Wormholes and black universes: phantoms and stability

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Black Holes and their Analogues: 100 years of General Relativity
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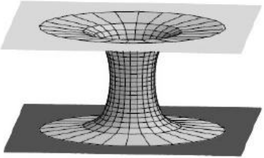
Motivation/Plan

- Search for non-singular BH-like solutions in classical gravity
- Exact solutions combining properties of BHs, WHs, and non-singular cosmological models
- Phantom scalar fields in the context of the accelerated cosmological expansion (estimates give $w \lesssim -1$, e.g., Planck-2015)
- Possible existence of a global primordial magnetic field ($\sim 10^{-15}$ G) causing correlated orientations of quasars distant from each other [R. Poltis and D. Stojkovic, Can primordial magnetic fields seeded by electroweak strings cause an alignment of quasar axes on cosmological scales? Phys. Rev. Lett. 105, 161301 (2010); arXiv: 1004.2704.] Possible global anisotropy
- Different geometric and causal structures and their connection with possible cosmological scenarios
- Phantom fields are not observed $\Rightarrow$ "trapped ghost" and "invisible ghost" concepts. Stability problem for a "trapped ghost"
- Stability problem in a more general context

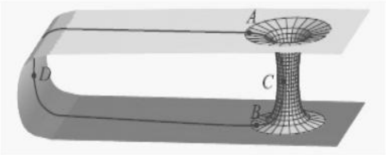
What is a wormhole?

Everybody knows

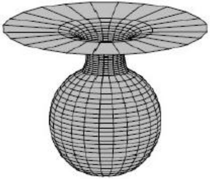
Wormholes (spherical vs. cylindrical)



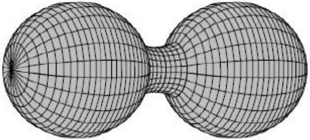
Sph: twice asymptotically flat wormhole (topology $S^2 \times R$).
Cyl: topology $S^1 \times R \times R$



A wormhole as a "handle", a **shortcut** between remote parts of the Universe (or a **time machine** if times at A and B are essentially different)



A "hanging drop" wormhole.
Topology:
R^3 for both **Sph** and **Cyl**

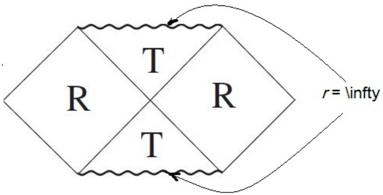


A "dumbbell" wormhole
Sph: topology S^3
Cyl: topology $S^2 \times R$

What is a black universe?

A black universe (BU) is a regular black hole where, beyond the horizon, instead of a singularity, there is an expanding, asympt. isotropic space-time.

The simplest BU: Schwarzschild-like structure



It combines the properties of the following objects:

- A black hole (BH) — a Killing horizon separating static and non-static spacetime regions;
- A wormhole (WH) — no center and a regular minimum of the area of coordinate spheres
- A nonsingular cosmological model — cosmological evolution begins from a horizon (Null Big Bang). At large times the nonstatic region reaches a de Sitter (dS) mode of isotropic expansion.

Phantom matter: *pro et contra*

Both WHs and black universes need exotic (phantom) matter as a source (at least in GR). We here assume their existence as a working hypothesis.

Main objection: if quantized, such a field leads to a catastrophe due to runaway particle production.
If it interacts with another, usual field, its particle-antiparticle pair creation adds negative energy to the phantom field itself $\Rightarrow$ again a catastrophe.

Way to avoid: (i) assume that there is no direct interaction with other field except classical gravity; (ii) no quantization, it should be an effective classical field originating from some underlying theory.

They are not observed in usual conditions (in cosmology — not surely). Maybe they exist only in a strong field region?
In addition, phantoms appear:

- in some string theory models [A. Sen. 2002]
- in supergravities [H. Nilles, 1984]
- in high-D models ($D > 11$) [N. Khviengia et al., 1998]

The model

Action[1]: $S = \frac{1}{2}\int \sqrt{-g}\, d^4x \Big[R + 2\epsilon g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - F^{\mu\nu}F_{\mu\nu} - 2V(\phi)\Big]$

- $F_{\mu\nu}$ — electromagnetic field tensor;
- ϕ — phantom scalar field ($\epsilon = -1$) with a potential $V(\phi)$.

General (formally) static sph. symm. metric in quasiglobal gauge ($g_{00}g_{11} = -1$):

$$ds^2 = A(u)dt^2 - \frac{du^2}{A(u)} - r^2(u)d\Omega^2$$

$u \in \mathbb{R}$ — quasiglobal coordinate, $\partial\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$,
$r(u)$ — "area function",
$A(u)$ — "redshift function",
$A(u) > 0 \quad \Rightarrow \quad$ R-region,
$A(u) < 0 \quad \Rightarrow \quad$ T-region.

[1] $\hbar = c = 8\pi G = 1$, signature $(+\ -\ -\ -)$

Properties of functions $A(u)$, $r(u)$ for WHs and BUs

- Both $A(u)$ and $r(u)$ should be regular, $r(u) > 0$ everywhere, and $r(\pm\infty) \to \infty$ there is a minimum of $r(u)$ — throat or bounce.
- Regions with $A > 0$ (R-regions) $\Rightarrow$ static, with $A < 0$ (T-regions) $\Rightarrow$ cosmological (Kantowski-Sachs).
- Flat, de Sitter or AdS asymptotic behavior as $u \to \pm\infty$:
  - **WH**: no horizons ($A(u) > 0$ everywhere), and flat or AdS asymptotics at both ends.
  - **BU**: flat or AdS asymptotic at one end; de Sitter asymptotic at the other end.

