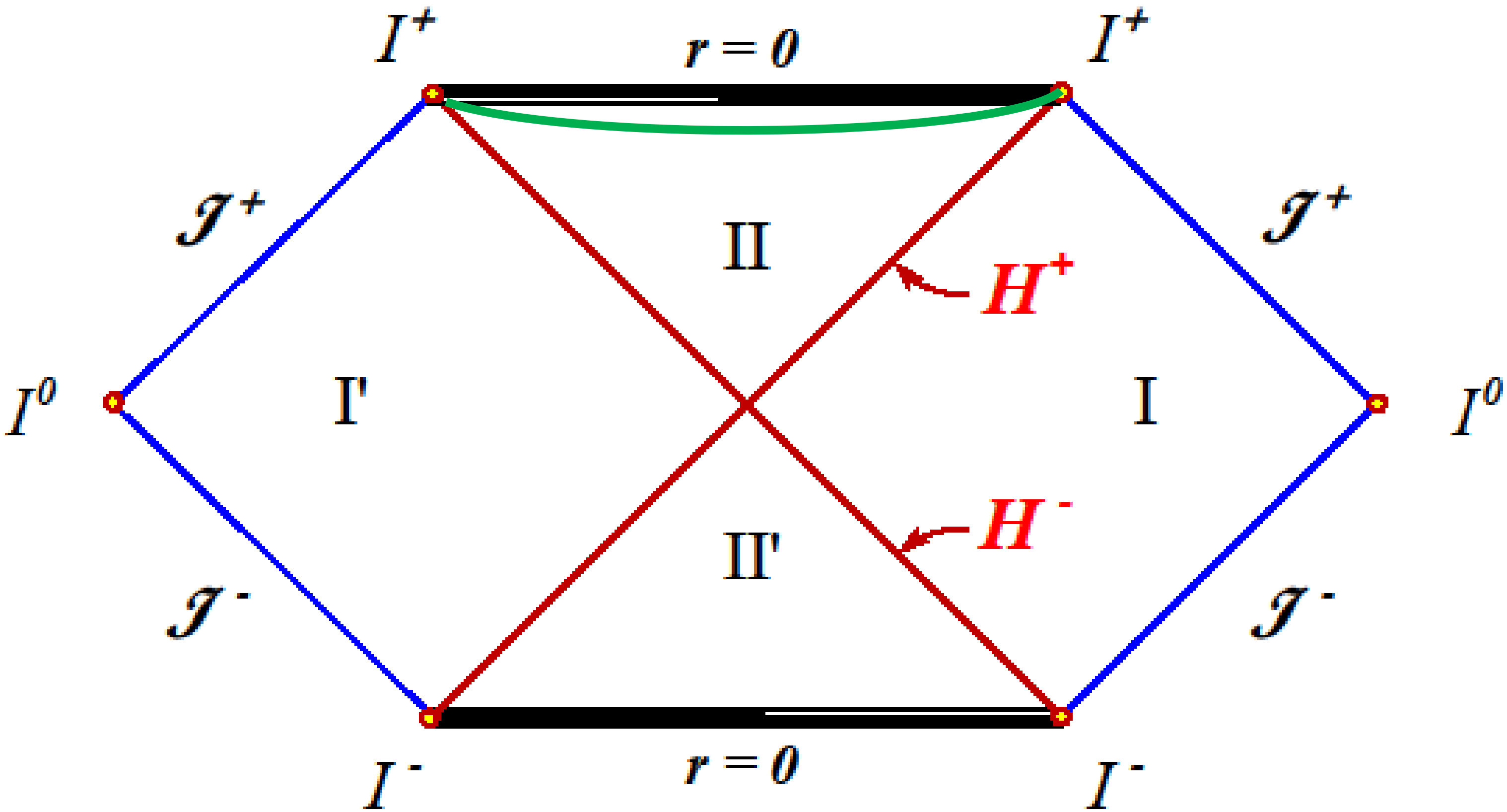
Contracting anisotropic universe

$$\mathbf{R}^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48M^2}{r^6},$$

$$\mathbf{R}^2 = \ell^{-4} \Rightarrow r_* = \ell\left(4\sqrt{3}\,\frac{M}{\ell}\right)^{1/3}$$

$$M \sim 4\times10^6 M_\odot, \quad \ell \sim 10^{-33}\,\mathrm{cm},$$

$$r_* \sim 10^{-7}\,\mathrm{cm} \gg \ell$$



I^+
$r = 0$
I^+
$\mathcal{J}^+$
II
$\mathcal{J}^+$
H^+
I^0
I'
I
I^0
H^-
$\mathcal{J}^-$
II'
$\mathcal{J}^-$
I^-
$r = 0$
I^-

