Beyond General Relativity

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Motivation for theories beyond General Relativity

Why do we need to go beyond GR + standard model of particle physics

Cosmology: unable to explain some observations, fine-tuning, tensions

Singularity: Black Hole and Big Bang

Beyond GR = EFT (they can model some effects or phenomenas)

Test of GR

Some History of the Universe

	Time	Energy	
Planck Epoch?	$< 10^{-43}$ s	10^{18} GeV	
String Scale?	$\gtrsim 10^{-43}$ s	$\lesssim 10^{18}$ GeV	
Grand Unification?	$\sim 10^{-36}$ s	10^{15} GeV	
Inflation?	$\gtrsim 10^{-34}$ s	$\lesssim 10^{15}$ GeV	
Baryogenesis?	$< 10^{-10}$ s	> 1 TeV	
Neutrino Decoupling	1 s	1 MeV	
BBN	3 min	0.1 MeV	
			Redshift
Matter-Radiation Equality	10^4 yrs	1 eV	10^4
Recombination	10^5 yrs	0.1 eV	1,100
Dark Ages	$10^5 - 10^8$ yrs		> 25
Reionization	10^8 yrs		$25 - 6$
Galaxy Formation	$\sim 6 \times 10^8$ yrs		~ 10
Dark Energy	$\sim 10^9$ yrs		~ 2
Solar System	8×10^9 yrs		0.5

arXiv:0907.5424

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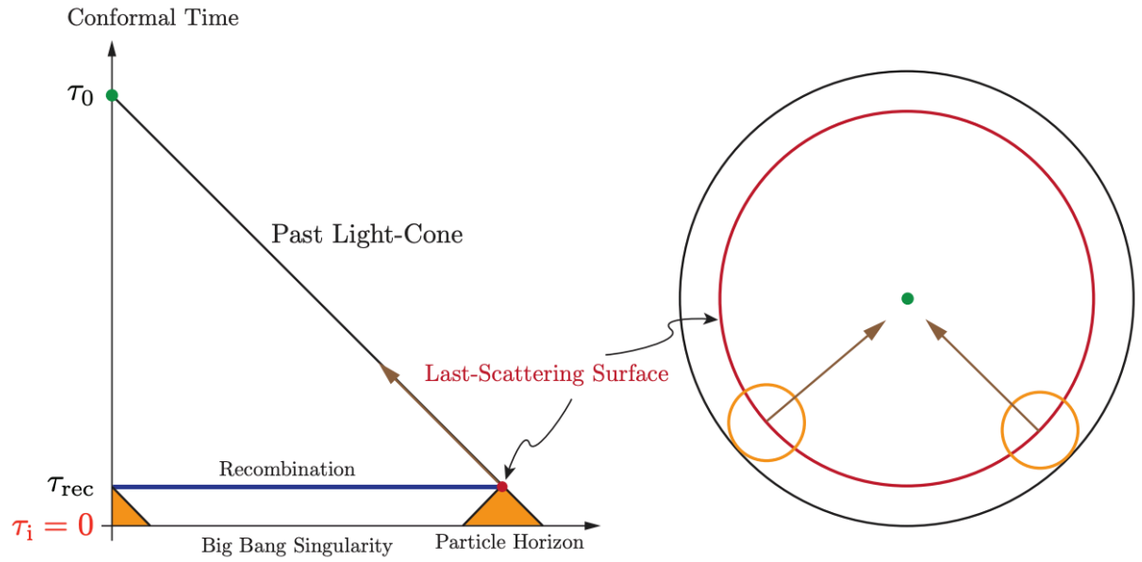
Early Universe (see Yi-Fu Cai lectures)

Horizon problem

Flatness problem



Fine-tuning problem more than inconsistencies of the theory

Inflation gives a nice solution to this fine-tuning arXiv: 0907.5424

But who is this inflaton (a new field?)

Brane model arXiv: hep-th/0403119

Supergravity arXiv: 1101.2488

String Theory arXiv: 1404.2601

Late acceleration of the Universe arXiv: astro-ph/9805201 arXiv: astro-ph/9812133

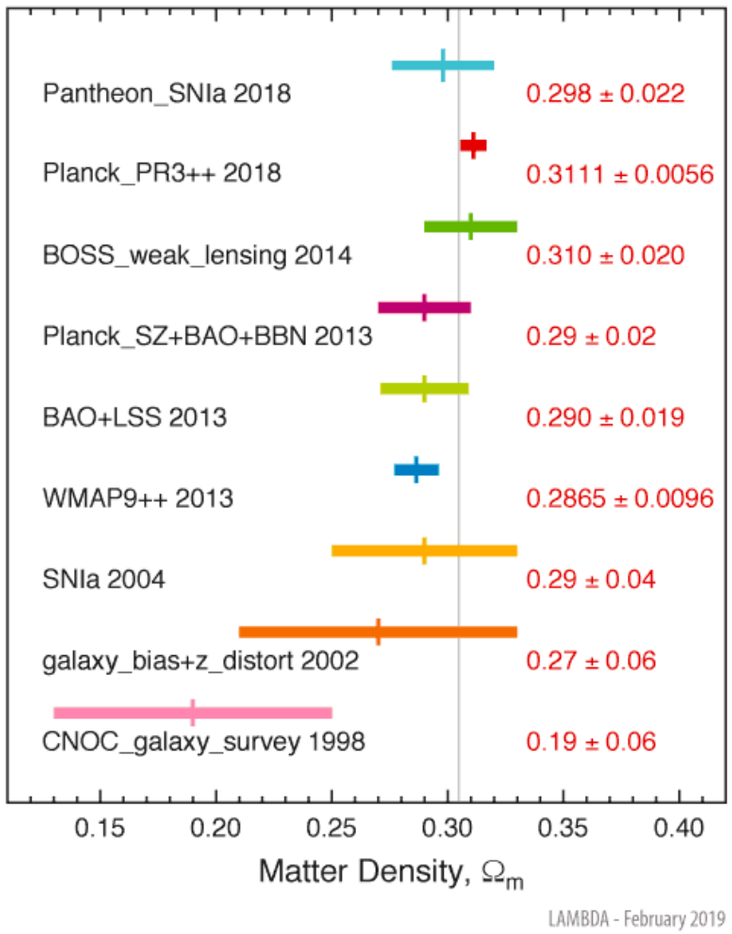
Balloon experiments (2000)

BOOMERanG arXiv: astro-ph/0004404
MAXIMA arXiv: astro-ph/0005124

Followed by WMAP in 2003 arXiv: astro-ph/0302209

And Planck in 2013 arXiv:1303.5076

$$\Omega_{k,0} \simeq 0$$



$$\Omega_{m,0} \simeq 0.3$$

FIRAS arXiv:0911.1955

$$\Rightarrow T_{CMB}$$

$$\Omega_{r,0} \simeq 5\ 10^{-5}$$

$$\boxed{\Omega_{\text{something},0} \simeq 0.7}$$

What is this “something”

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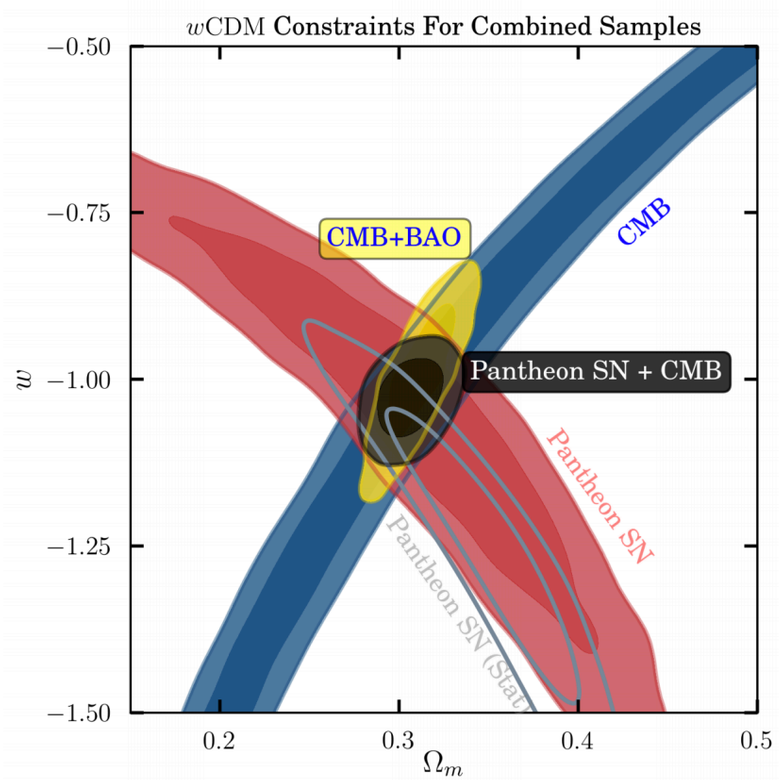
Late acceleration of the Universe



“Something” looks like an unclustered fluid with

$$P \simeq -\rho$$

That is the cosmological constant

$$\Lambda$$

But we have two problems

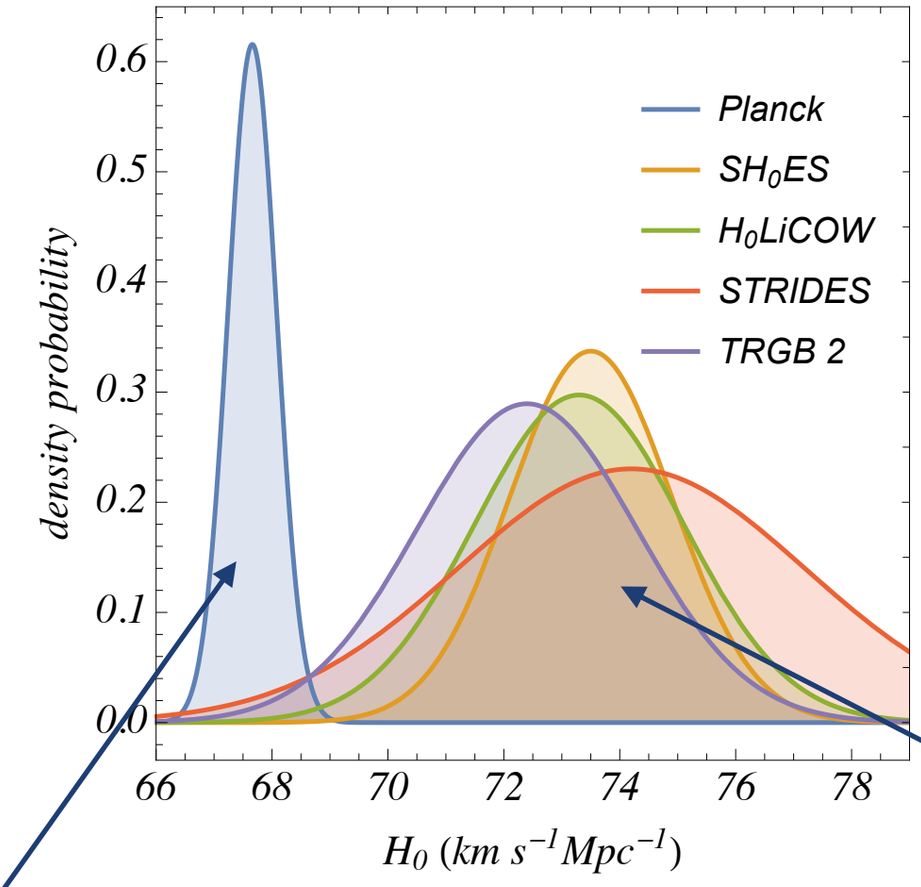
56 orders of magnitude difference between theory and observations arXiv:1205.3365

Coincidence problem $\rho_m \simeq \rho_\Lambda$

‘Most embarrassing observation in physics’ – that’s the only quick thing I can say about dark energy that’s also true
Edward Witten