- **Scalar field** $\phi(x)$:

$$T_\mu^\nu[\phi] = \epsilon A(u)\phi'(u)^2\,\mathrm{diag}(1,\ -1,\ 1,\ 1) + \delta_\mu^\nu V(u)$$

$\epsilon = -1$ — phantom field

- **Electromagnetic field** $F_{\mu\nu}(x)$ (conforms to Wheeler's idea of a "charge without charge"):

$$F_{01} = -F_{10}\ (\mathrm{electric}), \quad F_{01}F^{01} = -q_e^2/r^4(u)$$

$$F_{23} = -F_{32}\ (\mathrm{magnetic}), \quad F_{23}F^{23} = q_m^2/r^4(u)$$

$$T_\mu^\nu[F] = \frac{q^2}{r^4(u)}\,\mathrm{diag}(1,\ 1,\ -1,\ -1), \qquad q^2 = q_e^2 + q_m^2.$$

(Maxwell's equations have been solved)

Einstein and scalar field equations

There are three independent equations [Eqs. (4), (5) follow from (1)–(3)] for 4 functions $r(u)$, $A(u)$, $V(\phi)$, $\phi(u)$:

$$
\begin{aligned}
r''/r &= -\epsilon\phi'^2, && (1)\\
(A'r^2)' &= -2r^2 V + 2q^2/r^2, && (2)\\
A(r^2)'' - r^2 A'' &= 2 - 4q^2/r^2, && (3)\\
2(Ar^2\phi')' &= \epsilon r^2 dV/d\phi, && (4)\\
-1 + A'rr' + Ar'^2 &= r^2(\epsilon A\phi'^2 - V) - q^2/r^2. && (5)
\end{aligned}
$$

Possible approaches:

- Specify the potential $V(\phi)$, find the functions r, A, ϕ (very hard technically).
- Specify $r(u)$, find the functions V, A, ϕ (*inverse problem method*)

To find examples of solutions possessing particular properties, the inverse problem method is quite suitable.

Solution

- Choose a function $r(u)$ that can provide WH and BU solutions:

$$r = (u^2 + b^2)^{1/2} \equiv b\sqrt{1 + x^2}, \quad x \equiv u/b, \quad b = \mathrm{const} = 1$$

- Integrate Eq. (3) twice, denoting $B(x) \equiv A/r^2$:

$$B(x) = B_0 + \frac{1 + q^2 + px}{1 + x^2} + \left(p + \frac{2q^2 x}{1 + x^2}\right) \arctan x + q^2 \arctan^2 x$$

- Fix the integration constants B_0 and p using the asymptotic flatness condition $\lim\limits_{x \to +\infty} B = 0$ and comparing asymptotic of $A(x)$ with the Schwarzschild-like one, $A(x) \simeq 1 - 2m/x$:

$$B_0 = -\frac{\pi p}{2} - \frac{\pi^2 q^2}{4}, \qquad p = 3m - \pi q^2$$

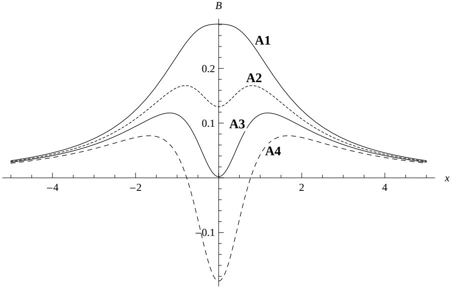
Solution 2

- There is a two-parametric family of curves $B(x, q, m)$ with the parameters q, m.
- For the scalar field and potential from Eqs. (1), (2) we have:

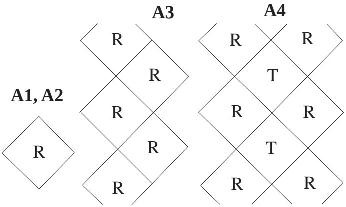
$$\phi = \pm\sqrt{2}\arctan x + \phi_0;$$

$$V = \frac{q^2}{(1+x^2)^2} - \frac{1}{2(1+x^2)}\Big[2q^2\arctan^2 x\,(3x^2+1) + \arctan x\Big(18x^2 m - 6\pi x^2 q^2 + 12xq^2 - 2\pi q^2 + 6m\Big) + q^2\left(\frac{3}{2}\pi^2 x^2 - 6\pi x + \frac{1}{2}\pi^2 + 6\right) - m(9\pi x^2 - 18x + 3\pi)\Big].$$

A: Symmetric asympt. flat configurations ($p = 0$)