Phenomenological description

(i) There exists the critical energy scale parameter μ. The corresponding fundamental length is $\ell = \dfrac{\hbar}{\mu c}$;

(ii) In the domain where $\Re \ll \ell^{-2}$ the metric obeys the Einstein equations with small corrections;

(iii) In the domain where $\Re \sim \ell^{-2}$ the Einstein equations should be modified;

(iv) Limiting curvature condition: $|\Re| \leq \dfrac{C}{\ell^2}$. C is a universal constant, defined by the theory and independent of the parameters of the solution.

[Markov, JETP Lett. 36, 265 (1982)]

Remark on inflation theory.

(i) An apparent horizon in a regular metric cannot cross $r = 0$.

(ii) It has two branches: outer- and inner-horizons.

(iii) Non-singular BH model with a closed apparent horizon

[V.F. and G.Vilkovisky, Phys. Lett., 106B, 307 (1981)]

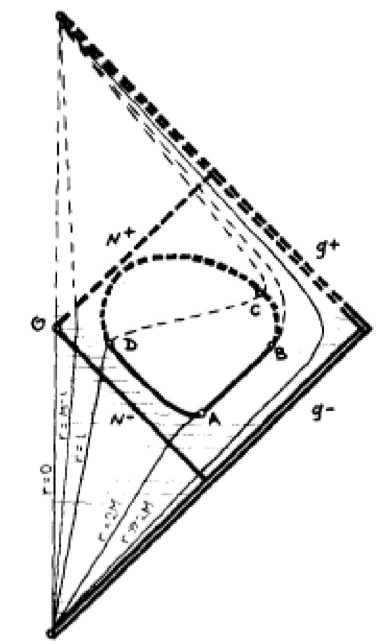


Fig. 1. Penrose diagram for the collapse of the null shell ($M \gg 1$). Solid (dashed) lines are used for the known (hypothetical) details of the picture. The shaded region is the region of validity of the obtained asymptotic solution. The line $N^- \cup N^+$ is the world line of the null shell. The closed and dashed bold line $ABCD$ is the apparent horizon. The light lines are the level lines r = const.

Non-singular model of black hole

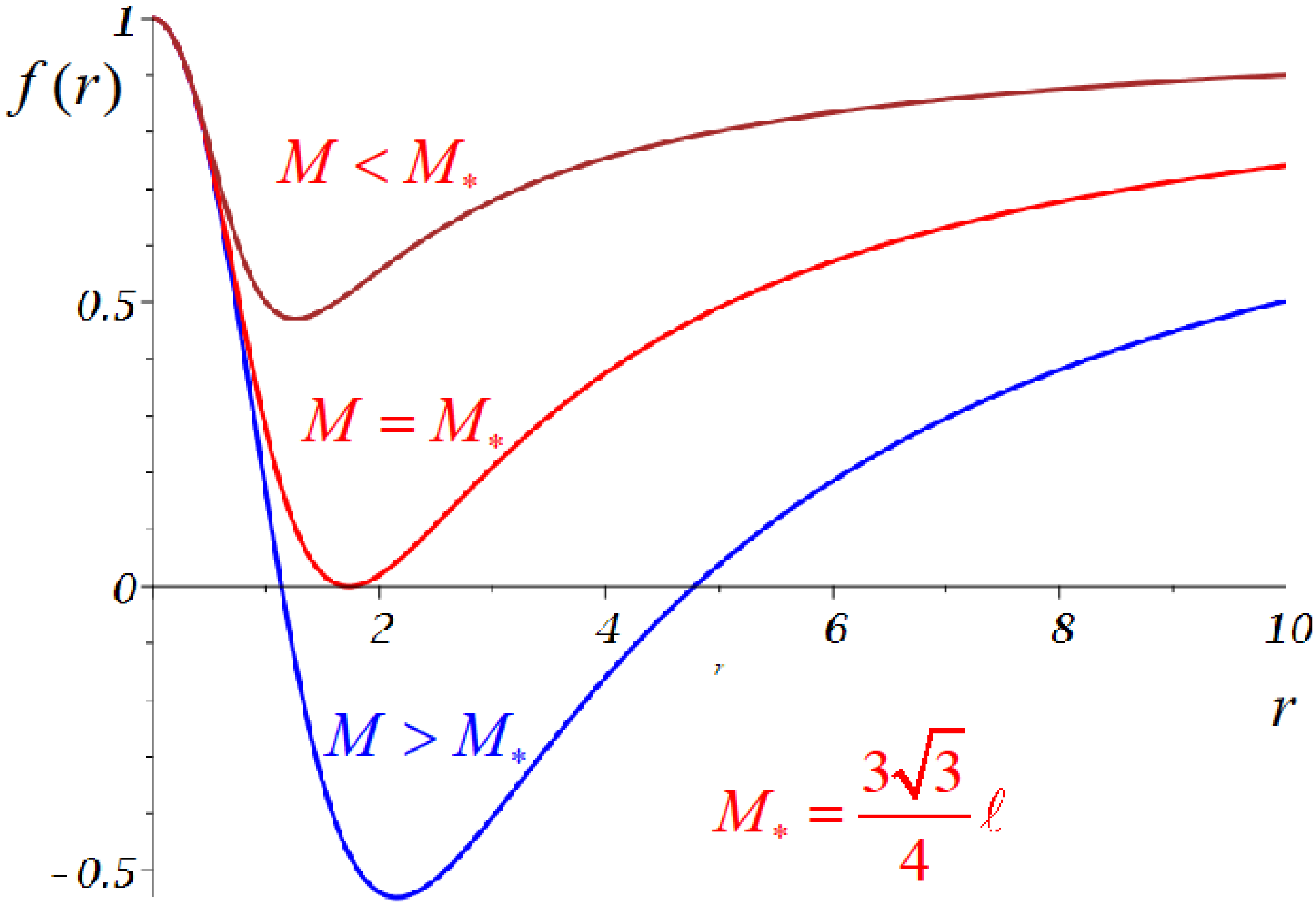
$$dS^2 = -\alpha^2 f dv^2 + 2\alpha dv dr + r^2 d\omega^2,$$