Fine-tuning problem more than inconsistencies of the theory

Hubble tension



Early universe

Late universe

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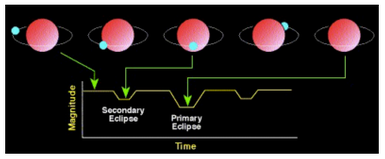
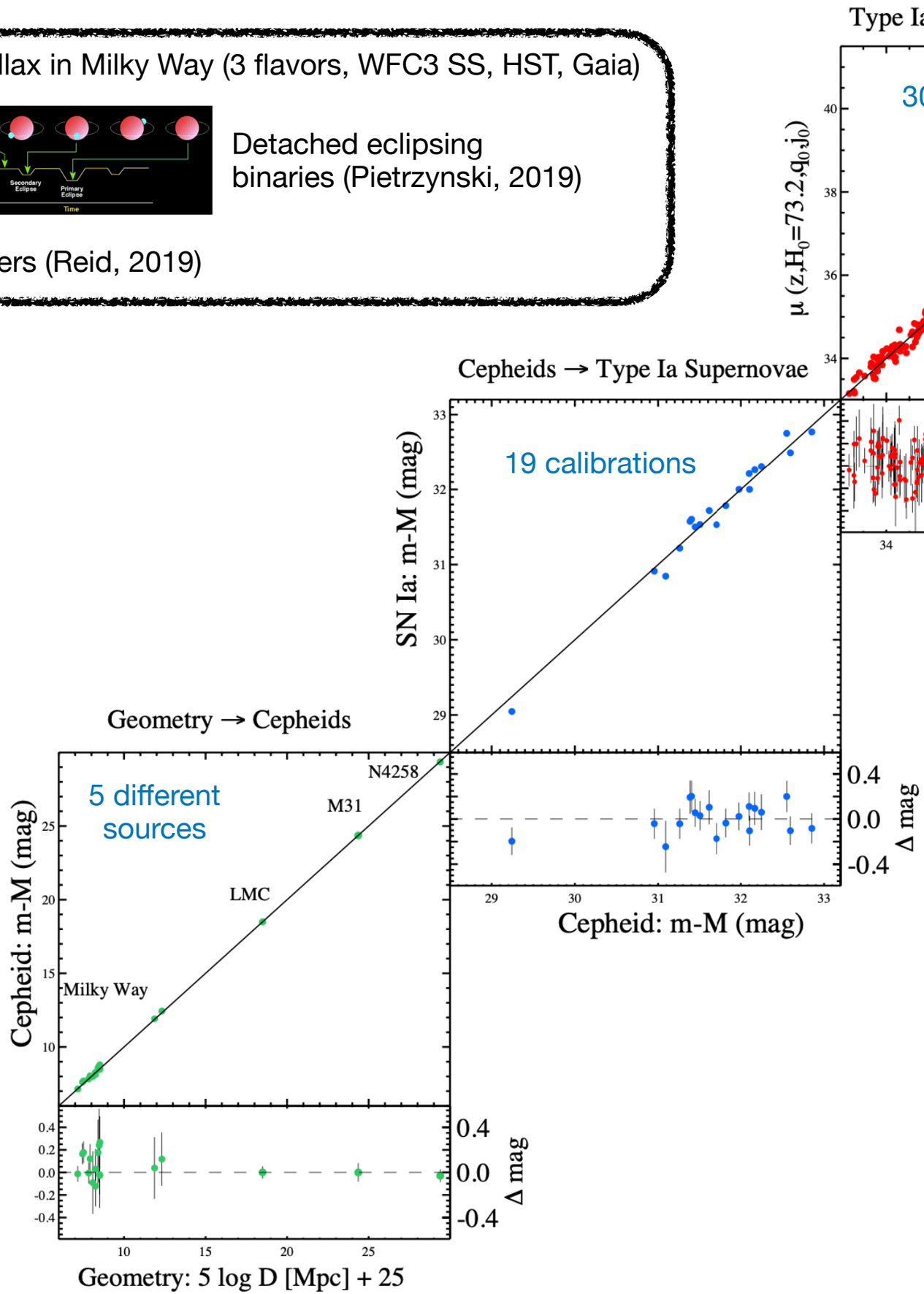
Parallax in Milky Way (3 flavors, WFC3 SS, HST, Gaia)



Detached eclipsing binaries (Pietrzynski, 2019)

Masers (Reid, 2019)



$$H_0 = 73.5 \pm 1.4$$

Geometric Source - Anchor	H_0
Masers (Reid-2019)	72.0
Detached eclipsing binaries (Pietrzynski-2019) + cepheids (Riess-2019)	74.2
Milky Way (10) HST, short P. (Benedict-2007)	76.2
Milky Way (8) HST, long P., spatial scan (Riess-2018)	75.7
Milky Way (50) Gaia+HST, long P., spatial scan (Riess-2018)	73.7

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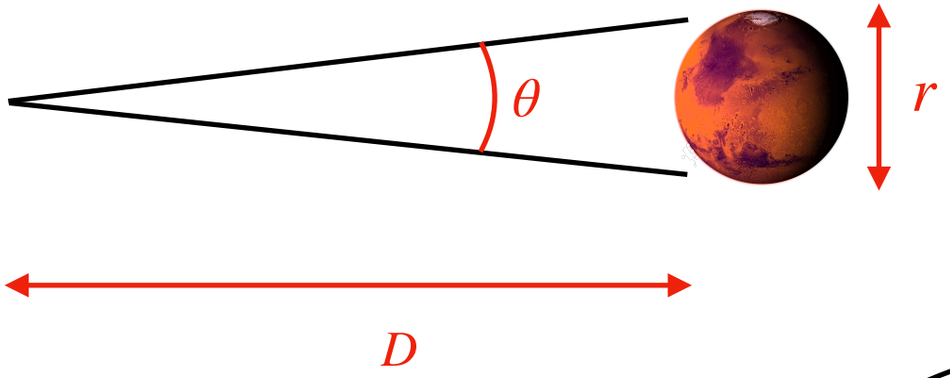


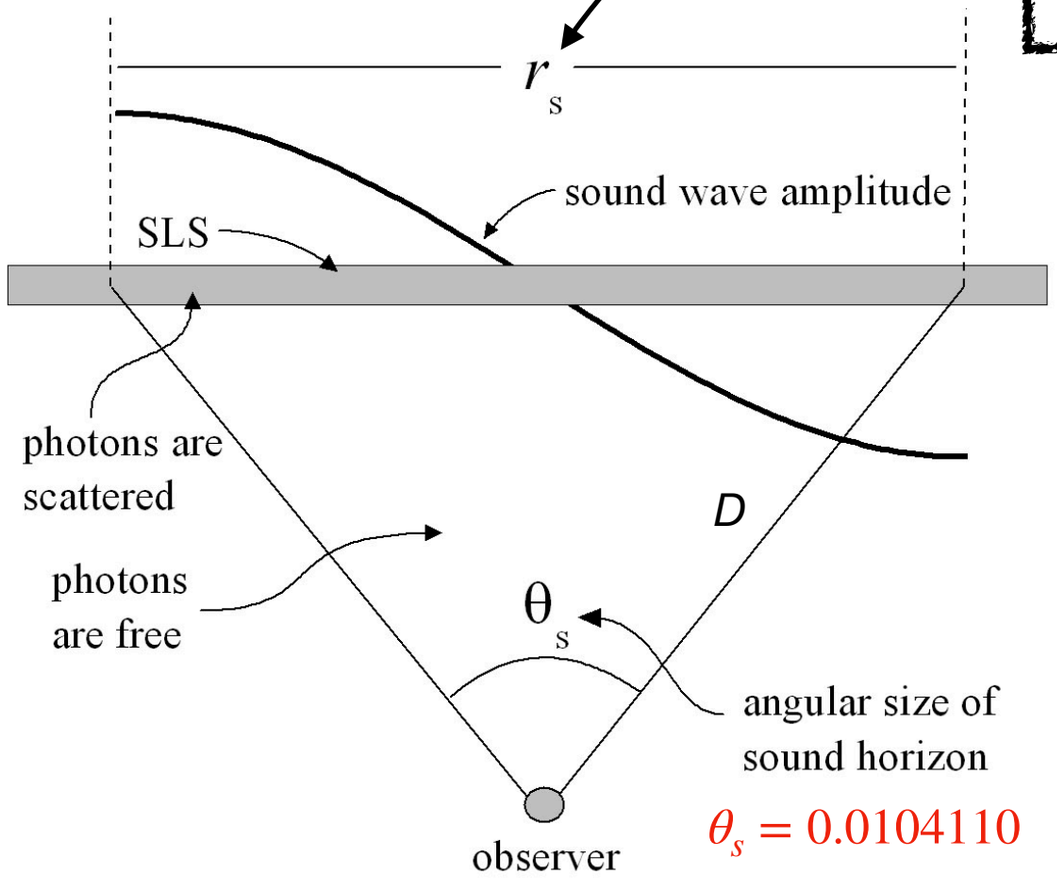
Model dependent

General Relativity
Initial conditions
Homogeneous and isotropic
Dark matter
Baryons
Photons
Neutrinos …

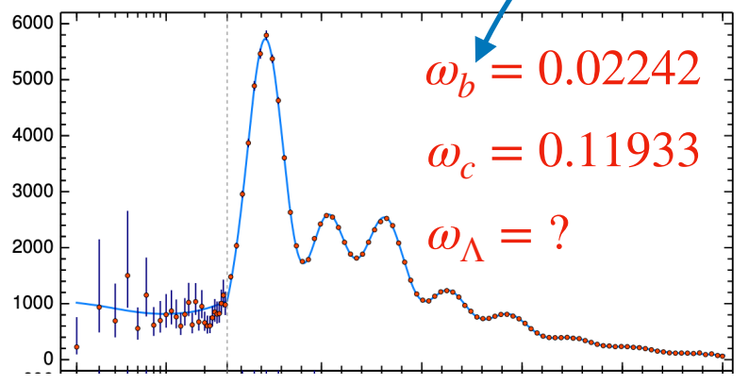
$$\sum_i \Omega_i = 1$$

$$\sum_i \omega_i = h^2$$

$$\omega_i = \Omega_i\, h^2, \quad h = \frac{H_0}{100}$$



$\theta_s = 0.0104110$



$\omega_b = 0.02242$

$\omega_c = 0.11933$

$\omega_\Lambda = ?$

$\Rightarrow$ $r_s = 144.57\ Mpc$

$\Downarrow$

$D = 13886.3\ Mpc$

$$D = c\int_0^{1089} \frac{dz}{\sqrt{(\omega_b + \omega_c)(1+z)^3 + \omega_r(1+z)^4 + \omega_\Lambda}}$$

$\omega_\Lambda = 0.315$

$\Downarrow$

$H_0 = 67.66 \pm 0.42$

Singularities

They always form - under some assumptions (Penrose singularity theorem)

Global condition: hyperbolicity (weak condition)

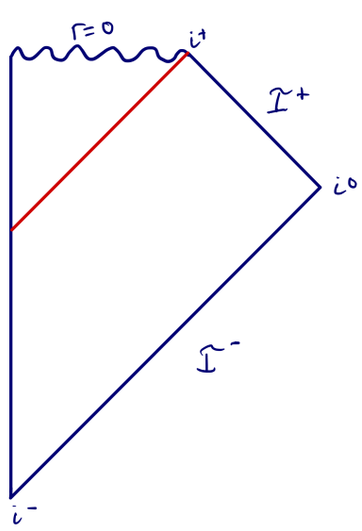
Trapping region (reasonable condition)

Energy condition: null energy condition $\rho + P \geq 0$ (for perfect fluid)

They could be avoided with extra fields

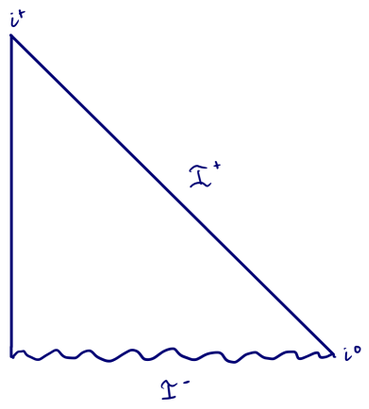
Not really an inconsistency because singularities are not observable (cosmic censorship)

Future singularity



Big Bang singularity is more problematic for the Cauchy problem

Past singularity



Modified gravity as an EFT (Effective Field Theory)

