Lecture III: Ashtekar variables for general relativity

(Courses in canonical gravity)

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1 The Palatini formulation of general relativity

The Palatini action for general relativity, is simply the Einstein-Hilbert action rewritten so that it is not a function of metric, but instead a function of a "connection" ω and a "frame field" e:

$$S[e,\omega] \;=\; \int dx^4 \sqrt{-\det g}\; R[\omega]\ .$$

This formalism provides first-order field equations for general relativity.

1.1 The tetrad formalism

A frame field (triad and tetrad) provides a way to specify geometries alternative but equivalent to metrics or line elements.

Suppose that, M is an oriented n-dimensional manifold diffeomorphic to $\mathbb{R}^n$. Physically, we can think of M as a small open set of space-time (since every manifold can be covered with charts diffeomorphic to $\mathbb{R}^n$). Since the tangent bundle of $\mathbb{R}^n$ is trivial, so is TM. A trivialization of TM, recall, is a vector bundle isomorphism

$$\begin{aligned} e:\ M\times\mathbb{R}^n &\longrightarrow TM \\ \{p\}\times\mathbb{R}^n &\longmapsto T_pM \end{aligned} \tag{1.1}$$

The trivialization e is also called a *frame field*, since for each $p\in M$ it sends the standard basis of $\mathbb{R}^n$ (the Minkowski space) to a basis of tangent vectors at p, or *frame*: If M is a 3-dimensional manifold, then, a frame field on M is called a *triad*; and if M is a 4-dimensional manifold, a frame field on M is called a *tetrad*.

The idea of Palatini formalism is to do a lot of work on the trivial bundle $M\times\mathbb{R}^n$, which serves as a kind of substitute for the tangent bundle. We can pass and forth between $M \times \mathbb{R}^n$ and TM by using the frame field e and its inverse

$$e^{-1} : \; TM \;\longrightarrow\; M \times \mathbb{R}^n \; . \tag{1.2}$$

Let us suppose that we are in the n-dimensional Lorentzian case (other case works similarly). A section of $M \times \mathbb{R}^n$ is just an $\mathbb{R}^n$-valued function on M, so, for any point $p \in M$, there is a natural basis of sections $\xi_0, \cdots, \xi_n$:

$$\begin{aligned} \xi_0(p) &= (1,0,0,...) \\ \xi_1(p) &= (0,1,0,...) \\ &. \\ &. \\ \xi_n(p) &= (0,0,...,1) \end{aligned} \tag{1.3}$$

Then, we can write any section s as $s = s^I \xi_I$, where we use Einstein summation to sum over I. In this way, $\mathbb{R}^n$ is often called the *internal space*. (We also use upper-case Latin letters $I, J, ..$ for internal indices associated to the basis of sections ξ_I, to avoid mixing with the space-time indices associated to the coordinate vector field ∂_μ on a chart.) Recall that the vector field e in (1.1) is a map from sections of $M \times \mathbb{R}^n$ to vector fields on M. Applying this map to the sections ξ_I, we get a basis of vector fields $e(\xi_I)$ on M, and in a chart we can write these as

$$e(\xi_I) \;=\; e_I^a \partial_a \; , \tag{1.4}$$

where the components e_I^a are functions on M. In relativity, it is typical to abbreviate $e(\xi_I)$ as just e_I, so we will do this. Furthermore, since either the coefficients e_I^a or the vector fields $e_I = e(\xi_I)$, are enough to determine the frame field e, it is common to call either of these things the frame field.

Given two sections s and s' of $M \times \mathbb{R}^n$, as a kind of 'imitation tangent bundles', one can define their canonical inner product $\eta(s, s')$ by

$$\eta(s, s') \;=\; \eta_{IJ} s^I s'^J \; , \tag{1.5}$$

where is copied after Minkowski metric:

$$\eta_{IJ} \;=\; \begin{pmatrix} -1 & 0 & 0 & .. \\ 0 & 1 & 0 & .. \\ & & ... & \\ 0 & 0 & .. & 1 \end{pmatrix} . \tag{1.6}$$

This is also called the *internal metric*. It should be noted that, we can raise and lower internal indices with η_{IJ} and its inverse η^{IJ}, just as we raise and lower space-time indices using a metric. In fact, what we are doing thereby, is mapping $\mathbb{R}^n$ to its dual (or vice versa) using the internal metric.

Suppose that, M has a Lorentzian metric g on it. Thus, we can take the inner products of vector fields on M by

$$g(v, v') \;=\; g_{ab} v^a v'^b. \tag{1.7}$$

A frame field is *orthonormal* if the vector fields e_I are orthonormal, that is

$$g(e_I, e_J) \;=\; \eta_{IJ} \;. \tag{1.8}$$

If the frame field in orthonormal, for any sections s and s' of $M \times \mathbb{R}^n$, the metric g on M is nicely related to the internal metric η, as follows:

$$\begin{aligned} g\left(e(s), e(s')\right) \;&=\; g\left(\eta(s^I \xi_I), \eta(s'^J \xi_J)\right) \\ &=\; s^I s'^J g(e_I, e_J) \\ &=\; \eta_{IJ} s^I s'^J \\ &=\; \eta(s^I \xi_I, s'^J \xi_J) \\ &=\; \eta(s, s') \;. \end{aligned} \tag{1.9}$$

In the Palatini formalism, we work with orthonormal frame fields e rather than metrics g on M. Given an orthonormal frame e, the results above implies that, the metric on M can be written in terms of the inverse frame field by

$$g(v, v') \;=\; \eta(e^{-1}v, e^{-1}v') \;. \tag{1.10}$$

Conversely, since we are assuming M is diffeomorphic to $\mathbb{R}^n$ (which we can always arrange by taking M to be a small open subset of space-time), one can show that, every metric on M admits some orthonormal frame field.