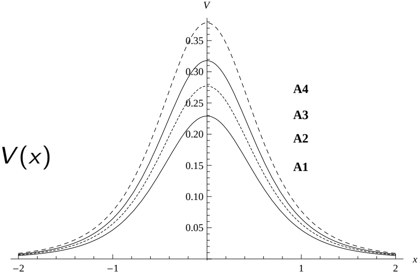


B(x) curves

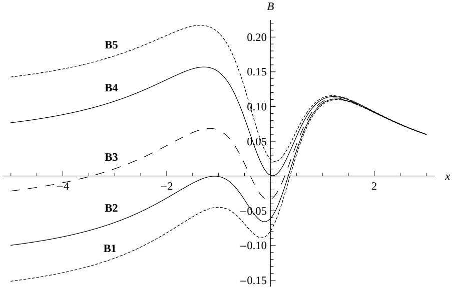


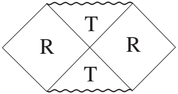
Carter-Penrose diagrams

Potentials $V(x)$

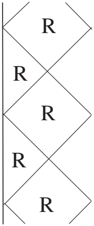


Asymmetric asymptotically flat configurations (B) 1



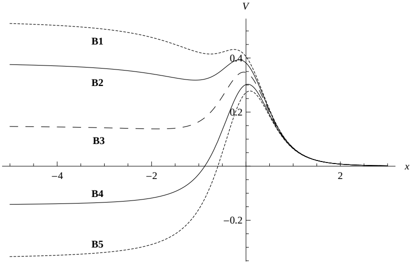


B1

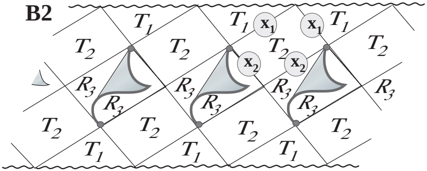


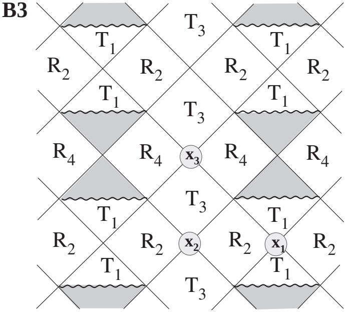
B4

Potentials $V(x)$



Asymmetric asymptotically flat configurations (B) 2

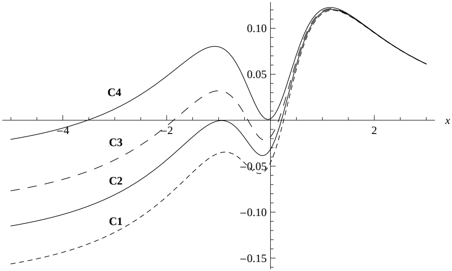


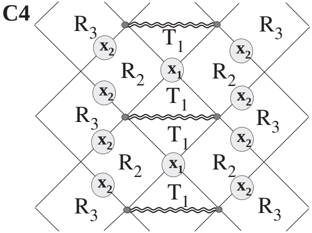


Here and further on: indices near R and T count regions along the x axis in the corresponding plot of $B(x)$ (left to right).

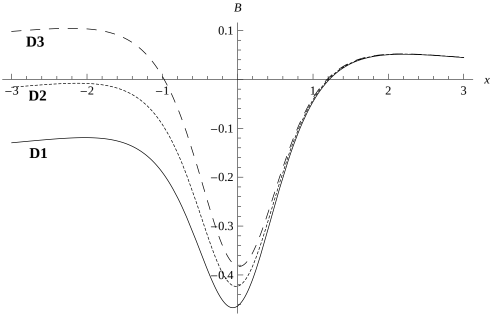
x_i are roots of $B(x) \mapsto$ horizons (left to right).

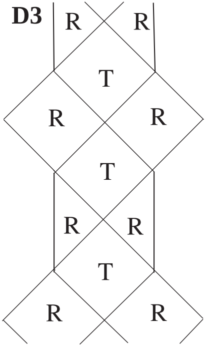
Asymmetric asymptotically flat configurations (C)



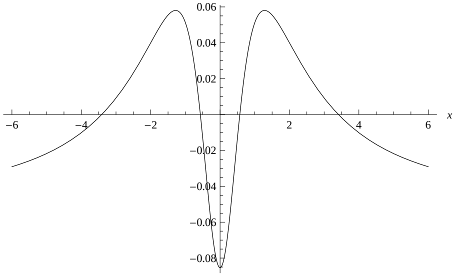


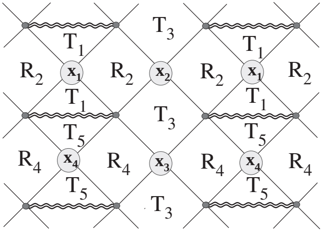
Asymmetric asymptotically flat configurations (D)





Example of a symmetric asymptotically non-flat (dS–dS) configuration with 4 horizons





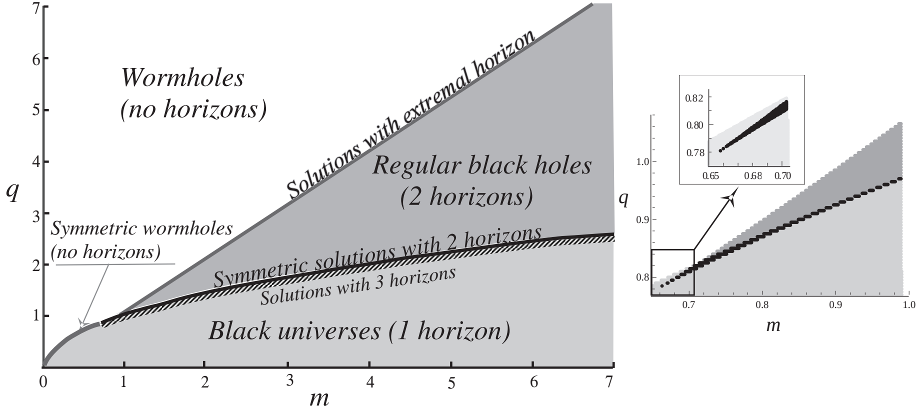
Asymptotically flat solutions with $m > 0$