We can model phenomena and test them with our models

Lorentz invariance: Einstein-aether arXiv: 0801.1547

Long range force: Massive gravity arXiv: 1105:3735

Coupling to matter: Screening mechanisms

Vainshtein, 1972 arXiv:1304.7240

Chameleon arXiv: astro-ph/0309300 arXiv: astro-ph/0309411

Symmetron arXiv:1001.4525

Einstein frame vs Jordan frame

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[F(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big]$$ Jordan frame

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\bar{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$$
$$\sqrt{-\bar{g}} = \Omega^4\sqrt{-g}$$
$$\Gamma^\alpha_{\mu\nu} = \bar{\Gamma}^\alpha_{\mu\nu} - \frac{1}{\Omega}\Big(\delta^\alpha_\mu\Omega_{,\nu} + \delta^\alpha_\nu\Omega_{,\mu} - \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\Omega_{,\beta}\Big)$$
$$R = \Omega^2\Big(\bar{R} + 6\frac{\bar{\Box}\phi}{\Omega} - 12\frac{(\bar{\partial}_\mu\Omega)^2}{\Omega^2}\Big)$$

$$\Big(F(\phi) = e^{-\phi}$$ String frame$\Big)$

arXiv:0806.2683

$$S = \int \mathrm{d}^4x\frac{\sqrt{-\bar{g}}}{\Omega^4}\Big[F(\phi)\Omega^2\Big(\bar{R} + 6\frac{\bar{\Box}\phi}{\Omega} - 12\frac{(\bar{\partial}_\mu\Omega)^2}{\Omega^2}\Big) - \frac{1}{2}\Omega^2\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big]$$

$$\Omega = \sqrt{F}$$

$$S = \int \mathrm{d}^4x\sqrt{-\bar{g}}\Big[\bar{R} - \frac{1}{2}\Big(\frac{1+3F'^2}{F^2}\Big)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big]$$ Einstein frame

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[F(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big] + S_m[g_{\mu\nu}, \Psi]$$ **Jordan frame**

$$S = \int \mathrm{d}^4x\sqrt{-\bar{g}}\Big[\bar{R} - \frac{1}{2}\Big(\frac{1+3F'^2}{F^2}\Big)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big] + S_m[F^{-1}g_{\mu\nu}, \Psi]$$ **Einstein frame**

$$\bar{m}(\phi) = \frac{m}{\sqrt{F}}$$

They are an infinite number of frames, are they equivalent?

Of course, it's just a field redefinition

But they are infinite debates on the physical frame, which one is physical

arXiv:astro-ph/9910176

$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$ $a(t)$ the Universe is expanding

How do we observe it?

We do not observe the expansion of the Universe

We observe cosmological redshift

$$E_{obs} = \frac{E_{lab}}{1+z}$$

So in the Jordan frame, we have expansion of the Universe

Let's work with conformal time $ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2)$

And perform a conformal transformation $g_{\mu\nu} = a^2\ \bar{g}_{\mu\nu}$

$$d\bar{s}^2 = -d\eta^2 + d\mathbf{x}^2$$

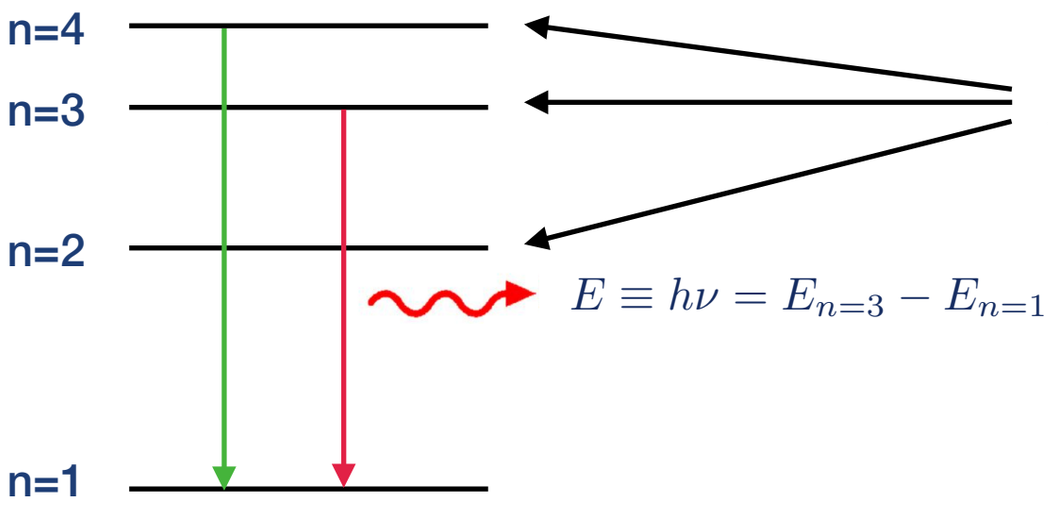
So in this frame, the Universe is static, we do not have expansion, we do not have redshift

Frames are not physically equivalent

All frames are physically equivalent

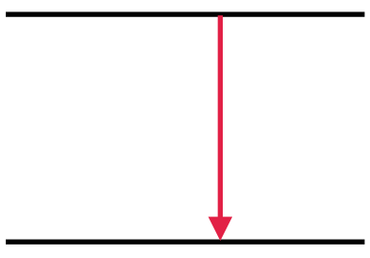
In the new frame, masses change with time

$$\bar{m}(\eta) = m\ a(\eta)$$

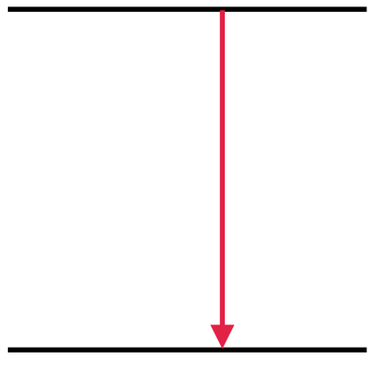


$$E_n = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}$$

Energy levels change with time



compared to



When photon was emitted

In the laboratory, today

The photon never lost energy,
our reference has changed

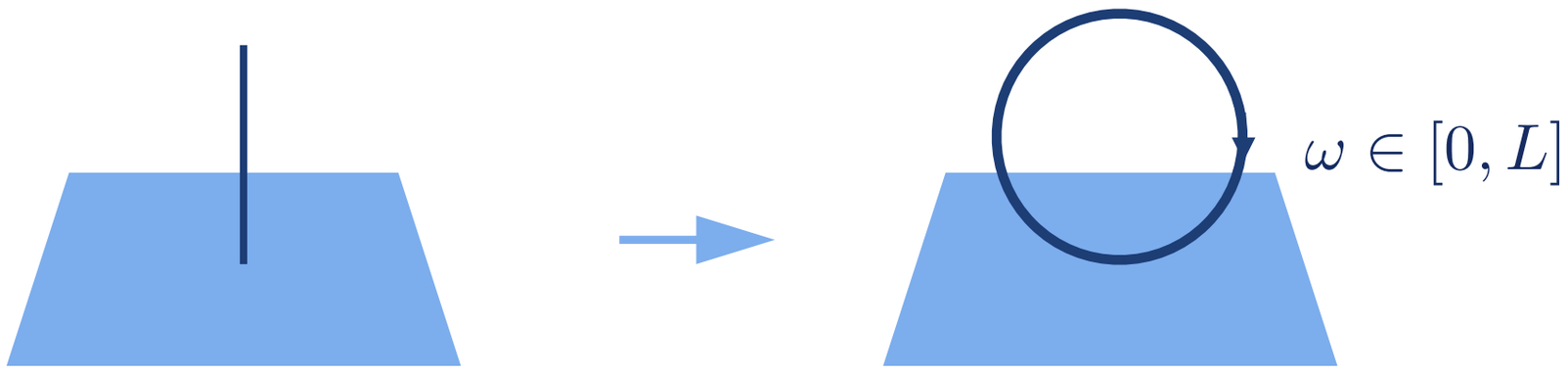
But we observe the same,
a redshift

$$E_{obs} = \frac{E_{lab}}{1+z}$$

Extra dimensions

compactified

Extra dimension → **Compactified: Kaluza-Klein**



$$f(x^\mu, \omega) = \sum_n e^{in\omega/L} f^{(n)}(x^\mu)$$

$$^{(5)}\Box f = 0 \qquad \Box f + \partial_\omega^2 f = 0$$

$$\Box f^{(n)} - \frac{n^2}{L^2} f^{(n)} = 0$$

$m_n = \frac{n}{L}$

$m_2 = \frac{2}{L}$

$m_1 = \frac{1}{L}$

$m_0 = 0$

Kaluza-Klein tower

If $L \ll 1$, only the massless mode survives $\quad f(x^\mu, \omega) \simeq f^{(0)}(x^\mu)$

$$S = \int \mathrm{d}^5 X \sqrt{-\gamma}\mathcal{R} \qquad\qquad X^A = (x^\mu, \omega)$$

$$\gamma_{AB} = \begin{pmatrix} e^{\phi/\sqrt{3}} g_{\mu\nu} + e^{-2\phi/\sqrt{3}} A_\mu A_\nu & e^{-2\phi/\sqrt{3}} A_\mu \\ e^{-2\phi/\sqrt{3}} A_\mu & e^{-2\phi/\sqrt{3}} \end{pmatrix}$$

$$\phi(x^\mu, \omega) \to \phi(x^\mu)$$
$$A(x^\mu, \omega) \to A(x^\mu)$$
$$g_{\mu\nu}(x^\mu, \omega) \to g_{\mu\nu}(x^\mu)$$

$$S = \int \mathrm{d}\omega \mathrm{d}^4 x \sqrt{-\gamma}\mathcal{R} = L \int \mathrm{d}^4 x \sqrt{-\gamma}\mathcal{R}$$

$$S = L \int \mathrm{d}^4 x \sqrt{-g} \Big[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-\sqrt{3}\phi} F_{\mu\nu} F^{\mu\nu} \Big]$$

dilaton

Einstein-Maxwell-dilaton

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$S = L \int \mathrm{d}^4x\sqrt{-g}\Big[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\sqrt{3}\phi}\underline{F_{\mu\nu}F^{\mu\nu}}\Big]$$

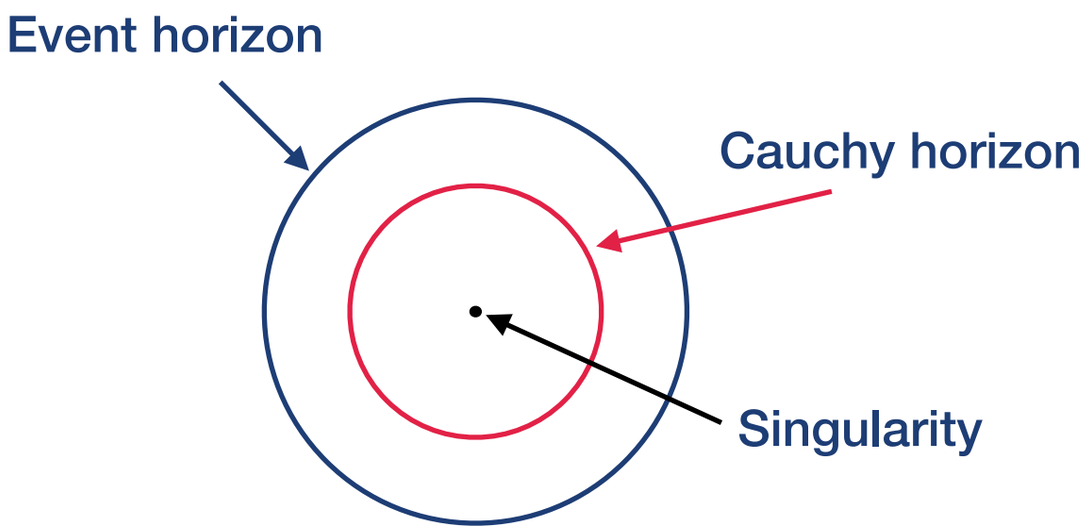
$= 0$ in cosmology

It's not interesting for cosmology but for BH

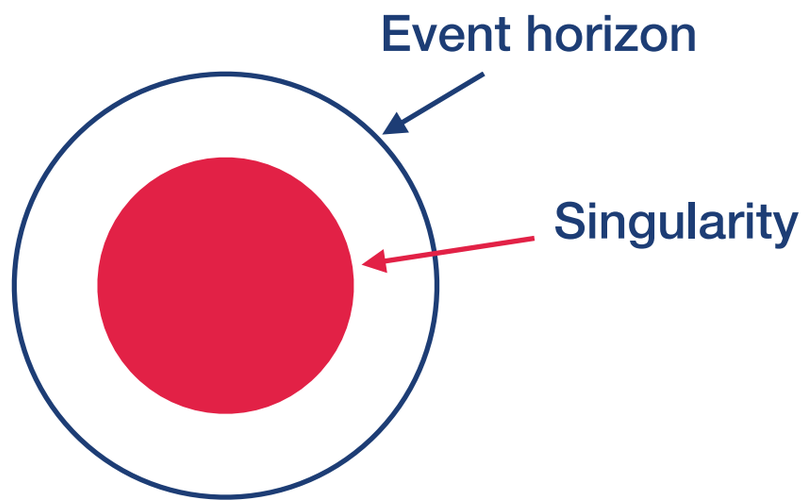
$$ds^2 = -\frac{1-\frac{r_+}{r}}{\sqrt{1-\frac{r_-}{r}}}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\frac{1-\frac{r_+}{r}}{\sqrt{1-\frac{r_-}{r}}}} + r^2\left(1-\frac{r_-}{r}\right)^{3/2}\mathrm{d}\Omega^2$$

$$(r\to\infty) \quad \longrightarrow -\left(1-\frac{r_+}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{r_+}{r}} + r^2\mathrm{d}\Omega^2$$