From (1.8), for an orthonormal frame field e we obtain

$$g(e_I, e_J) \;=\; g_{ab} e^a_I e^b_J \;=\; \eta_{IJ} \;. \tag{1.11}$$

Using the internal metric η_{IJ} in (1.11) we can contract the internal indices of tetrad:

$$e^I_a e^a_J \;=\; \delta^I_J \;. \tag{1.12}$$

It follows that, the inverse frame field is given by the following formula:

$$e^{-1} v \;=\; e^I_\alpha v^\alpha \xi_I \;. \tag{1.13}$$

Consider that $v = e(s)$ for some section $s \in M \times \mathbb{R}^n$, (1.13) gives

$$\begin{aligned} e^{-1} v \;&=\; e^I_\alpha v^\alpha \xi_I \\ &=\; e^I_\alpha e^\alpha_J s^J \xi_I \\ &=\; \delta^I_J s^J \xi_I \\ &=\; s^I \xi_I \\ &=\; s \;. \end{aligned} \tag{1.14}$$

The function e^I_α here is called the *co-frame field*: If M is 3-dimensional, a co-frame filed on M is also called a *co-triad*; if M is 4-dimensional, a co-frame field is also called a *co-tetrad*. From the result above, we can write a metric g on M in terms of co-frame field e^I_a as follows:

$$\begin{aligned} g_{ab} &= g(\partial_a, \partial_b) \\ &= \eta(e^{-1}\partial_a, e^{-1}\partial_b) \\ &= \eta(e^I_a \xi_I, e^J_b \xi_J) \\ &= \eta_{IJ} e^I_a e^J_b \ . \end{aligned} \tag{1.15}$$

For a given field $e_{bJ} := g_{bc} e^c_J$, we can contract the internal indices by using the inverse internal metric η^{IJ} as

$$\eta^{IJ} e^a_I e_{bJ} = \delta^a_b \ . \tag{1.16}$$

Raising the index b we obtain

$$\eta^{IJ} e^a_I e^b_J = g^{ab} \ , \tag{1.17}$$

which contains the inverse form of metrics in (1.15).

Equations (1.15) and (1.17) indicate that, the tetrads e^a_I and e^b_J contains all information found in the metric g_{ab}, since the latter can be constructed from them. Thus, the tetrad can be taken as a fundamental description of the geometry, with the metric as a derived concept. However, the tetrad (as a tensor without symmetry conditions) has more independent components than the metric. By applying the Lorentz transformation $e^a_I \to \Lambda_I{}^K e^a_K$ to each tetrad e^a_K, we have that

$$\begin{aligned} \eta^{IJ} e^a_I e^b_J \longrightarrow\ & \eta^{IJ}(\Lambda_I{}^K e^a_K)(\Lambda_J{}^L e^b_L) \\ =\ & \eta^{IJ} \Lambda_I{}^K \Lambda_J{}^L e^a_K e^b_L \\ =\ & (\Lambda^T \eta \Lambda)^{KL} e^a_K e^b_L \\ =\ & \eta^{KL} e^a_K e^b_L \\ =\ & g^{ab} \ . \end{aligned} \tag{1.18}$$

This shows that, applying the Lorentz transformation to the tetrad does not change the corresponding metric. Therefore, Lorentz transformations of the tetrad provide new gauge freedom; we will call this *internal gauge* to distinguish it from the space-time gauge that arises in any theory of space-time geometry.

1.2 Connections via tetrads

Beside the frame fields, the other ingredient in the Palatini formalism is a *connection* on the trivial bundle $M \times \mathbb{R}^n$. In general relativity, the covariant derivative for a vector field v^b in the direction v^a is defined as $\nabla_a v^b = \partial_a v^b + \Gamma^b_{ac} v^c$, by employing the Christoffel connection Γ^a_{bc}. By analogy of the definition of a metric-preserving connection, for a different type of vector field v^I arises on the vector bundle $M \times \mathbb{R}^n$, whose parallel transport or covariant derivative must be defined independently of that for space-time vector fields v^a; we define the covariant derivation $\mathcal{D}$, of the vector field v^I on $M \times \mathbb{R}^n$ as

$$\mathcal{D}_a v^I \;=\; \nabla_a v^I + \omega_a{}^I{}_J v^J \;, \tag{1.19}$$

where the the *connection 1-forms* $\omega_a{}^I{}_J$ are the vector potentials analogous to Γ^b_{ac}. For internal tensors, each index requires the addition (or subtraction) of a connection term in order to ensure the Leibniz rule. In the presence of a mixed tangent-space and internal tensor, such as tetrad, we apply ∇_a using the Christoffel connection for which internal indices do not matter, and a term containing $\pm\omega_a{}^I{}_J$ for every internal index. (Note that, it makes no sense if a connection on $M \times \mathbb{R}^n$ is torsion free. Moreover, there is thus no 'Levi-Civita connection' on $M \times \mathbb{R}^n$.)

For the Minkowski metric η_{IJ}, since there are no tangent-space indices, the internal metric looks like a scalar on tangent-space and takes the same values everywhere the covariant derivative of η_{IJ} on the tangent-space is zero; $\nabla_a \eta_{IJ} = 0$. However, η_{IJ} varies under covariant derivative $\mathcal{D}_a$ on the vector bundle $M \times \mathbb{R}^n$:

$$\begin{aligned}\mathcal{D}_a \eta_{IJ} &= \nabla_a \eta_{IJ} - \omega_a{}^K{}_I \eta_{KJ} - \omega_a{}^K{}_J \eta_{IK} \\ &= -\omega_a{}^K{}_I \eta_{KJ} - \omega_a{}^K{}_J \eta_{IK} \\ &= -\omega_{aIJ} - \omega_{aJI} \;.\end{aligned} \tag{1.20}$$

If $\mathcal{D}_a \eta_{IJ} = 0$, from (1.20) we obtain

$$\omega_{aIJ} \;=\; -\omega_{aJI} \;. \tag{1.21}$$

Thus, connection 1-forms, leaving the Minkowski metric invariant, must be *antisymmetric* in their internal indices. In this case, the connection $\mathcal{D}$ on $M \times \mathbb{R}^n$ is said to be a *Lorentz connection*; the parallel transport along a curve is a mapping which leaves the Minkowski space invariant, and amounts to a Lorentz transformation. The connection 1-forms, when contracted with a vector field v^a, provide an infinitesimal parallel transport $v^a \omega_a{}^K{}_I$ along the direction v^a. This is a way of saying that, ω, with the required symmetry properties, $v^a \omega_a{}^K{}_I \in so(3,1)$ thus it lives in the lie algebra of the Lorentz group $SO(3,1)$.

