Scalar Waves on a Naked Singularity Background

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Outline

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Einstein-Maxwell-Dilaton theory-I

1. In the low-energy limit of superstring theory in four dimensions, one has some coupling among the graviton, scalar fields (dilaton, axion) and gauge fields [Dine, Seiberg, PRL '85] . In the case of zero axion field, the action reads

$$S[g_{\mu\nu},\phi,A_\mu]=\int_{\mathcal{M}} d^4x\sqrt{g}(-R+2\partial_\mu\phi\partial^\mu\phi-W(\phi)F_{\mu\nu}F^{\mu\nu}). \tag{1}$$

2. The function $W(\phi)$ accounts for all kinds of couplings to the gauge field.

3. Field equations are [Gibbons-Maeda,NPB,'88]:

$$R_{\mu\nu}=2\partial_\mu\phi\partial_\nu\phi+W(\phi)[F_{\mu\beta}F_\nu^\beta-\frac{1}{2}g_{\mu\nu}F^2], \tag{2}$$

$$\nabla_\nu\Big(W(\phi)F^{\mu\nu}\Big)=0, \tag{3}$$

$$\Box\phi=\frac{1}{4}\frac{\partial W}{\partial\phi}F^2, \tag{4}$$

$$\nabla_{[\mu}F_{\rho\sigma]}=0. \tag{5}$$

Einstein-Maxwell-Dilaton theory-II

1. For $W(\phi) = e^{-2\alpha\phi}$ one recovers the bosonic sector of $SU(4)$ symmetric $N = 4$ supergravity theory in four dimensions [Cremmer-Scherk-Ferrara, PLB '78].
2. Black hole solutions in EMD theory were found by Gibbons-Maeda, [Gibbons-Maeda, NPB,'88]. Garfinkle-Horowitz-Strominger also reported charged black hole solutions [Garfinkle-Horowitz-Strominger, PRD, '91].
3. Charged rotating solutions appear in the context of Kaluza-Klein theory [Horne-Horowtiz, PRD '92].
4. GM solution can be written as [Gibbons-Maeda, NPB,'88]

$$ds^2 = f(r)dt^2 - f^{-1}dr^2 - R^2(r)d\Omega_2, \tag{6}$$

$$f = \frac{(r - r_+)(r - r_-)^{1+\gamma}}{r^{1+\gamma}}, \quad R^2 = r^2\left(1 - \frac{r_-}{r}\right)^{1-\gamma}, \tag{7}$$

$$e^{2\phi} = e^{2\phi_\infty}\left(1 - \frac{r_-}{r}\right)^{1-\gamma}, \tag{8}$$

$$F = \frac{Q}{r^2}e^{\phi_\infty}dr \wedge dt, \quad -1 \leq \gamma = \frac{1-\alpha^2}{1+\alpha^2} \leq 1 \tag{9}$$

Einstein-Maxwell-Dilaton theory-III

1. r_+ and r_- are given in terms of the Arnowitt-Deser-Misner mass M and electric charge Q:

$$M = \frac{(r_+ + r_-)}{2}, \quad Q^2 = e^{-2\alpha\phi_\infty}\frac{r_+ r_-}{2}(\gamma+1).$$

2. Notice that BHs in EMD theory can acquire scalar charge (hair), called dilaton charge [Garfinkle-Horowitz-Strominger, PRD '91] :

$$Q_D(\partial\mathcal{M}) = \frac{1}{4\pi}\int_{\partial\mathcal{M}:S_\infty} d^2\Sigma n^\mu \nabla_\mu \phi$$

.

3. Physically, one can think of Q_D as the monopole charge sourcing the scalar field.
4. Depending on the values taken by γ one may have different kinds of solutions.
   - $\gamma = 0 (\alpha = 1)$: BH in supergravity in D=4. $\gamma = 1(\alpha = 0)$: RN BH.
   - $\gamma = \frac{1}{2}(\alpha = \sqrt{3})$: BH in D=4 after dimensional reduction from KK in D=5.
   - $\gamma = -1(\alpha = \infty)$: Scalar field minimally coupled to gravity $[A_\mu = 0]$.