We have a singularity at $r = r_$



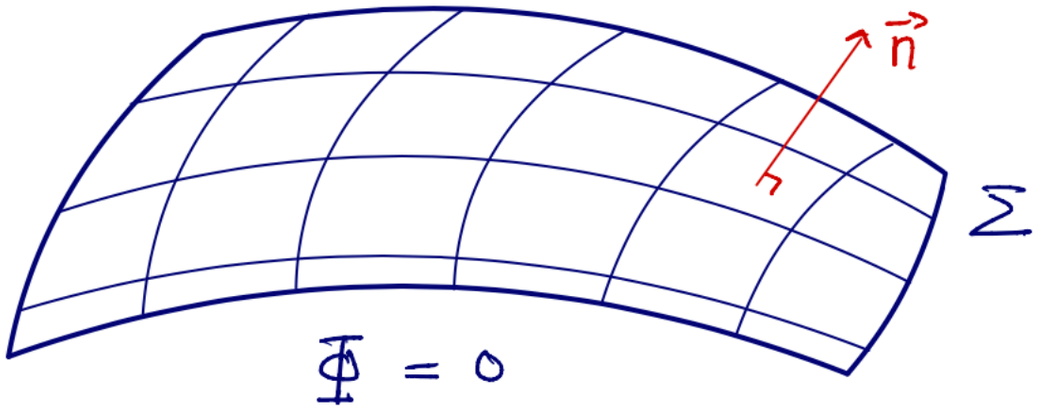
Reissner-Nordström



Outside is linearly stable arXiv:hep-th/9202014

Extra dimensions

Brane



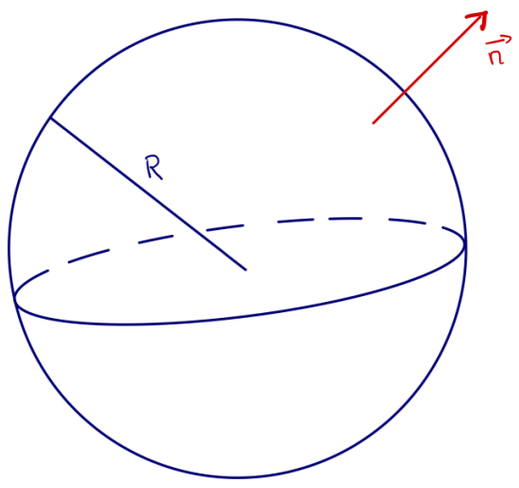
Equation of the hypersurface $\Phi(x^\mu) = 0$

$$n_\mu = \frac{\partial_\mu \Phi}{\sqrt{|g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi|}}$$

Normalization factor

If x^μ are the coordinates of the full space and y^a the coordinates on Σ

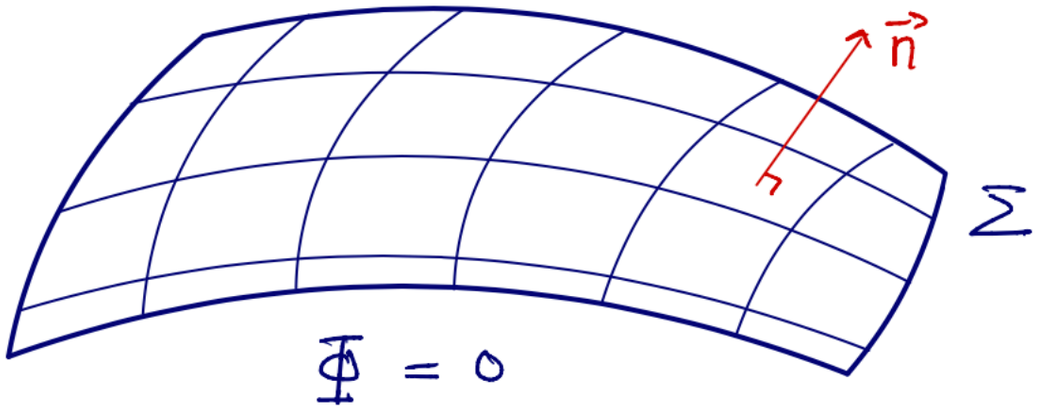
The induced metric on Σ is $h_{ab} = g_{\mu\nu}e^\mu_a e^\nu_b$ with $e^\mu_a = \dfrac{\partial x^\mu}{\partial y^a}$



$\Phi: \; x^2 + y^2 + z^2 - R^2 = 0$

$$n_\mu \propto \partial_\mu \Phi \quad \rightarrow \quad \begin{aligned} n_x &\propto \partial_x \Phi = 2x \\ n_y &\propto \partial_y \Phi = 2y \\ n_z &\propto \partial_z \Phi = 2z \end{aligned}$$

$$\vec{n} = \left(\frac{x}{R}, \frac{y}{R}, \frac{z}{R}\right) \equiv \vec{e}_r$$



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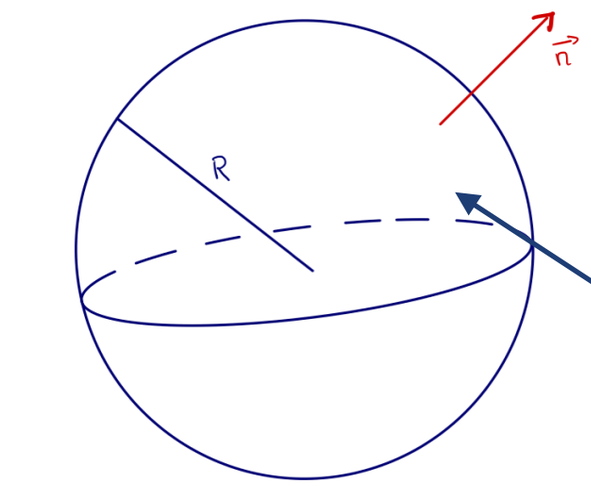
$$e^x_\phi = \frac{\partial x}{\partial \phi} = -R\sin\theta\sin\phi$$
$$e^y_\phi = \frac{\partial y}{\partial \phi} = R\sin\theta\cos\phi$$
$$e^z_\phi = \frac{\partial z}{\partial \phi} = 0$$



$$x^\mu = (x, y, z)$$

$$y^a = (\theta, \phi)$$

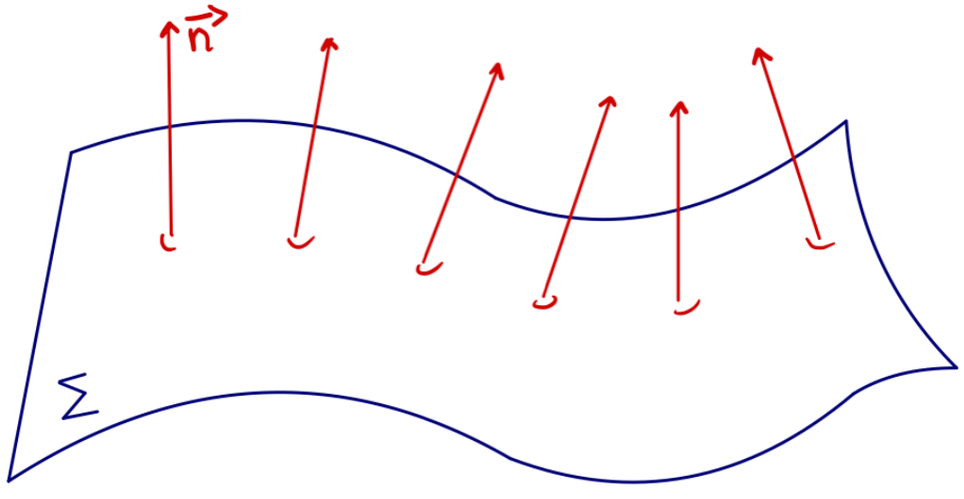
$$\Phi: \; x^2 + y^2 + z^2 - R^2 = 0$$

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta$$

$$e^x_\theta = \frac{\partial x}{\partial \theta} = R\cos\theta\cos\phi$$
$$e^y_\theta = \frac{\partial y}{\partial \theta} = R\cos\theta\sin\phi$$
$$e^z_\theta = \frac{\partial z}{\partial \theta} = -R\sin\theta$$

$$h_{\theta\theta} = g_{\mu\nu}e^\mu_\theta e^\nu_\theta = e^x_\theta e^x_\theta + e^y_\theta e^y_\theta + e^z_\theta e^z_\theta = R^2$$
$$h_{\phi\phi} = g_{\mu\nu}e^\mu_\phi e^\nu_\phi = e^x_\phi e^x_\phi + e^y_\phi e^y_\phi + e^z_\phi e^z_\phi = R^2\sin^2\theta$$
$$h_{\theta\phi} = \ldots = 0$$

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$$



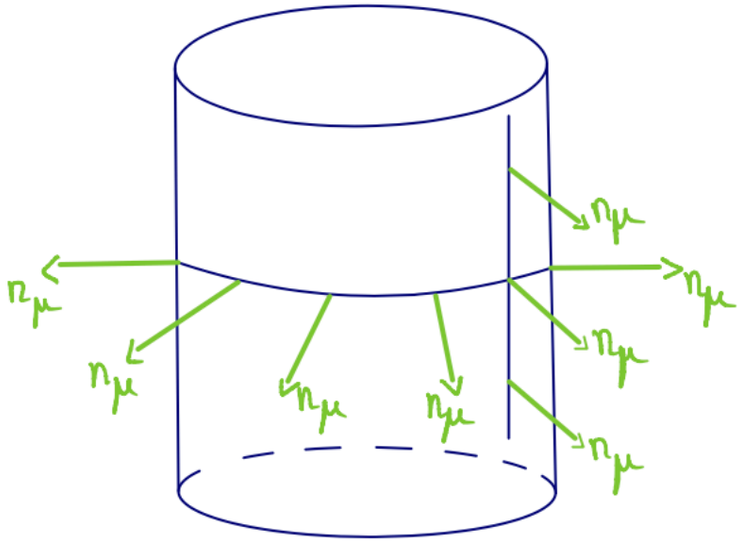
The vector $\vec{n}$ changes direction. The surface is curved

This is the extrinsic curvature

$$K_{ab} = e^{\mu}_{a} e^{\nu}_{b} \nabla_{\mu} n_{\nu} \qquad K = h^{ab} K_{ab}$$