The covariant derivative $\mathcal{D}_a$ then preserves the internal metric η_{IJ} as well as the space-time metric g_{ab} (because of the same property of ∇_a). In the tetrad formulation, however, we describe the space-time geometry not by any one of these tensors, but by the tetrad e^a_I or the co-tetrad e^I_a. A required condition for the co-tetrad to be covariantly constant can be specified by the definition

$$\omega_a{}^I{}_J \;:=\; e^{bI}\nabla_a e_{bJ} \;, \tag{1.22}$$

for the connection 1-forms. Using this equation for the connection 1-form, the covariant derivative of the co-tetrad can be obtained as

$$\begin{aligned} \mathcal{D}_a e_b^I &= \nabla_a e_b^I + \omega_a{}^I{}_J e_b^J \\ &= \nabla_a e_b^I + e^{cI}\nabla_a(e_{cJ})e_b^J \\ &= \nabla_a e_b^I - e^{cI}e_{cJ}\nabla_a e_b^J \\ &= \nabla_a e_b^I - \nabla_a e_b^I \;=\; 0 \;. \end{aligned} \tag{1.23}$$

Therefore, the covariant derivative $\mathcal{D}$ defined by the connection 1-forms (1.22) preserves the tetrads well as the space-time metric. It must further preserve the Minkowski metric $\eta^{IJ} = g^{ab}e_a^I e_b^J$ and its inverse η_{IJ}; so that the ω_{aIJ} defined by (1.22) are antisymmetric and satisfy equation (1.21).

1.3 Curvature via tetrads

Making all indices of the connection 1-forms internal, we have that

$$\omega_{IJK} \;=\; e_I^a \omega_{aJK} \;=\; e_I^a e_J^b \nabla_a e_{bK} \;. \tag{1.24}$$

The ω_{IJK} are called *Ricci rotation coefficients*. Similarly, all indices of the Riemann tensor R_{abcd} can be made internal:

$$\begin{aligned} R_{IJKL} &= R_{abcd}\; e_I^a e_J^b e_K^c e_L^d \\ &= e_I^a e_J^b e_K^c \left(\nabla_a\nabla_b - \nabla_b\nabla_a\right) e_{cL} \;. \end{aligned} \tag{1.25}$$

Then, using the relation

$$\begin{aligned} e_K^c \nabla_a\nabla_b e_{cL} &= \nabla_a\left(e_K^c \nabla_b e_{cL}\right) - \left(\nabla_a e_K^c\right)\left(\nabla_b e_{cL}\right) \\ &= \nabla_a \omega_{bKL} - \eta^{MN}\omega_{aNK}\omega_{bML} \;, \end{aligned}$$

in Eq. (1.25) we have that

$$\begin{aligned} R_{IJKL} &= e_I^a e_J^b \left[\nabla_a\omega_{bKL} - \nabla_b\omega_{aKL} - \eta^{MN}(\omega_{aNK}\omega_{bML} - \omega_{bNK}\omega_{aML})\right] \\ &= e_I^a\nabla_a\omega_{JKL} - e_J^a\nabla_a\omega_{IKL} - \eta^{MN}(\omega_{INK}\omega_{JML} - \omega_{JNK}\omega_{IML} \\ &\qquad + \omega_{INJ}\omega_{MKL} - \omega_{JNI}\omega_{MKL}). \end{aligned} \tag{1.26}$$

The internal Riemann tensor (1.26) can provides the internal Ricci tensor $R_{IJ} = \eta^{KL}R_{IJKL}$.

1.4 The palatini action

The spin connection ω in Eq. (1.22) is a one-form with values in the Lie algebra of the Lorentz group $so(3,1)$:

$$\omega^I{}_J(x) \;=\; \omega^I_{aJ}(x)dx^a \; . \tag{1.27}$$

It defines a gauge-covariant exterior derivative $\mathbf{D}$ on forms. For instance, it acts on any one-form u^I, with a Lorentz index, as

$$\mathbf{D}u^I \;=\; du^I + \omega^I{}_J \wedge u^J \; . \tag{1.28}$$

The relation for the connection one form (1.22), upon antisymmetrization, can be written as

$$\eta^{IJ} e_{I[a}\omega_{b]KJ} \;=\; \nabla_{[a}e_{b]K} \;=\; (d\mathbf{e}_K)_{ab} \; , \tag{1.29}$$

in terms of the exterior derivative of the one-form $\mathbf{e}_K$, taking values in the internal vector space. This exterior derivative, obtained after antisymmetrization, requires only partial derivatives for its computation, and thus, no knowledge of the Christoffel coefficients is required. The torsion two-form is defined as

$$T^I \;:=\; \mathbf{D}e^I \;=\; de^I + \omega^I{}_J \wedge e^J \; . \tag{1.30}$$

A tetrad field e determines uniquely a torsion-free spin connection $\omega = \omega[e]$, called compatible with e; thus, setting $T^I = 0$ we obtain

$$de^I \;=\; -\omega^I{}_J \wedge e^J \;=\; e^J \wedge \omega_J{}^I \; . \tag{1.31}$$

It shows that the connection one-forms ω can be determined completely from (1.31).

The internal curvature tensor $\mathbf{R}_{IJ}$ is the Lorentz algebra valued two-form:

$$\mathbf{R}_I{}^J \;=\; R_{Iab}^J \, dx^a \wedge dx^b \; , \tag{1.32}$$

in terms of the mixed tangent-space/internal-space Riemann tensor R_{Iab}^J. Then, Eq. (1.26) for the Riemann tensor in terms of connection one-forms takes the compact form:

$$\mathbf{R}_I{}^J \;=\; d\omega_I{}^J + \omega_I{}^K \wedge \omega_K^J \; . \tag{1.33}$$

Equations (1.31) and (1.33) are called the (Cartan) *first and second structure equations.* From Eq. (1.28) we can write

$$\mathbf{D}^2 u^I \;=\; u^J \wedge R_J{}^I \; , \tag{1.34}$$

from which, by setting $u^J = e^J$, we have that

$$\mathbf{D}^2 e^I \;=\; e^J \wedge R_J{}^I = 0 \; . \tag{1.35}$$

This implies a flat region where the curvature is zero.