Einstein-Maxwell-Dilaton theory-IV

1. Clement-Galtsov- Leygnac devised a deformation method to obtain a new solution from an old one [Clement- Galtsov- Leygnac, PRD, 2005], [Giddings-Strominger,PRD'92] :
   - Choose $r = r_- + \epsilon\bar{r}$ and $r_+ = r_- + \epsilon b$.
   - Re-scale variables as $t = \epsilon^{-1}\bar{t}$, $r_- = \epsilon^{-\alpha^2} r_0$, and $\phi_\infty = -\alpha \ln \epsilon$.
   - Take the limit $\epsilon \to 0$ and set $\gamma = 0$ ($\alpha^2 = 1$).
2. One arrives at a new solution that is not asymptotically flat neither AdS/dS:

$$ds^2 = \left(\frac{r-b}{r_0}\right) dt^2 - \left(\frac{r_0}{r-b}\right) dr^2 - r r_0 d\Omega_2, \tag{10}$$

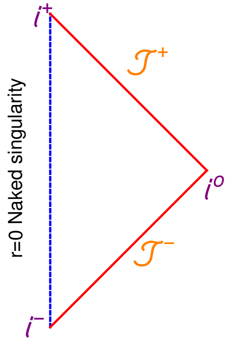
$$e^{2\phi} = \frac{r}{r_0}, \tag{11}$$

$$F = \frac{1}{\sqrt{2} r_0} dr \wedge dt. \tag{12}$$

3. Notice that for BHs, the Hawking's temperature is $T_H = 1/(4\pi r_0)$ and Bekenstein's entropy reads $S_B = \pi r_0 b$.

Clement- Galtsov- Leygnac solution

1. The mass parameter is $M = b/4$ whereas the charge is $Q = r_0/\sqrt{2}$.
2. If $b \geq 0$ then the solutions correspond to BHs. Here, we focus on the $b < 0$ case [Fig. 1] where the spacetime is endowed with a naked time-like singularity located at $r = 0$.



Some well-known results

1. One can proof that the analysis stability of a massless scalar field is equivalent to the metric perturbation for $\ell = 1$ [Clement- Galtsov-Leygnac ,PRD, 2005].

2. What are the theoretical results about the stability/instability of a NS background?

   - For the Schwarzschild background with negative mass parameter (NS), and under metric perturbations, Gibbons-Hartnoll-Ishibashi found that the NS is **stable!!!!**[Gibbons-Hartnoll-Ishibashi, PTP'2005].

   - The perturbation of the Schwarzschild background with negative mass parameter was re-examined by Gleiser-Dotti and proved GHI wrong, perturbations are linearly unstable [Gleiser-Dotti, CQG'2006].

   - Clement- Galtsov-Leygnac explored the linear estability of the naked singularity background in EMD theory [Clement- Galtsov-Leygnac ,PRD, 2005]. They stated that the solution is **stable!!!!**

3. We revisited the analysis of Clement-Galtsov-Leygnac [Clement- Galtsov-Leygnac ,PRD, 2005].

General Setup

1. Consider the problem of a massless scalar field propagating in a NS geometry [Ishibashi-Hosoya, PRD '99].

   - Klein-Gordon Eq. on $\mathcal{M}$, $\Box\phi = 0$, for $0 < r < +\infty$, $0 \leq t < +\infty$, $0 \leq \theta \leq 2\pi$, and $0 \leq \varphi < \pi$. Plus some boundary conditions on $\partial\mathcal{M} : \{r = 0\} \cup \{r = \infty\}$.
   - One can pose the problem as $\partial_{tt}\phi = -A\phi$ with $A = -\partial_x^2 + V(x)$, being x the tortoise coordinate, $0 < x < \infty$. Here ∂_t is a timelike (global) Killing vector in $\mathcal{M}$.
   - $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H} = L^2$ in an operator which acts on a Hilbert space $\mathcal{H}$ endowed with an inner product $\langle , \rangle$.
   - One looks for self-adjoint extension of A. The self-adjoint property is essential to guarantee a unitary evolution [Simon and Reed].
   - A symmetric operator that is self-adjoint satisfies $\mathcal{D}(A) = \mathcal{D}(A^\dagger)$.
   - Norm can be calculated using the natural (covariant) KG inner product, $||\phi||^2_{KG} \propto \int_{\Sigma_t} j^\mu d\Sigma_\mu$ [Wald's book].