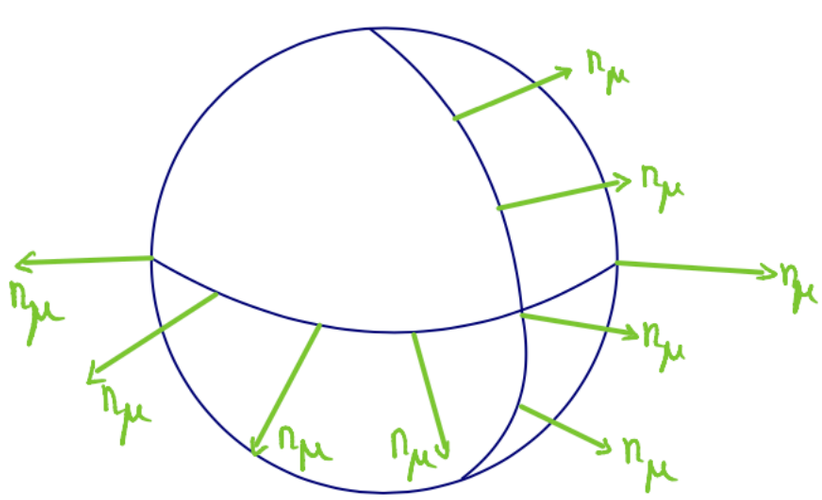
If $K > 0$, the surface is convex and if $K < 0$, the surface is concave

$$K_{ab} = K = 0$$



$$K_{ab} = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$$

$$K = \frac{1}{R}$$



$$K_{ab} = \begin{pmatrix} R & 0 \\ 0 & R\sin^2\theta \end{pmatrix}$$

$$K = \frac{2}{R}$$

$$S = \int \mathrm{d}^4x\sqrt{-g}R$$

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big)\delta g^{\mu\nu} + \int \mathrm{d}^4x\sqrt{-g}\nabla_\mu V^\mu$$

$$\oint \mathrm{d}^3y\sqrt{-h}n_\mu V^\mu$$

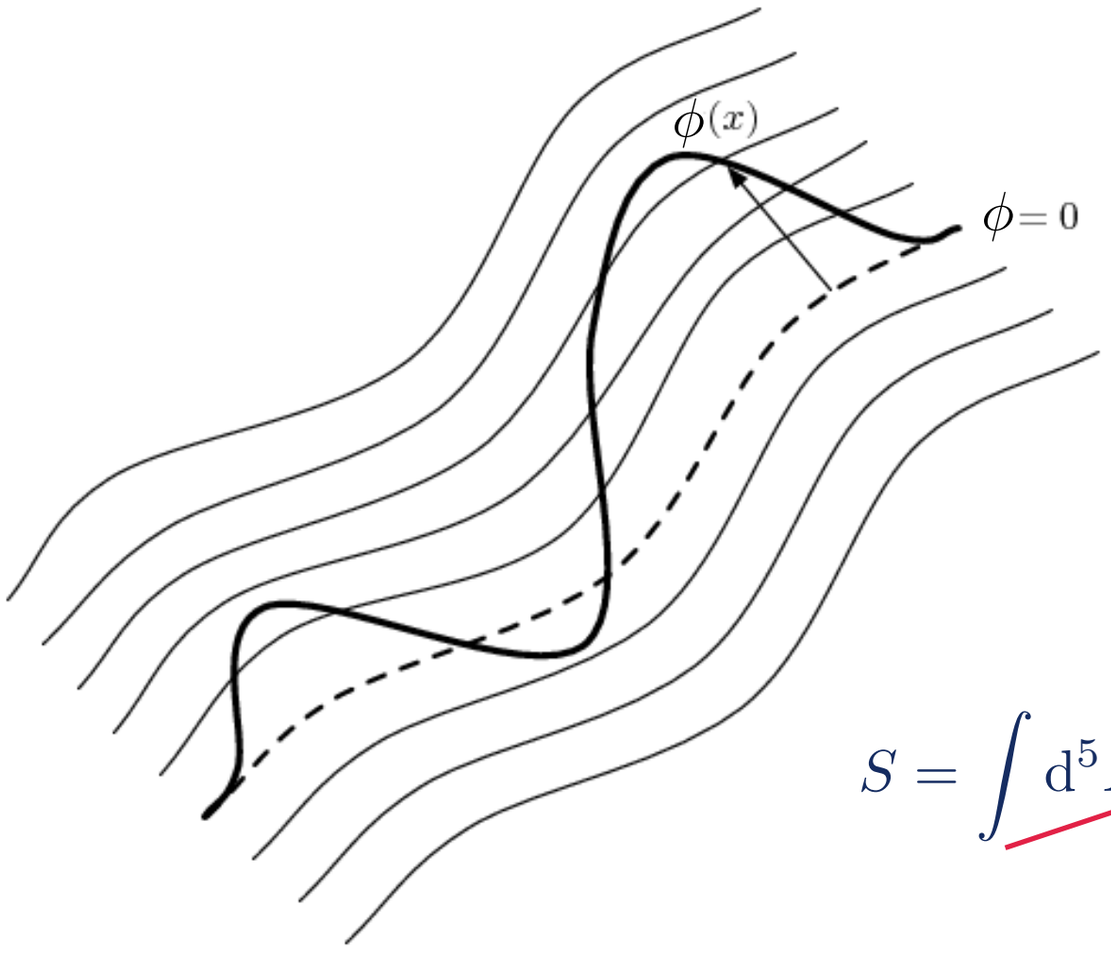
with $V^\mu = \Big(g^{\mu\beta}g^{\alpha\gamma} - g^{\mu\alpha}g^{\beta\gamma}\Big)\nabla_\alpha \delta g_{\beta\gamma}$

Variation principle means $\delta g_{\mu\nu} = \delta g^{\mu\nu} = 0$ but not $\nabla_\mu \delta g_{\alpha\beta}$

We need to add a surface term which cancels the last term

$$S = \int \mathrm{d}^4x\sqrt{-g}R + 2\int \mathrm{d}^3y\sqrt{-h}K$$

Gibbons-Hawking-York term



ϕ measures the brane position with respect to some chosen foliation

$$X^\mu = x^\mu, \quad X^4 = \phi(x^\mu)$$

Induced metric $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu = \eta_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi$

Extrinsic curvature $K_{\mu\nu} = -\dfrac{\partial_{\mu\nu}\phi}{\sqrt{1+(\partial\phi)^2}}$

$$S = \int \mathrm{d}^5X\sqrt{-h}\,{}^{(5)}\mathcal{R} + 2\int \mathrm{d}^4x\sqrt{-g}K + \int \mathrm{d}^4x\sqrt{-g}\Lambda$$

$$\sqrt{-g} = \sqrt{1+(\partial\phi)^2} \qquad K = \Box\phi - \frac{\partial_{\mu\nu}\phi\partial^\mu\phi\partial^\nu\phi}{1+(\partial\phi)^2}$$

$$S = 2\int \mathrm{d}^4x\sqrt{-g}K + \int \mathrm{d}^4x\sqrt{-g}\Lambda$$

$$S = \int \mathrm{d}^4x\left[2\sqrt{1+(\partial\phi)^2}\Box\phi - 2\frac{\partial_{\mu\nu}\phi\partial^\mu\phi\partial^\nu\phi}{\sqrt{1+(\partial\phi)^2}} + \Lambda\sqrt{1+(\partial\phi)^2}\right]$$

$(\partial\phi)^2 \ll 1$

$$S = \Lambda\int \mathrm{d}^4x\left[1 + \frac{1}{2}(\partial\phi)^2 + \frac{2}{\Lambda}(\partial\phi)^2\Box\phi\right]$$

Galileon

arXiv:1103.5745

GR is the unique theory in 4D with only the metric and with second order differential equation

In 5D, Einstein theory is not unique

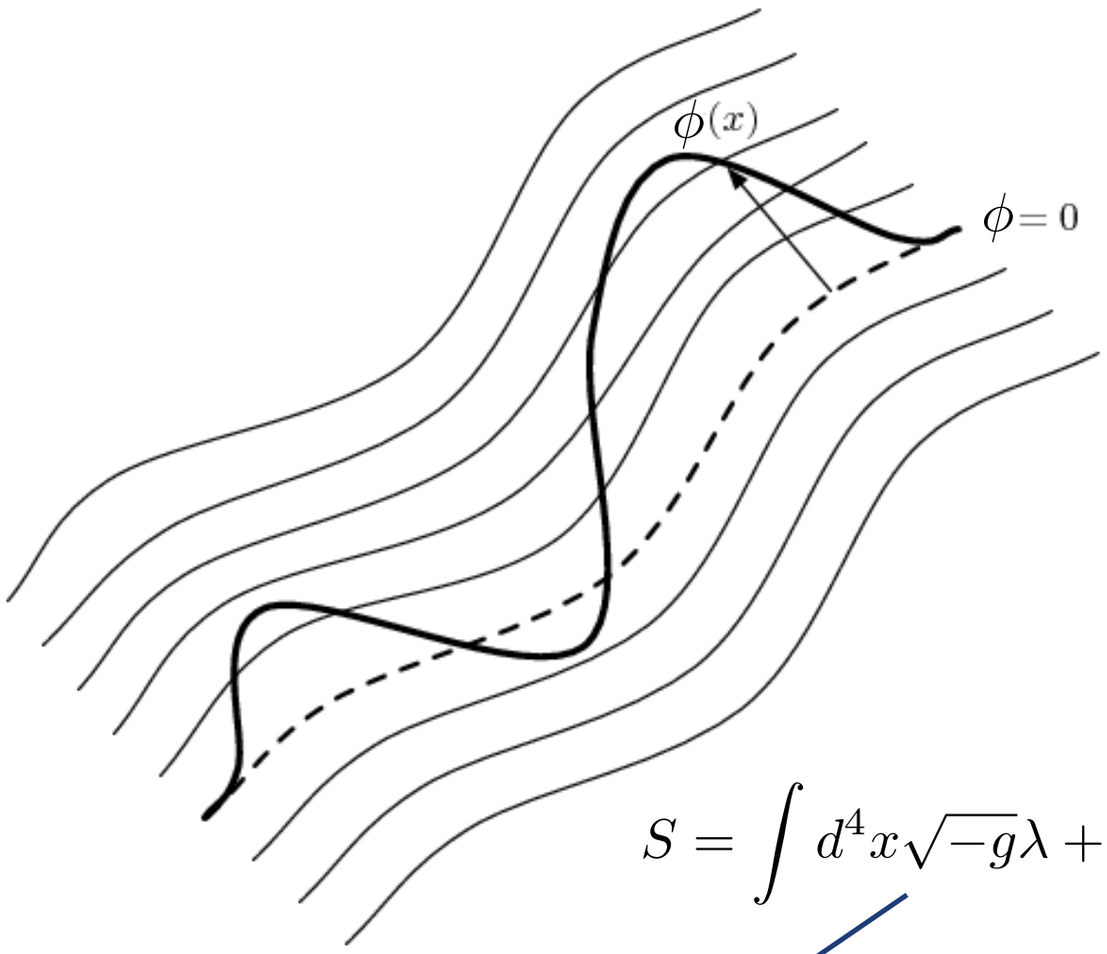
$$S = \int \mathrm{d}^5 X \sqrt{-h}\Big[R - 2\Lambda + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\Big]$$

Gauss-Bonnet

$$2\int \mathrm{d}^4 x\sqrt{-g}K$$

$$2\int \mathrm{d}^4 x\sqrt{-g}K_{GB}$$

arXiv:hep-th/0208205



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Extrinsic curvature $K_{\mu\nu} = -\dfrac{\partial_{\mu\nu}\phi}{\sqrt{1+(\partial\phi)^2}}$

$$S = \int d^4x\sqrt{-g}\lambda + \int d^4x\sqrt{-g}K + \int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}K_{GB}$$

$(\partial\phi)^2 \ll 1$

$\frac{1}{2}(\partial\phi)^2$

$X = -\frac{1}{2}(\partial\phi)^2$ $\quad K(\phi, X)$

$(\partial\phi)^2\Box\phi$

$G_3(\phi, X)$

$(\partial\phi)^2\Big((\Box\phi)^2 - (\partial_{\mu\nu}\phi)^2\Big)$

$G_4(\phi, X)$

$(\partial\phi)^2\Big((\Box\phi)^3 + 2(\partial_{\mu\nu}\phi)^3 - 3\Box\phi(\partial_{\mu\nu}\phi)^2\Big)$

$G_5(\phi, X)$

$$S = \int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i + S_m$$

$$\mathcal{L}_2 = K(\phi,\, X)$$

$$\mathcal{L}_3 = -G_3(\phi,\, X)\Box\phi$$

$$\mathcal{L}_4 = G_4(\phi,\, X)R + G_{4X}(\phi,\, X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]$$

$$\mathcal{L}_5 = G_5(\phi,\, X)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5X}(\phi,\, X)\left[(\Box\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right]$$

We can choose a model, i.e. (K, G_3, G_4, G_5)

Study the background, study the perturbations….