Let us introduce the *curvature two-form* F_{ab}^{IJ} defined in

$$F^{IJ} \;=\; F_{ab}^{IJ}\; dx^a \wedge dx^b \; , \tag{1.36}$$

to be the curvature of a general connection one-form ω_a^{IJ}, given by (1.19), on the internal space as:

$$\begin{aligned} F_{ab}^{IJ} \;&=\; \partial_a \omega_b^{IJ} - \partial_b \omega_a^{IJ} + [\omega_a, \omega_b]^{IJ} \\ &=\; 2\partial_{[a}\omega_{b]}^{IJ} \;+\; 2\omega_{[a}^{IK}\omega_{b]}^{LJ}\eta_{KL} \; . \end{aligned} \tag{1.37}$$

Notice that ω_a^{IJ} is not required to obey the first structure equation. Furthermore, if ω is a *Lorentz connection*, then the curvature two form F is *antisymmetric* with respect to the changing both the tangent and internal space indices: $F_{ab}^{IJ} = -F_{ba}^{IJ} = -F_{ab}^{JI}$.

Suppose that, we have both a frame field e and a Lorentz connection ω. One can transfer the Lorentz connection from the trivial bundle $M \times \mathbb{R}^n$ to the tangent bundle TM. When we do this, we obtain a connection $\tilde{\nabla}$ on TM given by

$$\tilde{\nabla}_a \partial_b \;=\; \tilde{\Gamma}_{ab}^c \partial_c \; , \tag{1.38}$$

where the coefficient $\tilde{\Gamma}_{ab}^c$ are defined by

$$\tilde{\Gamma}_{ab}^c \;=\; \omega_{aI}^J e_b^I e_c^J \; . \tag{1.39}$$

We will call $\tilde{\nabla}$ the *imitation Levi-Civita connection*, and call the $\tilde{\Gamma}_{ab}^c$ the *imitation Christoffel coefficients*. Note that, the imitation Christoffel coefficients are obtained by converting internal indices in the vector potential ω to space-time indices, using the tetrad and co-tetrad. Similarly, we can define an *imitation Riemann tensor* by

$$\tilde{R}^c{}_{ab}{}^d \;=\; F_{ab}^{IJ}\; e_I^c e_J^d \;\; , \tag{1.40}$$

from which we define the *imitation Ricci tensor* $\tilde{R}_{ab} = \tilde{R}^c{}_{acb}$ and an *imitation Ricci scalar* $\tilde{R} = \tilde{R}_a^a$.

Using the results we have already obtained, we are able to describe the *Palatini action*. This action is basically an Einstein-Hilbert action, instead is a function of a tetrad e and Lorentz connection ω. The Palatini action is given by

$$\boxed{S[e,\omega] \;=\; \frac{1}{16\pi G}\int_M d^4x \; |e| \; e_I^a \; e_J^b \; F_{ab}^{IJ}(\omega)} \tag{1.41}$$

where e^{-1} is the determinant of the space-time tetrad e^a_I. It should be noted that, in Palatini approach, the metric g on M is not a fundamental field; but is a function of the tetrads as $g_{ab} = \eta_{IJ} e^I_a e^J_b$. The independent fields in the action (1.41) are the tetrad e and the connection ω; this formalism in which e and ω are independent is called the *first-order formalism*.

Using the relation $\epsilon_{IJKL} e^I_a e^J_b e^K_c e^L_d = e\varepsilon_{abcd}$ and contracting it twice with the tetrad, we obtain

$$\varepsilon^{abcd}\epsilon_{IJKL} e^K_c e^L_d \;=\; 2ee^{[a}_I e^{b]}_J \; . \tag{1.42}$$

This identity allows us to rewrite the Palatini action (1.41) in differential-form notation:

$$\boxed{S[e,\omega] \;=\; \frac{1}{64\pi G}\int_M \epsilon_{IJKL}\; \mathbf{e}^K \wedge \mathbf{e}^L \wedge \mathbf{F}^{IJ}(\omega)} \tag{1.43}$$

On the other hand, one can show that, the Palatini action gives the Einstein's equation. More precisely, using variation of action with respect to both ω and e, the equation $\delta S[\omega, e] = 0$ implies that, the metric $g_{ab} = \eta_{IJ} e^I_a e^J_b$ satisfies the vacuum Einstein's equation. Notice that, in variation of the action (1.41), only the space-time connection drops out of the antisymmetrized covariant derivative, not the Lorentz connection. We begin by computing the variation with respect to the tetrad e, that is, computing δS assuming $\delta\omega = 0$. This lets us compute the variation of the action as follows:

$$\begin{aligned}
\delta S \;&=\; \frac{1}{16\pi G}\int_M dx^4 |e| \left[(\delta e^a_I) e^b_J F^{IJ}_{ab} + e^a_I (\delta e^b_J) F^{IJ}_{ab} - e^K_c (\delta e^c_K) e^a_I e^b_J F^{IJ}_{ab}\right] \\
&=\; \frac{1}{8\pi G}\int_M dx^4 |e| \left(e^b_J F^{IJ}_{ab} - \frac{1}{2} e^I_a e^c_K e^d_L F^{KL}_{cd}\right) \delta e^a_I \; .
\end{aligned} \tag{1.44}$$

Expressing this in terms of the imitation Ricci tensor and scalar, we obtain

$$\delta S \;=\; \frac{1}{8\pi G}\int_M dx^4 |e| \left(\tilde{R}_{ab} - \frac{1}{2} g_{ab}\tilde{R}\right) \eta^{IJ} e^b_J (\delta e^a_I) \; . \tag{1.45}$$

It follows that $\delta S = 0$ for an arbitrary variation of the tetrad e precisely when

$$\tilde{R}_{ab} - \frac{1}{2} g_{ab}\tilde{R} \;=\; 0 \; , \tag{1.46}$$

which looks a lot like Einstein's equation. In fact this is hold when the imitation Riemann tensor is equal to the Riemann tensor of g (or when $\tilde{\nabla} = \nabla$). In the following we show that, this case occurs when computing the δS with assuming that $\delta e = 0$.