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Ishibashi-Wald Prescription

1. Suppose that the spacetime is not globally hyperbolic [Ex. Ads, NS]. Is it possible to study the evolution of a scalar field propagating in those backgrounds in the sense of I.V.P? [Ishibashi-Wald, CQG 2003, 2004]

2. Consider some initial data with compact support $\mathcal{K}_0 = \mathrm{supp}(\phi_0) \cup \mathrm{supp}(\dot{\phi}_0)$ on a spacelike hypersurface, $(\phi_0, \dot{\phi}_0) \in C_0^\infty \times C_0^\infty$ on $\Sigma_0 \subset \mathcal{M}$.

3. Consider the values of $\phi_t \equiv \phi(x,t)|_{\Sigma_t}$ and $\dot{\phi}_t \equiv \partial_t \phi(x,t)|_{\Sigma_t}$. It is possible to assure a casual evolution of $(\phi_t, \dot{\phi}_t) \in C_0^\infty \times C_0^\infty$ on $\Sigma_t \subset \mathcal{M}$ for sufficiently small elapsed time t. [Ishibashi-Wald, CQG 2003, 2004]

4. The field can be written as $\phi(t,x) = \cos(\sqrt{A_E}t)\phi_0(x) + {A_E}^{-1/2}\sin(\sqrt{A_E}t)\dot{\phi}_0(x)$, where A_E stands for the SAE of A given by the Friedrich theorem [Simon and Reed].

5. Notice that $\cos(\sqrt{A_E}t)$ and ${A_E}^{-1/2}\sin(\sqrt{A_E}t)$ are both bounded operators-spectral theorem holds-[Simon and Reed]. Then, the natural norm on each Σ_t induced by the KG inner product, $||\phi_t||^2_{KG} \propto \int_{\Sigma_t} j^\mu d\Sigma_\mu = \int_{\Sigma_t} j^t d\Sigma_t$, is bounded as well [Ishibashi- Hosoya, PRD'99], [Ishibashi-Wald, CQG 2003, 2004].

Questions and Tools

1. What are the main issues to tackle in dealing with the stability/instability analysis?
    - Determine if the operator A has some self-adjoint extensions along with the domain of such a extension.
    - To study the boundary conditions associated to the scalar fields on Σ? (**Dirichlet**, Robin, Neumann)
    - To show whether the NS background in EMD is linearly stable or unstable? (Are there squared integrable modes of A with negative eingenvalue? [What about the spectrum of A?]
2. Tools?[Simon and Reed].
    - *Von Neumann* deficiency indices method allows us to determine the existence of SAE of A.
    - The boundary problem associated with A can be explored using the Weyl's theorem. Classify the endpoints of the domain as limit point (LP) or limit circle (LC).
    - Some useful theorems about the existence of a negative eigen-value for an operator A based on the specific behavior of the effective potential plus some reg. conditions.

FRDM approach-I

1. Consider the operator $A_n\psi = \left(-\partial_x^2 + V[r(x)]\right)\psi = \omega^2\psi$ with $0 < x < \infty$, where the potential reads

$$V_n(x) = \frac{1}{r_0^2}\left[\ell_{1/2}^2 + \frac{\ell(\ell+1)}{(e^{x/r_0}-1)} - \frac{1}{4(e^{x/r_0}-1)^2}\right], \tag{13}$$

2. Near $x = 0$, with the help of the Frobenius theorem, the solutions are both S.I in dx (LC)

$$\psi(x) = A\cos(\alpha)\sqrt{\frac{x}{r_0}}\left(1 + \ell_{1/2}^2\frac{x}{r_0}\right) + A\sin(\alpha)\sqrt{\frac{x}{r_0}}\left(\ln\left(\frac{x}{r_0}\right) + \ell_{1/2}^2\ln\left(\frac{x}{r_0} - 2\right)\frac{x}{r_0}\right),$$

3. At spatial infinity only $e^{-\kappa x}$ with $\kappa = \sqrt{|\omega^2 - V_\infty|}$ is square integrable in dx (LP).
4. There are many self-adjoint extension of the A_n operator parametrized by a unitary map and the deficiency indexes are necessarily given by $(n_+, n_-) = (1,1)$.