But a better formalism is arXiv:1404.3713

The model is not specified

So background and perturbation equations can be separated

We can choose the background we want

And perturbations will also be free to choose

They depend on 4 new functions

$$(\alpha_K, \alpha_B, \alpha_T, \alpha_M)$$

arXiv:1404.3713

	$\boldsymbol{\alpha}_{\mathrm{K}}$	$\boldsymbol{\alpha}_{\mathrm{B}}$	$\boldsymbol{\alpha}_{\mathrm{M}}$	$\boldsymbol{\alpha}_{\mathrm{T}}$
ΛCDM	0	0	0	0
quintessence	X	0	0	0
k-essence	X	0	0	0
galileon cosmology	X	$\frac{\alpha_k}{6} - \alpha_M$	X	0
metric $f(R)$	0	$-\alpha_{\mathrm{M}}$	X	0
f(Gauss-Bonnet)	0	X	X	X

Gravitational waves travel at the speed $c_T^2 = 1 + \alpha_T$

Neutron star inspiral GW170817 $c_T = 1 \pm 10^{-15}$ $\Rightarrow \alpha_T \simeq 0$

$$S = \int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i + S_m$$

$$\mathcal{L}_2 = K(\phi,\, X)$$

$$\mathcal{L}_3 = -G_3(\phi,\, X)\Box\phi$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}(\phi, X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5X}(\phi, X)\left[(\Box\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right]$$

$$\alpha_T \simeq 0 \Rightarrow G_5(\phi, X) = 0\,,\;\; G_4(\phi, X) = F(\phi)$$

Many models have been killed!!!

Not really

arXiv:1806.09417

Hubble constant $H_0 \simeq 68 km\ s^{-1}\ Mpc^{-1}$

$$k \simeq 3 \times 10^{-19} \mathrm{Hz}$$

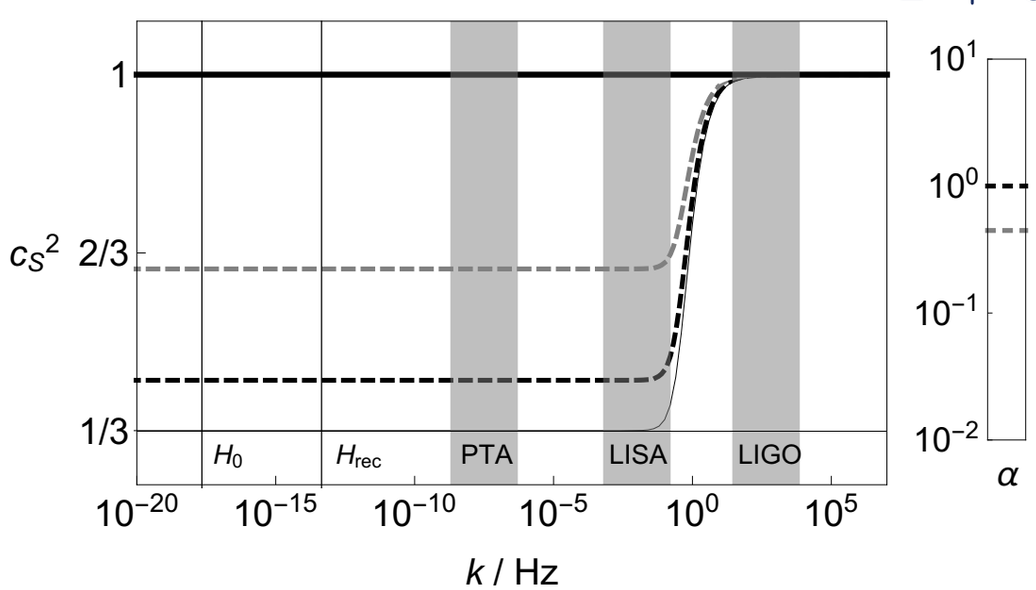
LIGO measured events for $10\ \mathrm{Hz} < k < 100\ \mathrm{Hz}$

If $c_T \equiv c(k)$ we might not have problems

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2\Lambda^4}(\partial\phi)^4 + \mathcal{O}\left(\frac{(\partial^2\phi)^2}{M^2}\right)$$

$$\phi = \alpha\Lambda^2 t \qquad c_T^2 = 1 - \frac{4\alpha^2}{1+6\alpha^2}$$

arXiv:1405.5213



arXiv:1806.09417

Cauchy problem

After all, these theories might have problems

As an EFT, we would like to test these theories at high energies like the merging of 2 BH

For that, simulations should be performed and therefore some mathematical consistency are required, such as a well posed Cauchy problem

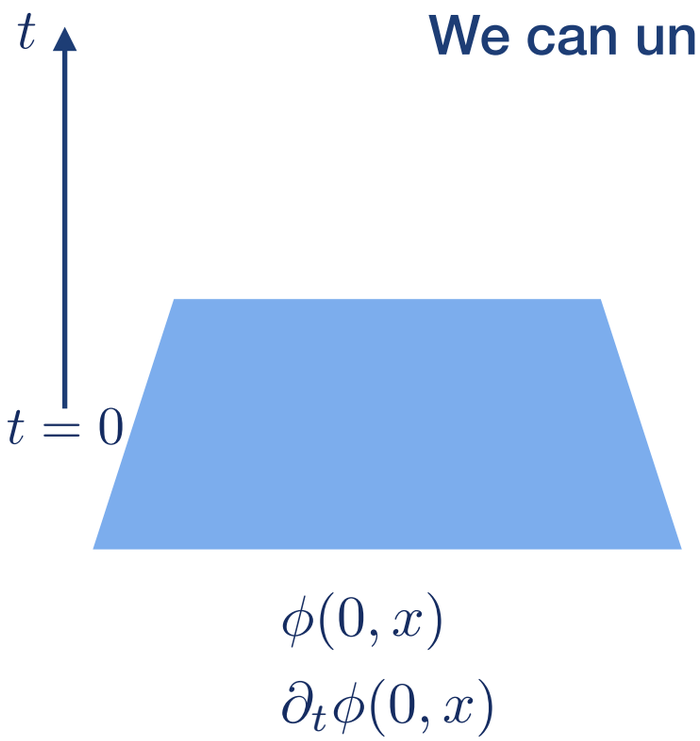
Because even small effects can accumulate over time and produce interesting effects arXiv:gr-qc/9602052

$$\ddot{q} = F(t, q, \dot{q}) \qquad\qquad q(0),\ \dot{q}(0)$$

We can uniquely determine $q(t)$ in the future — Cauchy-Lipschitz-Picard-Lindelöf theorem

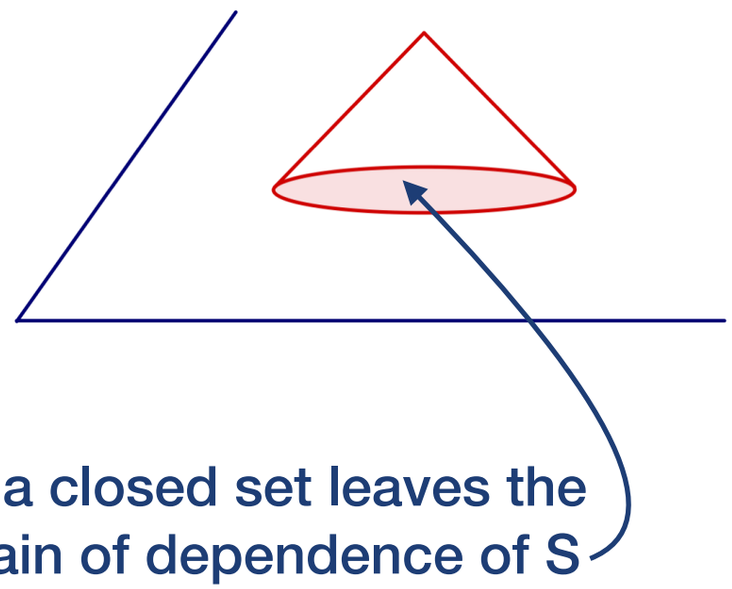
$$\mathcal{O}_2\phi = 0$$

$(\partial_t^2, \partial_x^2, \partial_t, ...)$



The initial value problem is well-posed if

-Unique solution
-Depends continuously on the data
-Variation of the initial data outside of a closed set leaves the solution unchanged in the future domain of dependence of S



To demonstrate the well-posedness of the equations, we usually show strong hyperbolicity

$$A(x,\phi)\dot{\phi} + B^i(x,\phi)\partial_i\phi + C(x,\phi) = 0$$

matrix ↗ $A(x,\phi)\dot{\phi}$ ↖ vector $B^i(x,\phi)\partial_i\phi + C(x,\phi) = 0$

$$\dot{\phi} = \underline{-A(x,\phi)^{-1}B^i(x,\phi)}\partial_i\phi - A(x,\phi)^{-1}C(x,\phi)$$

$M^i(x,\phi)$

The system is weakly hyperbolic, if for $\forall \xi_i$ with norm 1

$M^i(x,\phi)\xi_i$ **has only real eigenvalues**

The system is strongly hyperbolic, if for $\forall \xi_i$ with norm 1

$M^i(x,\phi)\xi_i$ **is diagonalizable with real eigenvalues and has complete set of linearly independent and bounded eigenvectors that depend smoothly on the variables (x, ϕ, ξ_i)**

The well-posedness problem of GR is a bit more subtle because of gauge invariance

At first, the GR equations are not strongly hyperbolic

But we can always add terms which are 0 when a gauge is fixed, we are adding just a gauge term

In such a way that the equations are strongly hyperbolic

If this gauge exists, the problem is well-posed

GR has a well-posed problem

Acta Math. 88 (1952) 141–225
Commun. Math. Phys. 14 (1969) 329–335

Lovelock and Horndeski are weakly hyperbolic
but might fail to be strongly hyperbolic

arXiv:1705.04370, arXiv:1710.10155
arXiv:2003.0839

The only model which are strongly hyperbolic are K-essence

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[F(\phi)R + K(\phi, X)\Big]$$

arXiv:1705.04370, arXiv:1710.10155

There is a no-hair theorem for the shift symmetric Horndeski $\phi \longrightarrow \phi + c$

arXiv:1202.1296

$$\mathcal{L}_2 = K(X)$$
$$\mathcal{L}_3 = -G_3(X)\Box\phi$$
$$\mathcal{L}_4 = G_4(X)R + G_{4X}(X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]$$
$$\mathcal{L}_5 = G_5(X)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5X}(X)\left[(\Box\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right]$$

$$\nabla_\mu J^\mu = 0$$

$$J^r \longrightarrow \partial_r(f(r)J^r) = 0 \longrightarrow f(r)J^r = \text{constant}$$