Varying the imitation Ricci scalar by ω provides that

$$\begin{aligned}
\delta\tilde{R} \;&=\; e^a_I \; e^b_J \; \delta F^{IJ}_{ab}(\omega) \\
&=\; e^a_I \; e^b_J \; \mathcal{D}_{[a}\delta\omega^{IJ}_{b]} \; .
\end{aligned} \tag{1.47}$$

Putting this in (1.41), variation of the action by ω, integrating by parts provides

$$\varepsilon^{abcd}\epsilon_{IJKL}\mathcal{D}_a(e_c^K e_d^L) \ = \ 0 \ . \tag{1.48}$$

which is equivalent to the compatibility condition

$$\mathcal{D}_{[a}e_{b]}^I \ = \ \partial_{[a}e_{b]}^I + \omega_{[a}{}^I{}_{|J|}e_{b]}^J \ = \ 0 \ . \tag{1.49}$$

This equation implies that the tetrad is covariantly constant with respect to the covariant derivative defined by ω.

1.5 The modified Palatini action

In previous section, it was seen that within the Palatini formalism, the form of constraint equations is much simpler than in Einstein-Hilbert approach we have discussed in the context of ADM formalism. However, their form is rather complicated. Furthermore, one can find that, the constraints are obtained in Palatini approach are not closed under Poisson brackets. This complicates the quantization of the theory in such a way that, the palatini formalism is little better than the Einstein-Hilbert one for the purposes of quantum gravity.

The new variables can be thought of as modification of the Palatini formalism that avoid this problem. The main idea is to take advantage of special features of 4-dimensional space-time and work with "self dual part" of the Lorentz connection. We will explain this notion in the following part.

1.5.1 The self-dual action

As with Maxwell's equations, using self-duality in gravity when metrics in Lorentzian requires working with complex-values fields. Thus, we define the *complexified tangent bundle* of M, written $\mathbb{C}TM$, to be the vector bundle whose fiber at each point $p \in M$ is the vector space $\mathbb{C} \otimes T_pM$ consisting of complex linear combinations of tangent vectors. There is also an 'imitation' complexified tangent bundle, namely the trivial bundle $M \times \mathbb{C}^4$. A *complex frame field* is then a vector bundle isomorphism:

$$e: \ M \times \mathbb{C}^4 \ \longrightarrow \ \mathbb{C}TM \ .$$

We define the internal metric η on $M \times \mathbb{C}^4$ by same formula as for $M \times \mathbb{R}^4$. This allows us to raise and lower internal indices. A connection ω on $M \times \mathbb{C}^4$ is an $End(\mathbb{C}^4)$-values 1-form on M. Its components are written ω_{aJ}^I, where a is space-time index and I, J are internal indices. Alternatively, we can raise an index and think of the connection as having components ω_a^{IJ}. We say that, ω is a Lorentz connection if $\omega_a^{IJ} = -\omega_a^{JI}$. Because of this antisymmetric property, we can think of a Lorentz connection as a $\Lambda^2\mathbb{C}^4$-valued 1-form. Recall that, the Hodge star operator maps 2-form to 2-form in 4 dimensions, which is the basis of duality symmetry.

There is an analogous *internal Hodge star operator* mapping $\Lambda^2\mathbb{C}^4$ to itself: denoting it by '$\star$', it is given by

$$\star T^{IJ} \;=\; \frac{1}{2}\epsilon^{IJ}{}_{KL}T^{KL}, \tag{1.50}$$

for any quantity with two antisymmetric raised internal indices, by analogy with formula for the usual Hodge star operator. In particular, we can define the 'internal Hodge dual' of a Lorentz connection by

$$(\star\,\omega)_a^{IJ} \;=\; \frac{1}{2}\epsilon^{IJ}{}_{KL}\omega_a^{KL}\;, \tag{1.51}$$

and we can write any Lorentz connection ω as a sum of self-dual and anti-self-dual parts:

$$\omega \;=\; \omega^+ + \omega^-, \qquad \star\,\omega^\pm \;=\; \pm i\;\omega^\pm. \tag{1.52}$$

Explicitly, we have

$$\omega^\pm \;=\; \frac{1}{2}(\omega \mp i\star\omega). \tag{1.53}$$

In the self-dual formulation of general relativity, one of the two basic fields is a self-dual Lorentz connection, that is, a Lorentz connection ω^+ on $M\times\mathbb{C}^4$ with

$$\star\omega^+ \;=\; i\omega^+. \tag{1.54}$$

The other basic field is a complex frame field:

$$e : M\times\mathbb{C}^4 \longrightarrow \mathbb{C}TM.$$

The action in the self-dual formulation is built using the curvature of the self-dual Lorentz connection, which is written F and given by

$$F_{\alpha\beta}^{IJ} \;=\; \partial_\alpha\omega_\beta^{+IJ} - \partial_\beta\omega_\alpha^{+IJ} + [\omega_\alpha^+,\omega_\beta^+]^{IJ}. \tag{1.55}$$

As in the Palatini formalism, one can use the frame field to define a metric g on M by

$$g_{\alpha\beta} \;=\; \eta_{IJ}e_\alpha^I e_\beta^J,$$

where the coefficients e_α^I are using the inverse frame field:

$$e^{-1}\partial_\alpha \;=\; e_\alpha^I\xi_I.$$

However, since the triad is complex, the metric g is complex as well. The *self-dual action* then is given by:

$$S_{\mathrm{SD}}[\omega^+,e] \;=\; \int_M e_I^\alpha\; e_J^\beta\; F_{\alpha\beta}^{\;IJ}\;\mathrm{vol}. \tag{1.56}$$