$$f = 1 - \frac{2Mr^2}{r^3 + 2M\ell^2},$$

Standard: α=1;

Modified: $\alpha = \dfrac{r^n + \ell^n}{r^n + \ell^n + (2M)^k \ell^{n-k}}.$

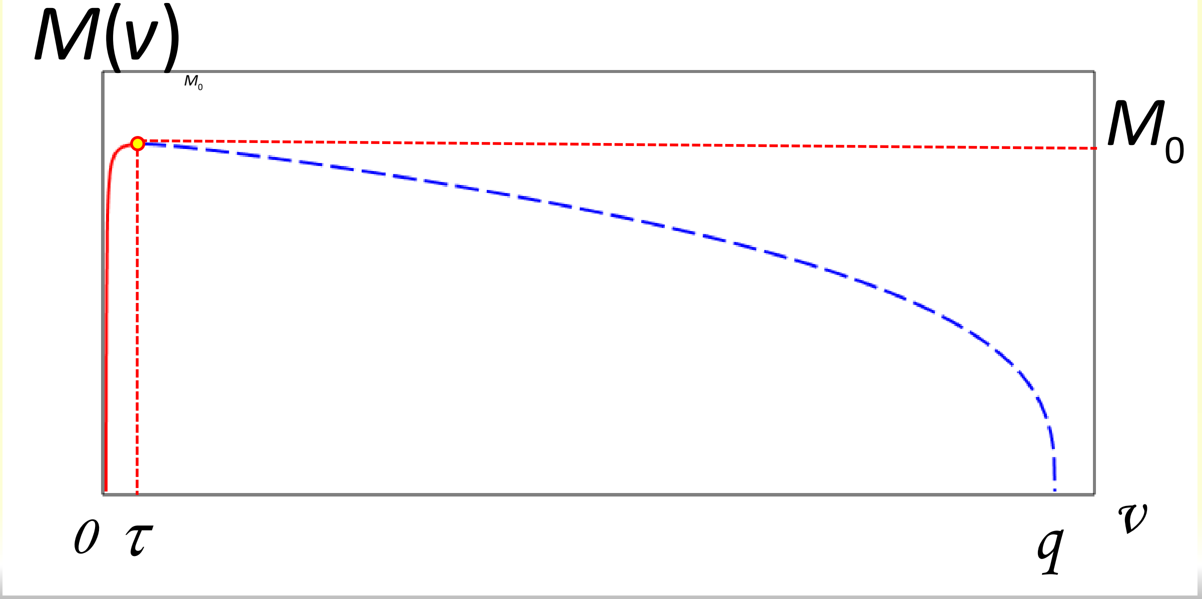
$$m_* = (27/4)^{1/6}\,\ell \Leftrightarrow M_* = \frac{3\sqrt{3}}{4}\,\ell$$