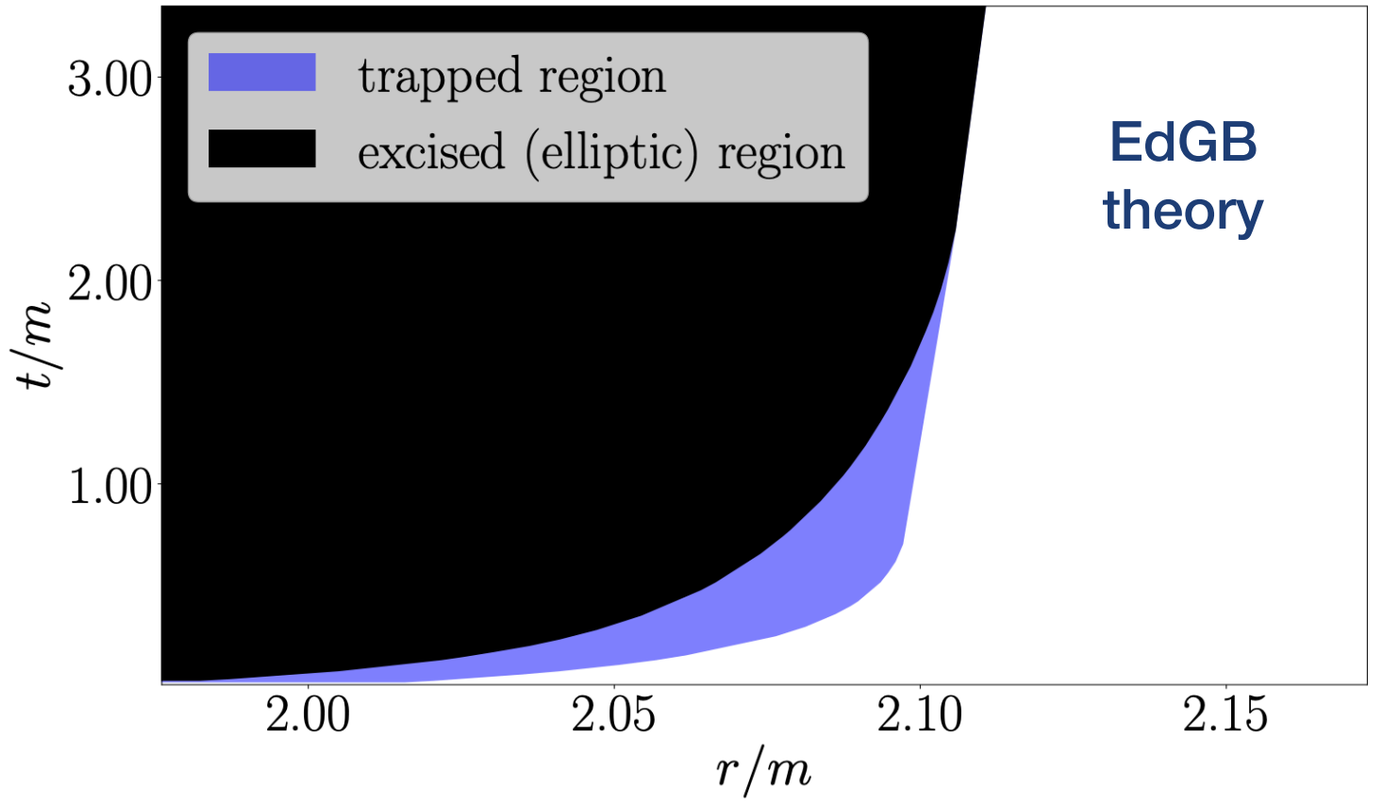
$$J^\mu J_\mu = (J^r)^2 g_{11} \longrightarrow J^r = 0 \quad \text{at the horizon} \longrightarrow J^r = 0 \quad \text{everywhere}$$

$$J^r(\phi', \text{metric}) = 0$$

There is a loop-hole $\quad S = \int \mathrm{d}^4x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + \alpha\phi R_{GB}\right]$ arXiv:1312.3622

Solutions were found perturbatively

Beyond General Relativity



The code always crashes after some time arXiv:1911.11027

K-essence

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[\frac{R}{2} + K(\phi, X)\Big] \qquad X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

$$G_{\mu\nu} = T_{\mu\nu} \equiv K_{,X}\partial_\mu\phi\partial_\nu\phi + Kg_{\mu\nu}$$

If $\partial_\mu\phi$ is timelike (i.e. $X > 0$), we can use a hydrodynamic approach

$$u_\mu = \frac{\partial_\mu\phi}{\sqrt{2X}} \qquad \textbf{Fluid velocity}$$

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu} \qquad \begin{aligned}\rho &= 2XK_{,X} - K\\ P &= K\end{aligned}$$

$$c_s^2 \equiv \frac{\partial P}{\partial\rho} = \frac{K_{,X}}{K_{,X} + 2XK_{,XX}} \qquad \textbf{Sound speed}$$

Variation wrt ϕ

$$\Big(\underline{K_{,X}g^{\mu\nu} - K_{,XX}\partial^\mu\phi\partial^\nu\phi}\Big)\partial_{\mu\nu}\phi - 2XK_{,\phi X} + K_{,\phi} = 0$$

$= \tilde{G}^{\mu\nu}(\phi, X)$ **effective metric**

This equation is hyperbolic if $\tilde{G}^{\mu\nu}$ has Lorentzian signature (Leray theorem, Wald p.251)

$\Leftrightarrow c_s^2 > 0$ **Hyperbolicity is identical to hydrodynamic stability**

Considering fluctuations around some given solution $\phi = \phi_0 + \delta\phi$

$$\frac{1}{\sqrt{-G}}\partial_\mu\left(\sqrt{-G}G^{\mu\nu}\partial_\nu\delta\phi\right) + M^2_{eff}\delta\phi = 0$$

arXiv:0708.0561

$$G^{\mu\nu} = \frac{c_s}{K^2_{,X}}\tilde{G}^{\mu\nu}, \quad G = \det(G^{-1}_{\mu\nu})$$

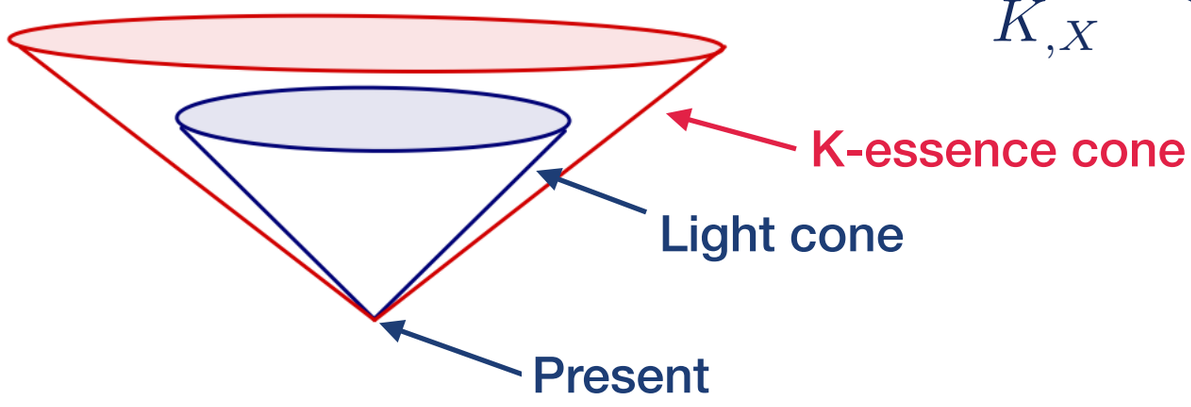
The fluctuations around the solution propagate in an emergent geometry

$$ds^2 = G^{-1}_{\mu\nu}dx^\mu dx^\nu$$

This influence cone is larger than those one determined by the metric $g_{\mu\nu}$ if

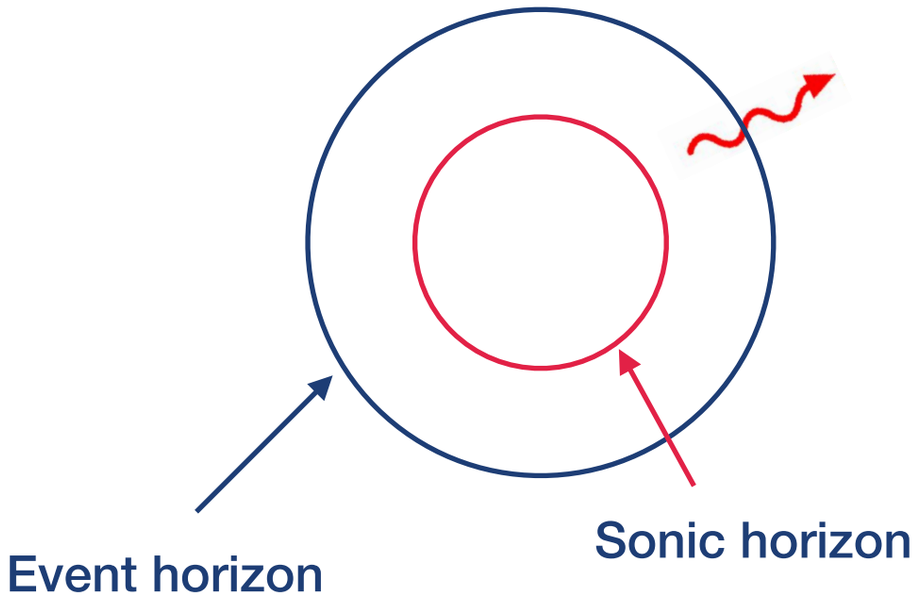
$$\frac{K_{,XX}}{K_{,X}} < 0$$

arXiv:gr-qc/0511158



Let's consider $$K(\phi, X) = X + \alpha X^2 \equiv -\frac{1}{2}(\partial\phi)^2 + \frac{\alpha}{4}(\partial\phi)^4$$

$\alpha < 0$ arXiv:1106.2054



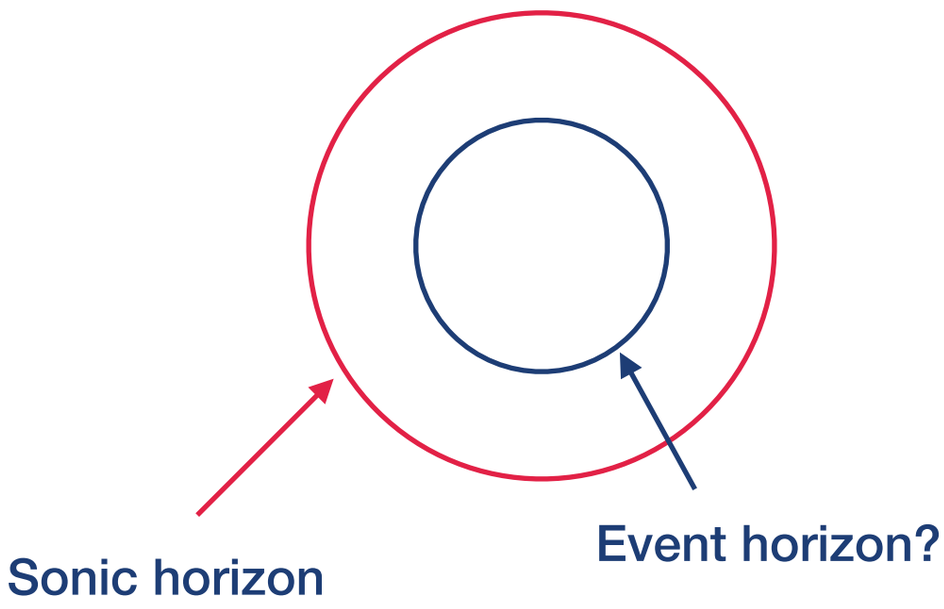
Superluminal propagation

But this model violates "positivity bounds" $(\alpha < 0)$

arXiv:1904.05874

Conditions to be UV completed

$\alpha > 0$ arXiv:2003.1373



The code crashes before the formation of the event horizon

Higher order corrections

1962: Utiyama, DeWitt J.Math.Phys. 3 (1962) 608-618

renormalization at one-loop demands that the Einstein–Hilbert action be supplemented by higher order curvature terms

1976: Stelle Phys.Rev.D 16 (1977) 953-969

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[R + aR^2 + bR_{\mu\nu}R^{\mu\nu}\Big]$$

is renormalizable but there is a massive ghost

1985: Zwiebach Phys.Lett.B 156 (1985) 315-317

$$R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

appears in string theory

It is natural to expect in QFT higher corrections of the following form

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[R + aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR^3 + dR\Box R + ...\Big]$$

The simplest is

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[R + aR^2 + bR^3 + ...\Big] \equiv \int \mathrm{d}^4x\sqrt{-g}f(R)$$

f(R)-gravity

$$S = \int \mathrm{d}^4x\sqrt{-g}f(R)$$

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[f(\psi) + f'(\psi)(R-\psi)\Big]$$