Solution type ($B(x)$ curve number)	Configuration $(x \to +\infty)$—$(x \to -\infty)$	Horizons: order n, R-T-region disposition
A1,A2	M - M WH	none [R]
A3	M - M extr. BH	1 hor., $n = 2$ [RR]
A4	M - M BH	2 hor., $n = 1$ [RTR]
B1, C1, C4, D1, D2	M - dS BU	1 hor., $n = 1$ [TR]
B2, C2	M - dS BU	2 hor., $n = 2; 1$ [TTR]
C4	M - dS BU	2 hor., $n = 1; 2$ [TRR]
B3, C3	M - dS BU	3 hor., $n = 1$, [TRTR]
D3	M - AdS BH	2 hor., $n = 1$ [RTR]
B4	M - AdS extr. BH	1 hor., $n = 2$ [RR]
B5	M - AdS WH	none [R]

Asymptotic behavior of $B(x, q, m)$ at $x \to -\infty$:

- $B(-\infty) = 0 \;\Rightarrow\;$ M
- $B(-\infty) < 0 \;\Rightarrow\;$ dS
- $B(-\infty) > 0 \;\Rightarrow\;$ AdS

Parametric map of asymptotically flat solutions on the (q, m) plane



Right: zoomed-in part of the map showing that configurations with 3 horizons are generic but occupy a very narrow band.

Estimates: BU and observational data

- Obs. limits on global magnetic fields: $10^{-9} \geq |\vec{B}| \geq 10^{-18}$ Gauss.
- Present scale factor $\approx 10^{28}$ cm, it corresponds to $r(u)$ in our metric.
- We take the conservative estimate: let $|\vec{B}| \geq 10^{-18}$ Gauss now.
  It evolves $\propto a^{-2} \Rightarrow$
  At recombination (since $a/a_0 \sim 10^{-3}$), $|\vec{B}| \sim 10^{-12}$ Gauss
  At baryogenesis ($a/a_0 \sim 10^{-12}$), $|\vec{B}| \sim 10^{6}$ Gauss
- Constraint on the energy density of a global magnetic field.
  Anisotropy at recombination (from CMB properties) $\sim 10^{-6}$.
  $\rho_{\rm CMB}/\rho_{\rm magn} = \rm const \Rightarrow$ at present $\rho_{\rm magn} \lesssim 10^{-39}$ g cm^{-3}
  $\Rightarrow |B| \lesssim 10^{-8}$ Gauss
  In reality it is much smaller (if any).
- Theoretical stability limit of a classical magn. field in Weinberg-Salam theory: $B \lesssim 10^{24}$ Gauss. Hence, the corresponding min $r(u) \approx 10^{7}$ cm ~ 100 km.

  B might be larger, but then its classical description fails.

Trapped ghosts

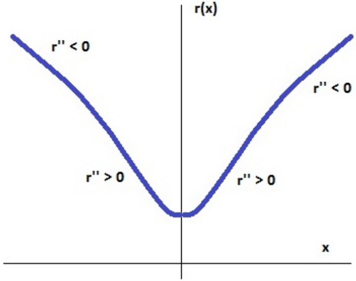
Problem: phantom fields are not observed.

Suggested solution: a field ϕ in the action

$$S = \frac{1}{2}\int \sqrt{-g}\, d^4x \Big[R + 2h(\phi) g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - F^{\mu\nu}F_{\mu\nu} - 2V(\phi)\Big],$$

($h(\phi)$ is a smooth function) which is phantom ($h(\phi) < 0$) in a strong-field region (say, $|\phi| > \phi_{\rm crit}$) and normal otherwise.

How to realize it in our problem setting? Recall that in our metric
$r'' > 0 \ \Rightarrow$ phantom,
$r'' > 0 \ \Rightarrow$ normal field



Trapped ghosts — 2

Field equations:

$$
\begin{aligned}
2(Ar^2 h\phi')' - Ar^2 h'\phi' &= r^2 dV/d\phi, && (6)\\
(A'r^2)' &= -2r^2 V + 2q^2/r^2; && (7)\\
r''/r &= -h(\phi)\phi'^2; && (8)\\
A(r^2)'' - r^2 A'' &= 2 - 4q^2/r^2, && (9)\\
-1 + A'rr' + Ar'^2 &= r^2(hA\phi'^2 - V) - q^2/r^2. && (10)
\end{aligned}
$$

Ansatz for $r(u)$ realizing the trapped ghost idea $[x := u/a]$:

$$r(u) = a\frac{x^2+1}{\sqrt{x^2+n}}, \quad n = \text{const} > 2, \quad a = \text{const} > 0. \tag{11}$$

Since $r''(x) = \dfrac{1}{a}\dfrac{x^2(2-n)+n(2n-1)}{(x^2+n)^{5/2}}$, we have

$r'' > 0$ at $x^2 < n(2n-1)/(n-2)$ and $r'' < 0$ at larger $|x|$,

as required; also, $r \approx a|x|$ at large $|x|$.