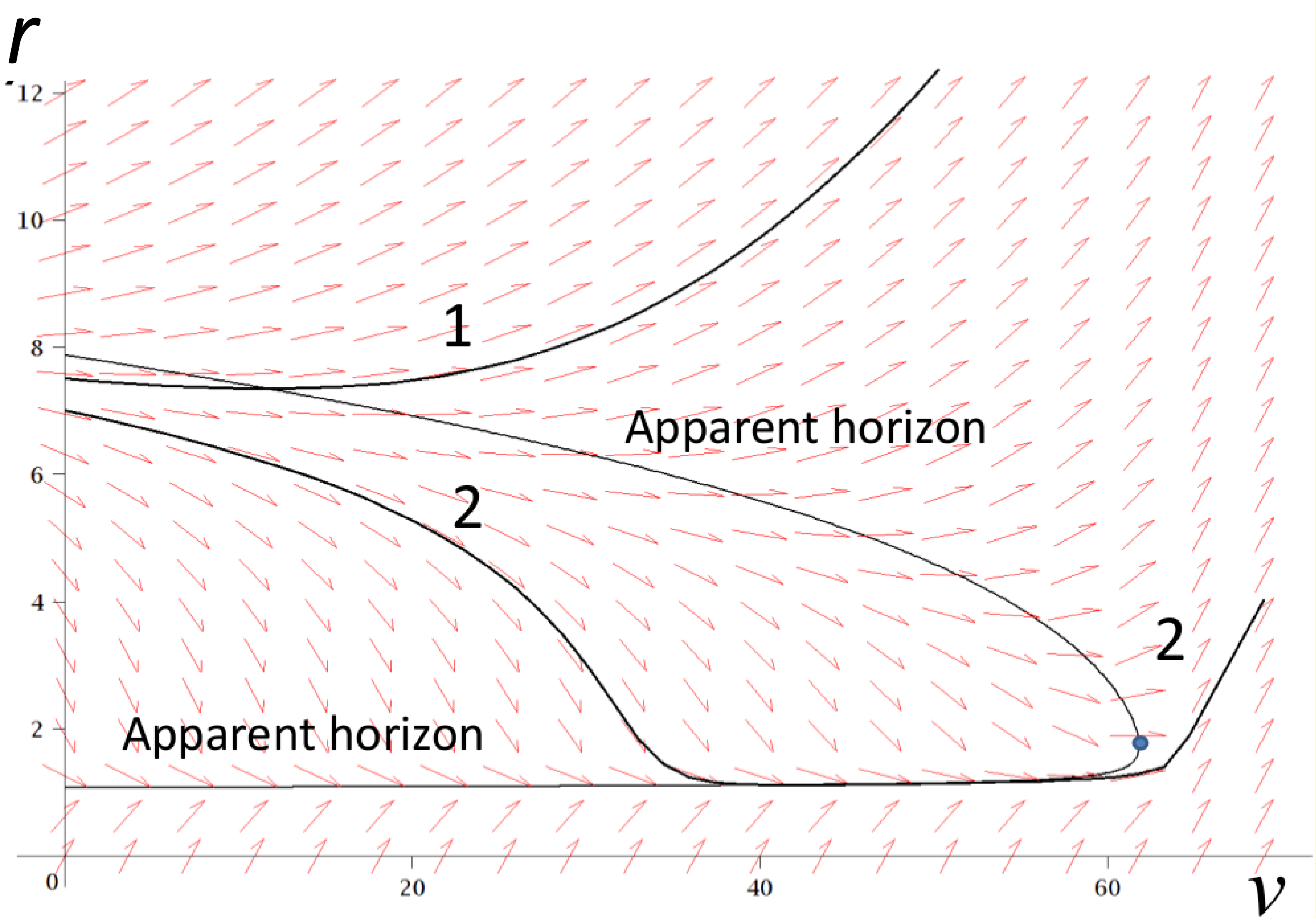


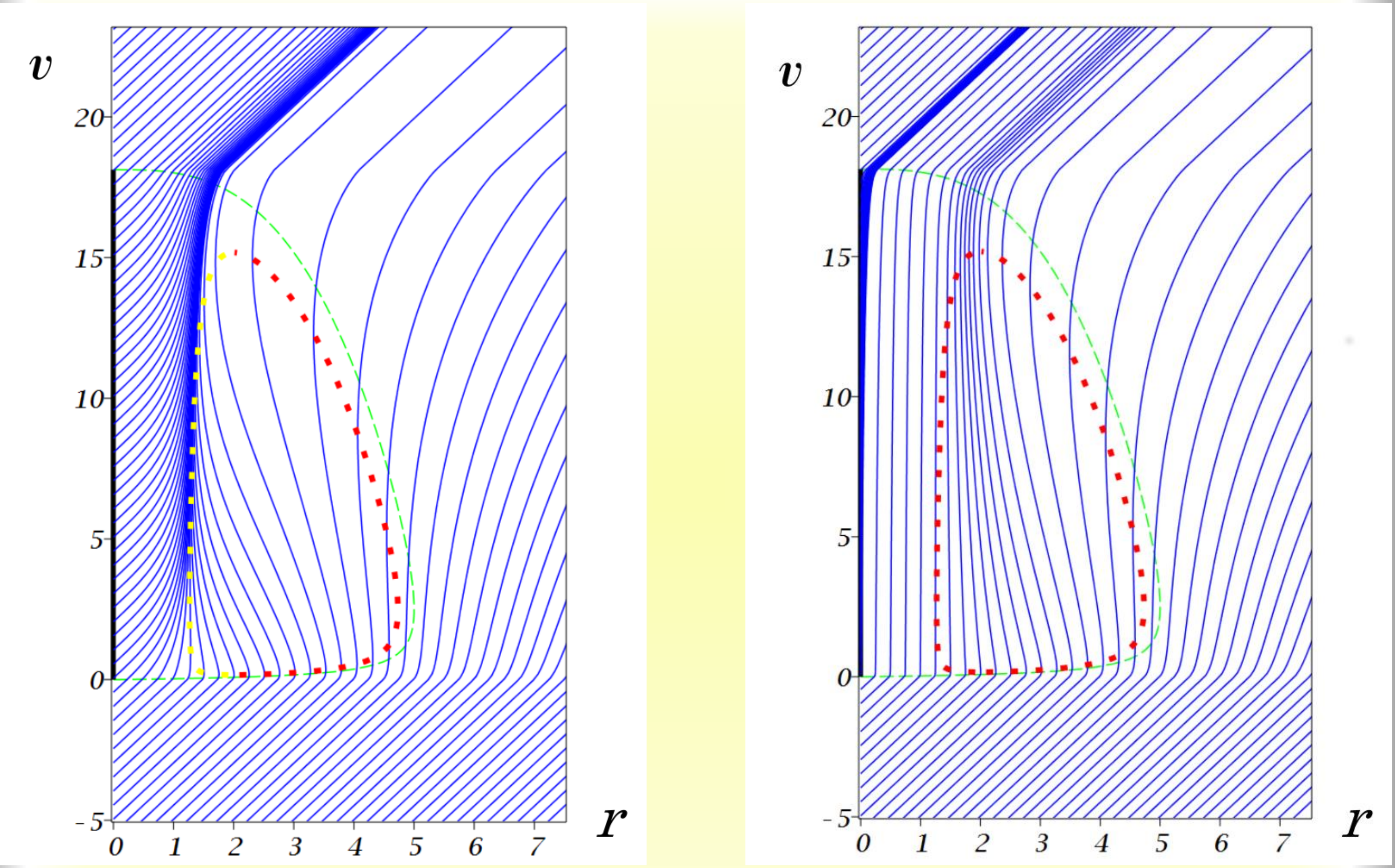
Non-singular model of an evaporating black hole

$$dS^2 = -f dv^2 + 2dvdr + r^2 d\omega^2 ,$$

$$f = 1 - \frac{2M(v)r^2}{r^3 + 2M(v)\ell^2} \quad (\alpha = 1)$$




r
12
10
8
6
4
2
0
20
40
60
v
1
2
2
Apparent horizon
Apparent horizon



$\mu_0 = 5, \quad \tau = 2.5, \quad C = 1, \quad \alpha = 1$ (left) and $\alpha = \dfrac{1+r^5}{1+r^5+\mu_0^3}$ (right)

Instead of summary

**Evolution of the black hole concept: formal solutions of Einstein equations->
physical objects with amazing properties ->
important elements of the Universe;**

The most powerful sources of energy;

`Rosetta stones' providing relations between different areas of physics;

Universal probes of new physics;

An `arena’ for interesting applications of modern mathematics;

Source of fundamental puzzles and problems;

Philosophical problems;

Role in the `very big picture’ of our Universe.