FRDM approach-II

1. Regarding the domain of the operator A_n ones notices that the linearity of the Schrödinger equation allows us to discard the constant A and fix the boundary conditions using a parameter $\alpha \in [0, \pi)$.
2. We define the subset $\mathcal{D}_\alpha \subset L^2((0,\infty), dx)$.
3. The spectrum of the operator $\sigma(A_n)$ and the dynamic of the scalar field essentially will depend on the chosen boundary condition, that is, the value taken by α [Fülop, SIGMA, 2007].
4. The domain of the operator is

$$\begin{aligned}\mathcal{D}_{\alpha=\pi/2} &= \{\psi \in L^2((0,\infty), dx) | A_n\psi \in L^2((0,\infty), dx), \ \cos(\alpha), \\ &W(\psi, \chi_1) + \sin(\alpha)\, W(\psi, \chi_2) = 0\}, \end{aligned} \tag{14}$$

5. We obtained squared integrable modes with negative eingenvalue for $\omega^2 = -k^2 < 0$–Unstable Mode–NS is linearly unstable. Same resutls for $\ell = 1$.

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CGL approach-I

1. CGL proposed to work with another coordinate defined as

$$y := \frac{1}{|b|} \ln \frac{r}{r+|b|}, \quad \text{with} \quad -\infty < y < 0 \tag{15}$$

2. They derived a master equation $A_c \phi_1 = 0$:

$$-\partial_y^2 \phi_1 + V_k[r(y)]\phi_1 = 0, \quad V_k[r(y)] = r\Big(2|b| + r(2 + k^2 r_0^2)\Big) \tag{16}$$

3. Given the fact that $V_k[r(y)] > 0$ then CGL concluded that the perturbations of a NS in EMD theory are stable.

4. It can be shown that $y = -\infty$ is a LC of A_c whereas $y = 0$ is a LP.

5. The operator must have necessarily deficiency indexes given by $(n_+, n_-) = (1, 1)$ and there must be a $\mathcal{U}(1)$ isometry with relates all the self-adjoint extensions.

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CGL approach-II

1. The domain for the operator A_c becomes

$$\mathcal{D}^c_{\alpha=0} = \{\phi_1 \in L^2((-\infty,0),dy) | A_c\phi_1 \in L^2((-\infty,0),\mu(y)dy) \\ \text{and}\cos(\alpha)W(\phi_1,\chi_1)+\sin(\alpha)W(\phi_1,\chi_2)=0\}. \tag{17}$$

2. Despite the fact that we are solving the same physical problem, there are some big differences between the two approaches:
   - CGL use a transformation of coordinates that hides the unstable mode.
   - Not only that, the analysis for BHs and NS in EMD gave the same results. This is, at least, a red flag.
   - We know that the BH Schawrzschild solution is stable under axial perturbations[Regge-Wheeler, PR 1957].
   - We also know that the Schawrzschild metric with negative mass parameter is unstable under axial perturbation [Gleiser-Dotti, CQG'2006].

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Some Interesting Points

1. CGL obtained:
   - At $r = 0$ the perturbation goes to a constant but diverges linearly near i^0 [breakdown of pert. approach].
   - The contrast of the gravitational energy at second order is zero at $r = 0$ and goes to zero near i^0.
2. FRDM obtained:
   - At $r = 0$ the perturbation behaves as the background but decays faster than the background near i^0. [breakdown]
   - The contrast of the gravitational energy diverges logarithmically at $r = 0$ and goes to zero near i^0.
3. In brief:
   - Check instability in other modes (S-T-V) [work in progress]. Check nonlinear evolution as well.
   - NS may not be able to form in nature.[Penrose]

Thank you for listening.