1.6 The Holst action for general relativity

Using the internal Hodge star operator we will be led to our next reformulation of the gravitational action by replacing F_{ab}^{KL} by

$$(\star\, F)_{ab}^{IJ} \;=\; \frac{1}{2}\epsilon^{IJ}{}_{KL}F_{ab}^{KL}, \tag{1.57}$$

in the Palatini action (1.41). It can be shown that the compatibility condition is satisfied, and we are thus still dealing with the connection 1-forms preserving the tetrad. Therefore, in the presence of the extra term $\epsilon^{IJ}{}_{KL}$, the solution space is extremely enlarged and contains all pairs of connections and tetrads compatible with each other. This allows us to generalize the Palatini action as

$$\boxed{S[e,\omega] \;=\; \frac{1}{16\pi G}\int_M d^4x\; |e|\; e_I^a\; e_J^b\; P^{IJ}{}_{KL}\; F_{ab}^{IJ}(\omega)} \tag{1.58}$$

with $P^{IJ}{}_{KL}$ being defined as

$$P^{IJ}{}_{KL} \;:=\; \delta_K^{[I}\delta_L^{J]} - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}\;. \tag{1.59}$$

In this case, the connection variation then provides the equation

$$\epsilon^{abcd}\epsilon_{IJKL}P^{KL}{}_{MN}\mathcal{D}_a(e_c^M\; e_d^N) \;=\; 0, \tag{1.60}$$

and still results in the compatibility condition. This equivalence to the previous conditions is most easily seen by noting that the matrix $P^{IJ}{}_{KL}$ interpreted as a mapping from the tensor product of two Minkowski spaces into itself. Varying by the tetrad provides an equation with $\epsilon^{Iabc}R_{abIL}$ added to Einsteins equation. Again, this extra term vanishes by symmetries of the Riemann tensor. Thus, irrespective of the value of γ, we produce the same equations of motion. The action used here is called the *Holst action*, and γ the Barbero-Immirzi parameter.

2 Ashtekar formulation of general relativity

We have seen in the ADM formulation of general relativity that, a space-time is foliated in a canonical formulation.

2.1 Space-time foliation; triad formalism

Similar to the spatial canonical metric h_{ab} we considered in ADM formalism, let us introduce now a spatial space-time tensor field

$$\mathcal{E}_I^a \;=\; e_I^a + n^a n_I\;, \tag{2.1}$$

with the unit normal n^a to spatial slice and $n_I := e^a_I n_a$; this field satisfies $\mathcal{E}^a_I n_a = \mathcal{E}^a_I n^I = 0$ and thus can be considered a *spatial triad.*

It should be noticed that, in the triad formalism for the herein canonical system, there is an additional condition to the usual decomposition of space-time tensors in normal and spatial parts; it is to split the internal directions of the tetrad in Minkowski time and space components. To do this, let us consider a partial gauge fixing of the internal $SO(3,1)$-transformations, the so-called time gauge. We then fix the boost part of internal Lorentz transformations by requiring $e^a_0 = n^I e^a_I = n^a$ to be the unit normal to the foliation (we assumed $n^I = \delta^I_0$ to be a timelike internal vector field). In this case, the tetrad e^a_I becomes the local internal frame of Eulerian observers; the partial gauge fixing is thus natural from the perspective of observables in a canonical formulation arising as those with respect to Eulerian observers. With the partial gauge fixing, internal Minkowski transformations are reduced to spatial rotations by requiring them to fix the chosen n^I. When directly referring only to the spatial rotation part of the group, we will use lower-case internal indices such as $\mathcal{E}^a_i$ for the spatial triad.

Let us now consider the unit normal vector n^a as

$$n^a \;=\; N^{-1}(t^a - N^a)\ . \tag{2.2}$$

Then, the tread e^a_I in terms of the spatial triad $\mathcal{E}^a_I$ and normal vectors reads

$$e^a_I \;=\; \mathcal{E}^a_I - N^{-1}(t^a - N^a)n_I\ .$$

Substituting these relations in the action (1.58), we can decompose the Palatini action in terms of spatial triads. Thus, by setting $|e| = N\sqrt{\det h}$ and noting the antisymmetry of $P^{IJ}{}_{KL}$, we obtain

$$\begin{aligned} S[e,\omega] \;=\; \frac{1}{16\pi G}\int d^4x\, \sqrt{\det h}\; P^{IJ}{}_{KL} F_{ab}{}^{KL}(\omega) \\ \times\left(N\mathcal{E}^a_I - 2n_I t^a + 2N^a n_I\right)\mathcal{E}^b_J\ . \end{aligned} \tag{2.3}$$

Comparing the action (2.3) with the ADM action we discussed previously, we expect that the first term in $N\mathcal{E}^a_I - 2n_I t^a + 2N^a n_I$ provides the Hamiltonian constraint, the last term the diffeomorphism constraint, and the middle term the symplectic structure with a derivative along t^a.

2.2 The Ashtekar-Barbero connection variables

Let start with an analysis of the symplectic term (i.e., the second term in (2.3)) to find the new canonical variables: We first introduce the purely spatial tensor:

$$\boxed{P^a_i \;:=\; \frac{\sqrt{\det h}}{8\pi\gamma G}\mathcal{E}^a_i} \tag{2.4}$$

Then, we can write the second term in (2.3) as

$$-\gamma\int d^4x\ P^{IJ}{}_{KL}F_{ab}{}^{KL}\Big(\frac{\sqrt{\det h}}{8\pi\gamma G}\mathcal{E}^b_J\Big)n_I t^a = -\gamma\int dx^4\ n_I t^a\ P^b_J\ P^{IJ}{}_{KL}F_{ab}{}^{KL}.$$

By replacing the most internal indices by the spatial ones, we obtain

$$\begin{aligned}-\gamma\int dx^4\ n_I t^a\ P^b_J\ P^{IJ}{}_{KL}F_{ab}{}^{KL} &= \gamma\int dx^4\ t^a\ P^b_j\Big(F_{ab}{}^{0j}-\frac{1}{2\gamma}\epsilon^{0j}{}_{KL}F_{ab}{}^{KL}\Big)\\ &= \gamma\int dx^4\ t^a\ P^b_j\Big(\partial_a\omega^{0j}_b-\partial_b\omega^{0j}_a+2\omega_{[a}{}^{0k}\omega_{b]k}^{\ \ j}\\ &\quad+\frac{1}{\gamma}\epsilon^j{}_{kl}\big(\partial_{[a}\omega^{kl}_{b]}+\omega_{[a}^{\ \ kK}\omega_{b]K}^{\ \ l}\big)\Big).\end{aligned}$$