Trapped ghosts — solutions

Solutions ($n = 3$ for definiteness):

$$B(x) = B_0 + \frac{26 + 24x^2 + 6x^4 + 3px(69 + 100x^2 + 39x^4)}{6(1+x^2)^3} + \frac{39p}{2}\arctan x$$

$$+ \frac{q^2[107 + 383x^2 + 375x^4 + 117x^6 + 6x(69 + 169x^2 + 139x^4 + 39x^6)\arctan x]}{9(1+x^2)^4}$$

$$+ 13q^2 \arctan^2 x,$$

where p and B_0 are integration constants. Asymptotic flatness $\Rightarrow$

$$B_0 = -\frac{13}{4}\pi(3p + \pi q^2). \qquad p = m - \frac{2}{3}\pi q^2.$$

Thus B is a function of x and two parameters, m = mass and q = charge. Other unknowns are, as before, found from the field equations. Using the arbitrariness in ϕ definition, it is convenient to assume

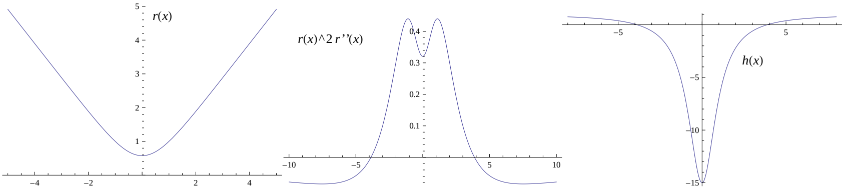
$$\phi(x) = \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}},$$

so that ϕ has a finite range: $\phi \in (-\phi_0, \phi_0)$. Then,

$$h(\phi) = \frac{x^2 - 15}{x^2 + 1} = \frac{3\tan^2(\sqrt{3}\phi) - 15}{3\tan^2(\sqrt{3}\phi) + 1}.$$

The Null Energy Condition is violated only where $h(\phi) < 0$.

Trapped ghosts — solutions 2



Plots of $r(x)$ (left), $r^2\, r''(x)$ (middle) and $h(x)$ (right) for $n = 3$.

The diversity of configurations described by the solutions, depending on the parameters m and q, is similar to that with a "visible" ghost.

As before, shifts in B_0 allow for obtaining non-asymptotically flat configurations.

Invisible ghosts

Another possible explanation of the unobservability of ghosts is their short range, i.e., a sufficiently rapid decay at large distances.

Example with two fields, normal (ϕ) and phantom (ψ):

$$\mathcal{S} = \frac{1}{16\pi}\int \sqrt{-g}d^4x\Big[R + 2h_{ab}(\partial\phi^a, \partial\phi^b) - 2V(\phi^a) - F_{\mu\nu}F^{\mu\nu}\Big],$$

in the same static, spherically symmetric space-time, where $\{\phi^a\}$ is a set of scalar fields, $h_{ab} = h_{ab}(\phi^a)$ is a nondegenerate target space metric, $(\partial\phi^a, \partial\phi^b) \equiv g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$, and $V(\phi^a)$ is an interaction potential.
The same static, spherically symmetric problem as before but now with two fields: $\phi^1 = \phi$ and $\phi_2 = \psi$, with $h_{ab} = \mathrm{diag}(1, -1)$.
One of the equations reads $r''/r = -\phi'^2 + \psi'^2$.
We choose the same $r(x)$ as for a trapped ghost($x = u/a$):

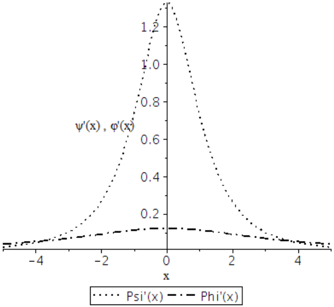
$$r(u) = a\frac{x^2+1}{\sqrt{x^2+n}}, \quad n = \mathrm{const} > 2, \quad a = \mathrm{const} > 0$$

Then we obtain the same set of geometries as before. Different is the nature and behavior of the scalar fields.

For the scalar ϕ (using the arbitrariness) we assume

$$\phi(x) = K \arctan(Lx),$$

where K and L are adjustable constants. We choose K and L in such a way as to make the phantom field ψ decay at large x more rapidly than ϕ. Thus, for $n = 4$ we should take $K = 2/\sqrt{23}, \qquad L = \sqrt{2/23}$, then $\psi' \sim x^{-4}$ while $\phi' \sim x^{-2}$ at large $|x|$.



The quantities ϕ' and ψ' in the strong and weak field regions

Perturbations

Problem: possible troubles at $\phi = \phi_{\rm crit}$ where $h(\phi) = 0$: *instability*.
Let us begin with a slightly more general action than before:

$$\mathcal{S} = \frac{1}{16\pi} \int \sqrt{-g} d^4x \Big[R + 2h(\phi)(\partial\phi)^2 - 2V(\phi) - S(\phi) F_{\mu\nu} F^{\mu\nu} \Big], \qquad (12)$$

with an arbitrary function $S(\phi) > 0$, characterizing a dilatonic-type interaction.

Ansätze for the metric and fields:
Metric: $ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} du^2 - e^{2\beta} d\Omega^2$ (arbitrary radial coordinate u);

$$\alpha = \alpha(u) + \delta\alpha(u,t); \qquad \beta = \beta(u) + \delta\beta(u,t); \qquad \gamma = \gamma(u) + \delta\gamma(u,t),$$

Scalar: $\phi = \phi(u) + \delta\phi(u,t)$;
Electromagn. field, admissible vector potential $A_\mu = \delta^0_\mu A_0 + \delta^3_\mu q_m \cos\theta + \partial_\mu \Phi$,
$\Phi = \Phi(x^\mu)$ = arbitrary function (usual gauge invariance).

Electromagn. field equations give for the electric field

$$S(\phi)\, e^{\alpha(u,t) + 2\beta(u,t) + \gamma(u,t)} F^{10} = q_e = \text{electric charge}. \qquad (13)$$

Let there be a static "background": $\alpha(u)$, $\beta(u)$, $\gamma(u)$, $\phi(u)$, $F_{\mu\nu}(u)$;
consider small perturbations described by "deltas".