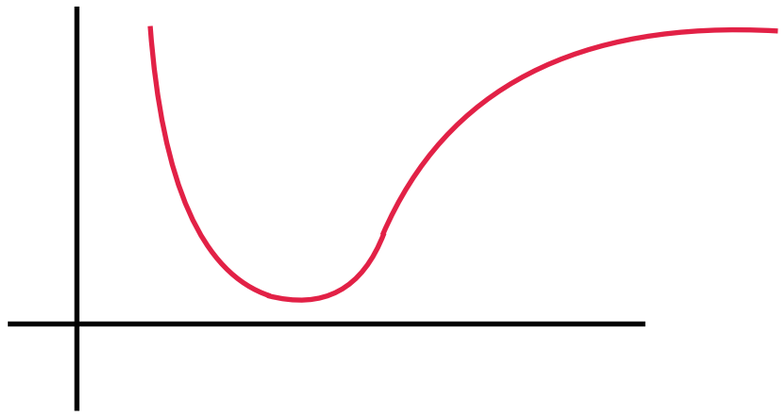
$$S = \int \mathrm{d}^4x\sqrt{-\bar{g}}\Big[\bar{R} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big]$$

$$V(\phi) = \frac{Rf'(R) - f(R)}{(f'(R))^2} \qquad \phi = \sqrt{3}\ln f'(R)$$

Starobinsky model $f(R) = R + \alpha R^2$

$$R = \frac{e^{\phi/\sqrt{3}} - 1}{2\alpha}$$

$$V(\phi) = \frac{\left(1 - e^{-\phi/\sqrt{3}}\right)^2}{4\alpha}$$



Conditions that should satisfy any model

$$f'(R) > 0$$
$$f''(R) > 0$$

Beyond General Relativity

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[R + aR^2\Big] + S_m$$

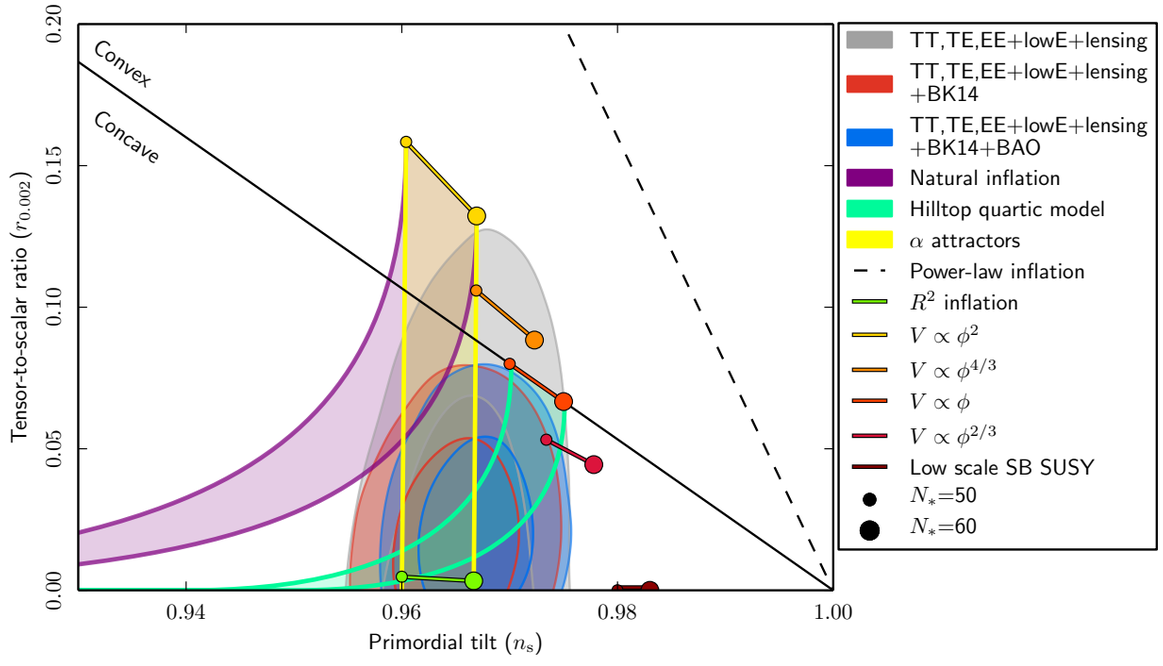
1980: Starobinsky Phys.Lett.B 91 (1980) 99-102

Many models as Higgs inflation are equivalent to Starobinsky model arXiv:1312.115

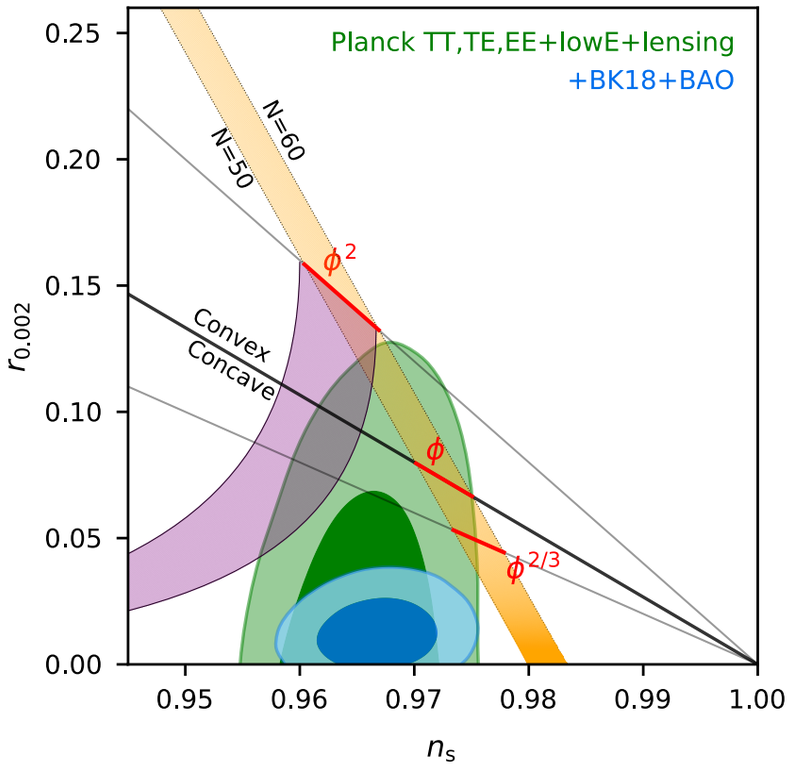
Can be easily embedded into supergravity

https://indico.ipmu.jp/event/28/contributions/1031/attachments/845/935/Ketov.pdf

Planck 2018 arXiv:1807.06211



BICEP/Keck arXiv:2110.00483

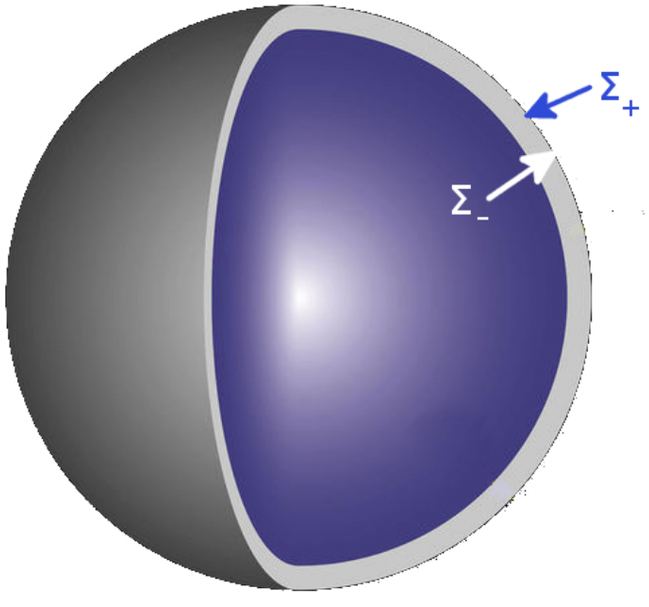


$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[R + aR^2\Big] + S_m$$

When $\alpha = 0$, we "recover" GR

So it should have more solutions than GR

No

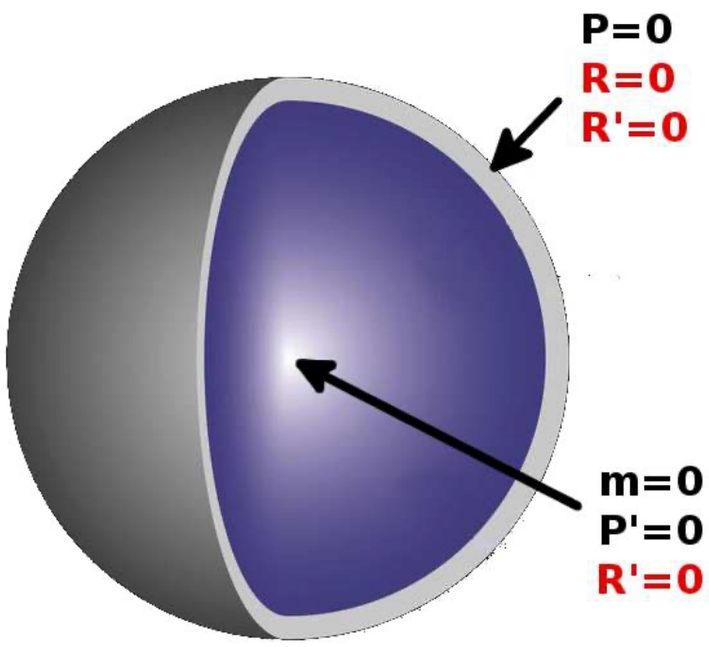


Generalized Israel Junction conditions arXiv:0711.1150

$$[h_{\mu\nu}]_\Sigma = 0$$
$$[K_{\mu\nu}]_\Sigma = 0$$
$$[R]_\Sigma = 0$$
$$[\nabla_\mu R]_\Sigma = 0$$

No Jebsen-Birkhoff theorem
Realistic astrophysical compact objects consistent with solar system
Should define a well defined black hole with regular horizon
BH no-scalar-hair theorems (Bekenstein1995,Sudarsky1995)

$\Rightarrow$ The exterior must be the Schwarzschild spacetime

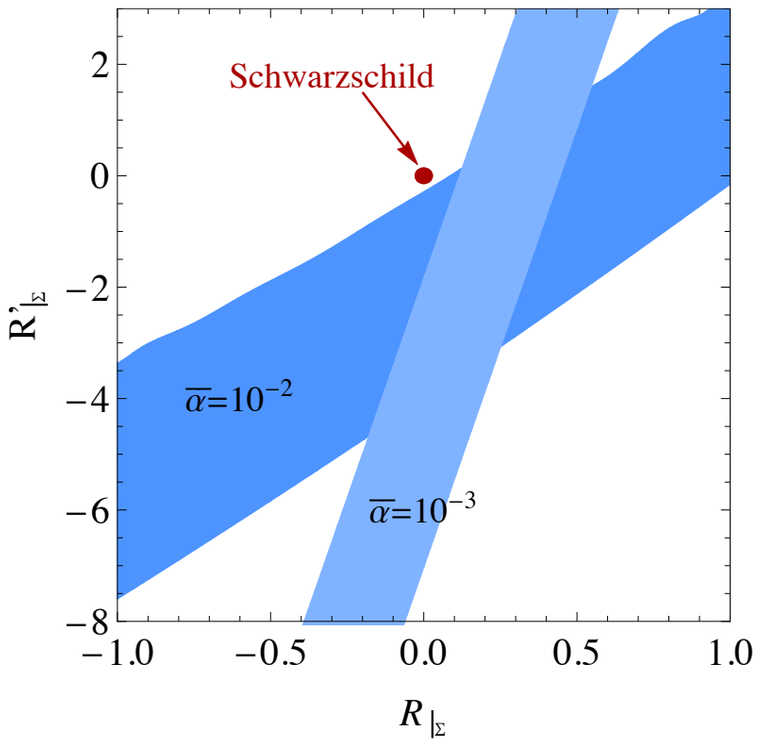


$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[R + aR^2\Big] + S_m$$

For $\alpha \ll 1$ we have a singular problem

It is difficult to satisfy all the conditions arXiv:1309.3279

confirmed in arXiv:1906.08954



Neutron stars do not exist in f(R) gravity?

$$\rho = AP^{\gamma}$$

Beyond General Relativity

Modified gravity models are easy to build

They can have a rich phenomenology

But many problems seem to exist, mostly in the non-linear strong regime

It seems better to constrain these theories in these regimes than in cosmology