Notice that, we have used $n_I=\eta_{IJ}n^J=\eta_{I0}=-\delta^0_I$ and $\epsilon_{0123}=1$. Simplifying this relation we find that the second term, after integrating by parts produce

$$\mathcal{L}_t\omega_b{}^{0j}\ =\ t^a\ \partial_b\omega^{0j}_a+\omega^{0j}_a\partial_b t^a\ ,$$

in combination with the first term. Similarly, the second line produce the Lie derivative of $\frac{1}{2\gamma}\epsilon^j{}_{kl}\omega^{kl}_b$. Since these are the only time derivatives appearing in the action, and since both of them are multiplied with γP^b_j, we can write the variable canonically conjugate to P^b_j as

$$A^i_a\ :=\ \frac{1}{2}\epsilon^i{}_{kl}\omega^{kl}_a+\gamma\omega^{0i}_a\ . \tag{2.5}$$

In order to interpret these components, let us consider the spatial covariant derivatives

$$\mathcal{D}_a v^i\ =\ \nabla_a v^i+h^b_a\ \omega_b{}^i{}_j v^j\ =\ \nabla_a v^i-\epsilon^i{}_{jk}\Gamma^j_a v^k\ , \tag{2.6}$$

where

$$\Gamma^i_a\ :=\ \frac{1}{2}\epsilon^i{}_{kl}\omega^{kl}_a\ , \tag{2.7}$$

is called the *spin connection.* The spin connection is an $so(3)$ connection that transforms in the standard inhomogeneous way under local $SO(3)$ transformations.

The second term, $K^i_a:=\omega^{0i}_a$, can be directly be computed from the compatible connection one-forms (1.22) (and using $e^0_d=\eta^{0I}g_{dc}e^c_I=-g_{dc}e^c_0=-n_d$):

$$\mathcal{E}_{ci}K^i_a\ =\ -h^b_a\ \mathcal{E}_{ci}\ \omega^{i0}_a\ =\ h^b_a\mathcal{E}_{ci}\ e^{di}\nabla_b e^0_d\ =\ h^b_a h^d_c\nabla_b n_d=K_{ac}\ ,$$

which is the extrinsic curvature contracted with spatial triad. Therefore, the canonical variable (2.5) can be written as

$$\boxed{A_a^i \;:=\; \Gamma_a^i + \gamma K_a^i} \tag{2.8}$$

This is called the *Ashtekar-Barbro connection*. This variable is also an $so(3)$ connection as adding a quantity that transforms as a vector to a connection gives a new connection. The remarkable fact regarding this new variable is that, it is in fact conjugate to P_a^i. Then, our canonical variables for general relativity are (A_a^i, P_j^b) and preserve the Poisson brackets:

$$\{P_j^a(x), A_b^i(y)\} \;=\; 8\pi G \delta_b^a \delta_j^i \delta(x,y), \tag{2.9}$$

$$\{P_j^a(x), P_i^b(y)\} \;=\; \{A_a^j(x), A_b^i(y)\} \;=\; 0\ . \tag{2.10}$$

Now, let us continue decomposition of the action, by writing the remaining terms in the contribution containing t^a:

$$\begin{aligned}
&\int dx^4\ t^a \Big[\gamma \omega_a^{0j}\Big(\partial_b P_j^b + \omega_{bj}{}^k P_k^b - \frac{1}{\gamma}\epsilon^k{}_{jl}\omega_{b0}^{\ l} P_k^b\Big) \\
&\qquad + \frac{1}{2}\epsilon^j{}_{kl}\omega_a{}^{kl}\Big(\partial_b P_j^b - \frac{1}{2}\epsilon_j{}^{nm}\epsilon_{nqp}\omega_b{}^{qp}P_m^b + \gamma \epsilon_{nj}{}^m \omega_b{}^{0n}P_m^b\Big)\Big] \\
&= \int dx^4\ t^a \Big(\Lambda^j \mathcal{D}_b^{(A)} P_j^b + (1+\gamma^2)\epsilon_{jm}{}^n \omega_t{}^{0j}\omega_b{}^{0m}P_n^b\Big),
\end{aligned}$$

where $\mathcal{D}^{(A)}$ is the covariant derivative using Ashtekar-Barbero connection, and Λ^j is introduced as

$$\boxed{\Lambda^j \;:=\; \frac{1}{2}\epsilon^j{}_{kl}\ \omega_t^{kl} + \gamma \omega_t{}^{0j}} \tag{2.11}$$

The components Λ^j and $\omega_t{}^{0j}$ do not appear with time derivatives in the action; their momenta are thus constrained to vanish, and they provide Lagrange multipliers of secondary constraints. These secondary constraints are the *Gauss constraints*:

$$\boxed{G_j \;:=\; \mathcal{D}_b^{(A)} P_j^b \;=\; 0} \tag{2.12}$$

and

$$S_j \;:=\; \epsilon_{jm}{}^n\ \omega_b{}^{0m} P_n^b \;=\; \epsilon_{jm}{}^n K_b^m P_n^b \;=\; 0\ , \tag{2.13}$$

which ensures that $K_{ab} := K_a^i \mathcal{E}_{bi}$ satisfies

$$0 \;=\; \epsilon^{ijk} K_b^i P_j^b \;=\; K_{ab}\epsilon^{ijk}\mathcal{E}_i^a P_j^b \;=\; \frac{1}{8\pi\gamma G} K_{ab}\varepsilon^{abc}\mathcal{E}_c^k\ . \tag{2.14}$$