Perturbations: effective potential

The only dynamic degree of freedom is connected with the scalar ϕ. Accordingly, metric perturbations $\delta\alpha$, $\delta\beta$, $\delta\gamma$ are excluded using the field equations $\Rightarrow$ wave equation for the varable $\psi(x,t) = \delta\phi\, \mathrm{e}^{\eta}$, $\eta = \beta + \frac{1}{2}\log|h|$:

$$\partial^2\psi/\partial t^2 - \partial^2\psi/\partial x^2 + V_{\rm eff}(x)\psi = 0, \tag{14}$$

master equation for radial perturbations,

$$V_{\rm eff}(x) = \mathrm{e}^{2\gamma-2\alpha}[U + \eta'' + \eta'(\eta' + \gamma' - \alpha')] \tag{15}$$

is the effective potential (the prime $= d/du$, the index ϕ means $d/d\phi$);

$$U \equiv \frac{1}{2}\phi'^2\left(\frac{h_\phi^2}{h^2} - \frac{h_{\phi\phi}}{h}\right) + \mathrm{e}^{2\alpha}\left[\frac{2h\phi'^2}{\beta'^2}(P - \mathrm{e}^{-2\beta}) + \frac{2\phi'}{\beta'}P_\phi + \frac{h_\phi}{2h^2}P_\phi - \frac{P_{\phi\phi}}{2h}\right].$$

$$P(\phi) := V(\phi) + Q(\phi)\,\mathrm{e}^{-4\beta}, \qquad Q(\phi) := q_e^2/S(\phi) + q_m^2 S(\phi).$$

Singularities in $V_{\rm eff}$ usually lead to instabilities (exponentially growing linear perturbations). Here, in addition to a "usual" singularity related to a throat ($\beta' = 0$), there emerge singularities where $h(\phi) = 0$.

Perturbations: gauge invariants

There are two independent forms of arbitrariness: (i) in choosing a radial coordinate u, (ii) perturbation gauge freedom $\equiv$ freedom to choose a reference frame in perturbed space-time by imposing a relation between $\delta\alpha$, $\delta\beta$, $\delta\phi, \ldots$

The above calculations used the gauge $\delta\beta = 0$ (simplest technically).

To be sure that we deal with real perturbations rather than purely coordinate effects, we must show that the results are gauge-invariant.

Small coordinate transformations $x^a \mapsto x^a + \xi^a$ in the (t, u) subspace:

$$t = \bar{t} + \Delta t(t,u), \qquad u = \bar{u} + \Delta u(t,u), \tag{16}$$

with small Δt and Δu. Any scalar with respect to such transformations, such as, e.g., $\phi(t,u)$ acquires an increment:

$$\Delta\phi = \dot{\phi}\Delta t + \phi'\Delta u \approx \phi'\Delta u \tag{17}$$

in linear approx. since both $\dot{\phi}$ and Δt are small. The radius r, also being a scalar in (t,u) subspace (a 2-scalar), behaves in the same way. The perturbations $\delta\phi$ and δr are transformed by (16) as follows:

$$\delta\phi \mapsto \overline{\delta\phi} = \delta\phi + \phi'\Delta u, \qquad \delta r \mapsto \overline{\delta r} = \delta r + r'\Delta u. \tag{18}$$

Hence the combination

$$\Psi \equiv r'\delta\phi - \phi'\delta r, \tag{19}$$

is invariant under the transformation (16), or gauge-invariant.

Perturbations: gauge invariants — 2

We have shown that the combination

$$\Psi \equiv r'\delta\phi - \phi'\delta r,$$

is gauge-invariant. But any combination constructed like that from any 2-scalars (for example, such as e^{ϕ} and $\beta = \ln r$, or two different linear combinations of ϕ and r) are also gauge-invariant. The same for Ψ multiplied by any 2-scalar or/and any combination of background quantities since they are known and fixed. In particular, ψ from the master equation (14):

$$\psi(x,t) = \delta\phi\, e^{\eta} = e^{\eta}(\delta\phi - \phi'\delta\beta/\beta')\Big|_{\delta\beta=0} = \Psi_1,$$

that is: ψ is the representative of the gauge invariant Ψ_1 in the gauge $\delta\beta = 0$. Other quantities in Eq. (14), including $V_{\rm eff}$, are made from background functions.

$\Rightarrow$ the master equation is gauge-invariant.

Stability of scalar-vacuum space-times

Scalar-vacuum space-times: the simplest among those described above.
Lagrangian:

$$L = \sqrt{-g}\Big(R + \epsilon g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - 2V(\phi)\Big), \tag{20}$$

$\epsilon = 1 \to$ normal scalar field, $\epsilon = -1 \to$ phantom scalar field.
Field equations:

$$\epsilon \Box\, \phi + V_\phi = 0, \qquad V_\phi \equiv dV/d\phi; \tag{21}$$

$$R_\mu^\nu = -\epsilon\phi_{,\mu}\phi^{,\nu} + \delta_\mu^\nu V(\phi). \tag{22}$$

As before, general spherically symmetric metric, arbitrary radial coordinate u:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \mathrm{e}^{2\gamma}dt^2 - \mathrm{e}^{2\alpha}du^2 - \mathrm{e}^{2\beta}d\Omega^2, \tag{23}$$