The diffeomorphism constraint follows from terms proportional to N^a in the action (2.3):

$$\begin{aligned}
N^a C_a^{\text{grav}} &= -\gamma n_I N^a P_j^b \; P^{IJ}{}_{KL} F_{ab}{}^{KL} \\
&= \gamma N^a P_j^b \left(F_{ab}{}^{0j} - \frac{1}{2\gamma} \epsilon^{0j}{}_{kl} F_{ab}{}^{kl} \right) \\
&= 2\gamma N^a P_j^b \left[\partial_{[a} \omega_{b]}{}^{0j} + \omega_{[a}{}^{0k} \omega_{b]k}{}^{j} + \frac{1}{2\gamma} \left(2\partial_{[a} \Gamma_{b]}^j + \epsilon^j{}_{kl} \omega_{[a}{}^{kL} \omega_{b]}{}^{l}{}_{L} \right) \right] \\
&= N^a P_j^b \left[2\partial_{[a} A_{b]}^j - \gamma \epsilon^j{}_{mk} \Gamma_{[a}^m \; K_{b]}^k - \frac{1}{2} \epsilon^j{}_{kl} \left(\Gamma_{[a}^k \; \Gamma_{b]}^l - K_{[a}^k \; K_{b]}^l \right) \right] \\
&= N^a P_j^b \left(\mathcal{F}_{ab}^j + (1+\gamma^2) \epsilon^j{}_{kl} K_a^k K_b^l \right),
\end{aligned} \tag{2.15}$$

where $\mathcal{F}_{ab}^j$ is the curvature of the Ashtekar-Barbero connection:

$$\mathcal{F}_{ab}^i \; = \; \partial_a A_b^i - \partial_b A_a^i - \epsilon^i{}_{kl} A_a^k A_b^l \; . \tag{2.16}$$

Notice that, the curvature $\mathcal{F}_{ab}^j$ in terms of the curvature of the spin connection, F_{ab}^i, can be written as

$$\begin{aligned}
\mathcal{F}_{ab}^i &= 2\partial_{[a} A_{b]}^i - \epsilon^i{}_{kl} A_a^k A_b^l \\
&= 2\partial_{[a} (\Gamma_{b]}^i + \gamma K_{b]}^l) - \epsilon^i{}_{kl} (\Gamma_a^k + \gamma K_a^k)(\Gamma_b^l + \gamma K_b^l) \\
&= F_{ab}^i + 2\gamma \mathcal{D}_{[a} K_{b]}^i - \gamma^2 \epsilon^i{}_{kl} K_a^k K_b^l \; ,
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
F_{ab}^i \; := \; \frac{1}{2} \epsilon^i{}_{kl} F_{ab}{}^{kl} &= 2\partial_{[a} \Gamma_{b]}^i + \epsilon^i{}_{kl} \epsilon^k{}_{mn} \epsilon^{ml}{}_j \; \Gamma_{[a}^n \; \Gamma_{b]}^j \\
&= 2\partial_{[a} \Gamma_{b]}^i - \epsilon^i{}_{kl} \Gamma_a^k \; \Gamma_b^l \; .
\end{aligned} \tag{2.18}$$

The Hamiltonian constraint follows from the term proportional to N in the action (2.3):

$$\begin{aligned}
NC^{\rm grav} &= -4\pi G\gamma^2 N\frac{P_i^aP_j^b}{\sqrt{\det h}}P^{ij}{}_{KL}F_{ab}{}^{KL} \\
&= -4\pi G\gamma^2 N\frac{P_i^aP_j^b}{\sqrt{\det h}}\Big(F_{ab}{}^{ij}-\frac{1}{2\gamma}\epsilon^{ij}{}_{KL}F_{ab}{}^{KL}\Big) \\
&= -4\pi G\gamma^2 N\frac{P_i^aP_j^b}{\sqrt{\det h}}\Big[2\partial_{[a}\omega_{b]}{}^{ij}+2\omega_{[a}{}^{iK}\omega_{b]}{}^{Lj}\eta_{KL} \\
&\quad +\frac{2}{\gamma}\epsilon^{ij}{}_k\Big(\partial_{[a}\omega_{b]}{}^{k0}+\omega_{[a}{}^{kl}\omega_{b]l}^{0}\Big)\Big] \\
&= -4\pi G\gamma^2 N\frac{P_i^aP_j^b}{\sqrt{\det h}}\Big(F_{ab}^{ij}+2K^i_{[a}\ K^j_{b]}-\frac{2}{\gamma}\epsilon^{ij}{}_k\mathcal{D}_{[a}K^k_{b]}\Big) \\
&= -4\pi G\gamma^2 N\frac{P_i^aP_j^b}{\sqrt{\det h}}\epsilon^{ij}{}_k\Big(\mathcal{F}^k_{ab}+(1+\gamma^2)\epsilon^k{}_{mn}K^m_aK^n_b \\
&\quad -2\frac{1+\gamma^2}{\gamma}\epsilon^{ij}{}_k\mathcal{E}^a_iP^b_j\mathcal{D}_{[a}K^k_{b]}\Big).
\end{aligned} \tag{2.19}$$

From the constraint equations (2.12), (2.13), (2.15) and (2.19), we can write the total Hamiltonian as a sum of constraints:

$$\begin{aligned}
H_{\rm grav}[A^i_a,P^b_j] &= \int dx^3\ \Big(-\Lambda^iG_i-(1+\gamma^2)\omega_t^{0j}S_j \\
&\qquad +NC_{\rm grav}+N^aC_a^{\rm grav}\Big).
\end{aligned} \tag{2.20}$$

After solving all second class constraints, the total gravitational Hamiltonian (2.20) reduces to

$$H_{\rm grav}[A^i_a,P^b_j] = \int dx^3\ \Big(-\Lambda^iG_i+NC_{\rm grav}+N^aC_a^{\rm grav}\Big). \tag{2.21}$$

which is written in terms of *only* first class constraints; the G_i is the *Gauss constraints* given by Eq. (2.12). The *Hamiltonian constraint* reads

$$\boxed{C^{\rm grav} = -4\pi G\gamma^2\frac{P_i^aP_j^b}{\sqrt{\det h}}\epsilon^{ij}{}_k\Big(\mathcal{F}^k_{ab}+(1+\gamma^2)\epsilon^k{}_{mn}K^m_aK^n_b\Big)} \tag{2.22}$$

and the *diffeomorphism constraint*:

$$\boxed{C_a^{\rm grav} = P^b_j\ \mathcal{F}^j_{ab}} \tag{2.23}$$

Now, we have seven (first class) constraints for the 18 phase space variables (A^i_a,P^b_j). In addition to imposing conditions among the canonical variables, first class constraints are generating functionals of (infinitesimal) gauge transformations. From the 18-dimensional phase space of general relativity, we end up with 11 fields necessary to coordinatize the constraint surface on which the above seven conditions hold. On that 11-dimensional constraints surface, the above constraint generate a seven-parameter-family of gauge transformations.

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