Effective potential near a throat

Perturbation equations reduce to the wave equation

$$\ddot{\psi} - \psi_{xx} + V_{\rm eff}(x)\psi = 0, \qquad (24)$$

where x is the "tortoise" coordinate ($\alpha + \gamma = 0$) in the static solution; the index $x \mapsto d/dx$. Effective potential:

$$V_{\rm eff}(x) = \mathrm{e}^{2\gamma-2\alpha}[U + \beta'' + \beta'^2 + \beta'(\gamma' - \alpha')],$$
$$U \equiv \mathrm{e}^{2\alpha}\left\{\epsilon(V - \mathrm{e}^{-2\beta})\frac{\phi'^2}{\beta'^2} + \frac{2\phi'}{\beta'}V_\phi + \epsilon V_{\phi\phi}\right\}. \qquad (25)$$

A further substitution $\psi(x,t) = y(x)\,\mathrm{e}^{i\omega t}$ (possible because the background is static), leads to the Schrödinger-like equation

$$d^2y/dx^2 + [\omega^2 - V_{\rm eff}(x)]y = 0. \qquad (26)$$

If (26) with proper bound. cond. leads to $\omega^2 < 0$, $\Rightarrow$ instability: $\psi \sim \mathrm{e}^{|\omega|t}$.

Near a throat $\beta' = 0$ (if any), $V_{\rm eff} \approx 2/(x - x_{\rm throat})^2$, a positive pole.
It is then impossible to consider the most "dangerous" perturbation, changing the throat radius $\Rightarrow$ regularization is necessary.

Regularized potential near a throat

Method: S-deformations of the potential $V_{\rm eff}$ [Ishibashi, Kodama, 2003; Gonzalez, Guzman, Sarbach, 2008]. Consider a wave equation

$$\ddot\psi - \psi_{xx} + W(x)\psi = 0, \qquad W(x) \text{ arbitrary} \tag{27}$$

(the above potential $V_{\rm eff}$ is an example). If we find a function $S(x)$ such that

$$W(x) = S^2(x) + S_x, \tag{28}$$

then Eq. (27) is rewritten as follows:

$$\ddot\psi + (\partial_x + S)(-\partial_x + S)\psi = 0. \tag{29}$$

Now, introduce the new function $\chi = (-\partial_x + S)\psi$ and apply the operator $-\partial_x + S$ to the l.h.s of Eq. (29) $\Rightarrow$ equation for χ:

$$\ddot\chi - \chi_{xx} + W_{\rm reg}(x)\chi = 0, \tag{30}$$

$$W_{\rm reg}(x) = -S_x + S^2 = -W(x) + 2S^2. \tag{31}$$

The new effective potential $W_{\rm reg}$ is indeed regular if $W(x) \approx 2/x^2$ near the throat $x = 0$, — and it is the case for $V_{\rm eff}$ (see above).

Main difficulty: solving the Riccati equation (28). Mostly numerical methods.

Static, spherical scalar-vacuum space-times

Metric: $ds^2 = A(x)dt^2 - dx^2/A(x) - r^2(x)d\Omega^2$.
Asympt. flatness at $x \to \infty$

- $V = 0$, $\epsilon = +1$: Fisher's solution (1948), naked singularity $r = 0$ at $x > 0$.
- $V = 0$, $\epsilon = -1$: "anti-Fisher" solutions [Bergmann, Leipnik, 1957].
  3 branches, all of them with a throat:
  - (A) Singularity at $x > 0$ with $A \to 0$, $r \to \infty$; "cold BHs" in special cases.
  - (B) Singularity at $x = 0$ with $A \to 0$, $r \to \infty$.
  - (C) Wormhole with another flat infinity at $x \to -\infty$ ("Ellis WH" as a special case).
- $V \neq 0$, $\epsilon = -1$, solutions with $r(x) = \sqrt{b^2 + x^2}$:
  - (i) Wormholes with AdS asympt. as $x \to -\infty$
  - (ii) Black universes with $m > 0$ (de Sitter as $x \to -\infty$).

In what follows: stability results for all of them.

Perturbations: Fisher; anti-Fisher, branch A

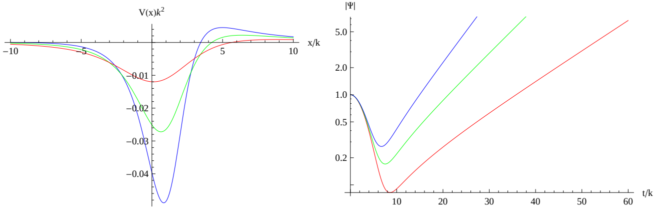
$$ds^2 = P(u)^a dt^2 - P(u)^{-a} du^2 - P(u)^{1-a} u^2 d\Omega^2, \qquad \phi = -\frac{C}{2k} \ln P(u),$$

where $P(u) := 1 - 2k/u$, with constants related by

$$m = ak, \qquad a^2 = 1 - \epsilon C^2/(2k^2), \quad k > 0.$$

Fisher: $\epsilon = +1 \quad \Rightarrow \quad |a| < 1$; the value $u = 2k \mapsto$ singularity, at which $x = 0$; $V_{\rm eff}(x) \approx -2/x^2$ — negative pole $\Rightarrow$ instability (under the natural boundary condition $|\delta\phi/\phi| < \infty$ as $x \to 0$).

Anti-Fisher A: $\epsilon = -1 \quad \Rightarrow \quad |a| > 1$; $u = 2k$ — sphere of infinite radius $(r \to \infty)$, singularity or horizon; a throat exists at some $u > 2k$.



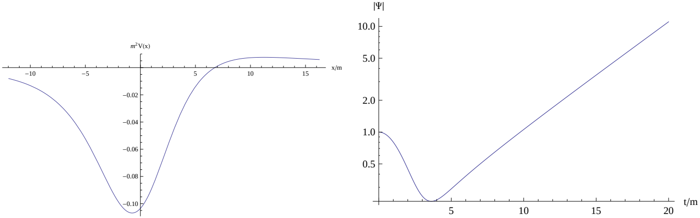
Regularized effective potential W_A (left) and the time-domain profiles (right) for Branch A solutions, showing the instability.

Perturbations: anti-Fisher, branch B

Branch B: $k = 0$, $u \in \mathbb{R}_+$,

$$ds^2 = \mathrm{e}^{-2m/u} dt^2 - \mathrm{e}^{2m/u}[du^2 + u^2 d\Omega^2], \quad \phi = C/u.$$

As before, $u = \infty$ is a flat infinity, while at the other extreme, for $m > 0$, $r \to \infty$ and all curvature invariants vanish as $u \to 0$. However, non-analyticity of the metric makes its continuation impossible. Throat: $u = m$.

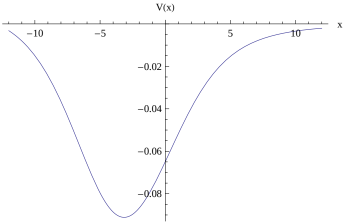


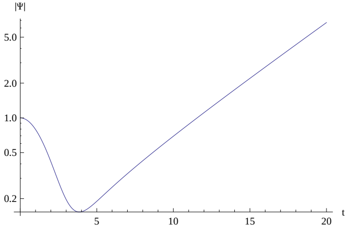
The potential W_B (left) and the time-domain profile (right) for Branch B solution $\Rightarrow$ instability.

Branch C, $k<0$: a wormhole with two flat asymptotics at $v=0$ and $v=\pi/|k|$. The metric has the form [H. Ellis 1973, K.B. 1973]

$$\begin{aligned} ds^2 &= \mathrm{e}^{-2mv}dt^2 - \frac{k^2\,\mathrm{e}^{2mv}}{\sin^2(kv)}\left[\frac{k^2\,du^2}{\sin^2(kv)} + d\Omega^2\right] \\ &= \mathrm{e}^{-2mv}dt^2 - \mathrm{e}^{2mv}[du^2 + (k^2+u^2)d\Omega^2], \end{aligned} \tag{32}$$

where $u \in \mathbb{R}$ and $|k|v = \cot^{-1}(u/|k|)$. If $m>0$, the wormhole is attractive for ambient test matter at the first asymptotic ($u\to\infty$) and repulsive at the second one ($u\to-\infty$), and vice versa in case $m<0$. For $m=0$: the simplest possible wormhole solution, sometimes called the Ellis wormhole, although Ellis discussed these solutions with any m.





The potential W_B (left) and the time-domain profile (right) for Branch C solution (example: $|k|=m=1$) $\Rightarrow$ instability. [Gonzalez et al., 2008]

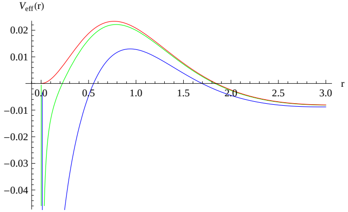
Perturbations: $V(\phi) \neq 0$, black universes

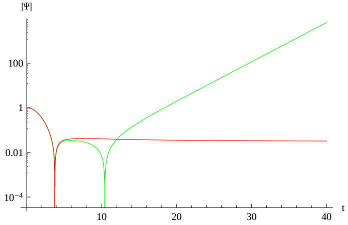
Solution: $ds^2 = A(u)dt^2 - du^2/A(u) - r^2(u)d\Omega^2$,

$$r(u) = (u^2 + b^2)^{1/2}, \quad \phi = \sqrt{2}\arctan(u/b), \quad b = \mathrm{const} > 0,$$
$$A(u) = (u^2 + b^2)\left[\frac{c}{b^2} + \frac{1}{b^2 + u^2} + \frac{3m}{b^3}\left(\frac{bu}{b^2 + u^2} + \arctan\frac{u}{b}\right)\right].$$

Black universe solutions (with a single simple horizon): $m > 0,\ c < 0$.

$c > -1 \quad \Rightarrow$ the throat is in the static (R) region (as in wormholes),
$c = -1 \quad \Rightarrow$ the throat coincides with the horizon,
$c < -1 \quad \Rightarrow$ no throat in R-region ($r = r_{\min}$ in T-region).





Left: Effective potentials for radial perturbations of a black universe with $m = 1/3$, $c = -1$ (red), $c = -1.001$ (green), $c = -1.01$ (blue). Right: Time evolution for $c = -1$ (red) $\Rightarrow$ stability, and $c = -1.001$ (green) $\Rightarrow$ instability.

$c = -1$: the only example of stable solution.

Conclusions

- Analytical exact solutions have been obtained in GR with an electromagnetic field and different kinds of phantom fields. These are globally regular static, spherically symmetric solutions describing traversable wormholes (with flat and AdS asymptotics) and regular black holes, in particular, black universes.
- The configurations obtained are quite diverse and contain different numbers of Killing horizons, from zero to four. This substantially enriches the list of known structures of regular BH configurations.
- Such models can be of interest both as descriptions of local objects (black holes and wormholes) and as a basis for building non-singular cosmological scenarios.
- Numerical estimates concerning a possible global magnetic field are compatible with a BU model.
- Phantom fields are not observed under usual conditions. This circumstance is accounted for by the concepts of trapped or (preferably) invisible ghosts.
- Almost all models with phantom fields (studied up to now) are unstable under radial perturbations. Exception: a black universe where the horizon coincides with the throat.

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THANK YOU!