Causal global structure of Kerr spacetime through a double null foliation

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Content

Why is Kerr geometry essential?

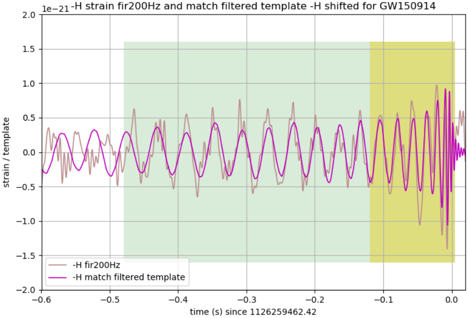
- Normally one thinks of black holes as the result of the collapse of a previous system.

  The probability that in the collapse, the initial system has some angular momentum is nearly 1.

  The vacuum solution of the Hilbert-Einstein equation with angular momentum is the Kerr geometry.

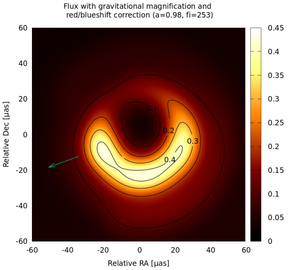
  The Kerr metric[Kerr(1963)] is one of the most important geometries in general relativity.

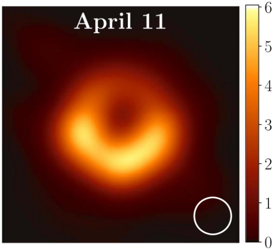
- All binary black hole systems, whose gravitational waves are detected[Abbott et al.(2016a), Abbott et al.(2016b)], are supposed to settle down to the Kerr geometry.



Strain of Hanford detector after a lowpass 200Hz FIR-filter("Convenient filtering techniques for LIGO strain of the GW150914 event", Osvaldo M. Moreschi, JCAP04(2019)032)[Moreschi(2019)], along with its adjusted by LIGO script template, after filtering, close to the time of the event.

- It has also become relevant with the first image ever created[Akiyama et al.(2019)] of a black hole.

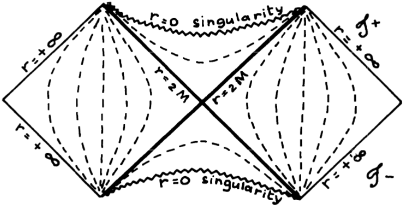




Comparison between our image("Strong gravitational lens image of the M87 black hole with a simple accreting matter model", Ezequiel F Boero and Osvaldo M Moreschi, Monthly Notices of the Royal Astronomical Society, Volume 507, Issue 4, November 2021, Pages 5974–5990, https://doi.org/10.1093/mnras/stab2336)[Boero and Moreschi(2021)], on the left, with the EHT image of April 11 of 2017, on the right. Our model is based on a two temperature thin disk with an accreting flow, in Kerr geometry, in prograde motion plus an asymmetric bar.

Introduction: IV

Causal structure in spherically symmetric spacetimes: In spherically symmetric spacetimes, it is natural to consider diagrams in which each point corresponds to one of the spheres of symmetries, and one arrives at the usual causal diagrams of the spacetime, as for example[Carter(1973)]:



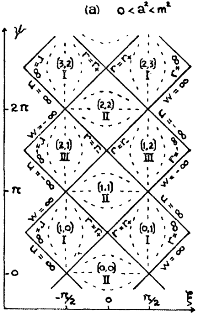
Let us note that each point can be characterized as $r =$constant, or $r_* =$constant, where $r_* = r + 2m \ln(\frac{r}{2m} - 1)$ is the tortoise coordinate.

This diagram is the main tool for understanding the global causal structure of Schwarzschild spacetime.

The symbol $\mathscr{I}^+$ denotes future null infinity; which is the asymptotic region reached by outgoing future directed null geodesics.

Introduction: V

Causal diagrams for Kerr spacetime: From Carter's work[Carter(1966)] and text books[Chandrasekhar(1983)] one is accustomed to see the diagram:



which is constructed only for the points on the axis $\theta = 0$ or $\theta = \pi$, of the spacetime.

Global causal diagrams for Kerr spacetime: We will present the necessary tools to construct global causal diagrams for the Kerr spacetime.

In Boyer-Lindquist coordinates
The Kerr metric can be expressed in a variety of forms:

$$\begin{aligned} ds^2 =& dt^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2\right) \sin^2(\theta) d\phi^2 \\ & - \varPhi(dt - a \sin^2(\theta) d\phi)^2 \\ =& (1 - \varPhi)\, dt^2 + 2\, \varPhi a \sin^2(\theta) dt d\phi \\ & - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \varPhi\, a^2 \sin^2(\theta)\right) \sin^2(\theta) d\phi^2 \\ =& (1 - \varPhi)\, dt^2 + 2\, \varPhi a \sin^2(\theta) dt d\phi \\ & - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{\Upsilon}{\Sigma} \sin^2(\theta) d\phi^2; \end{aligned} \tag{1}$$

where

$$\Sigma = r^2 + a^2 \cos^2(\theta), \tag{2}$$

$$\Delta = r^2 + a^2 - 2mr, \tag{3}$$

$$\varPhi = \frac{2mr}{\Sigma}, \tag{4}$$

The Kerr metric II

and
$$\Upsilon = (r^2 + a^2)^2 - \Delta\, a^2 \sin^2(\theta). \tag{5}$$
The inverse metric is then given by
$$\begin{aligned}\left(\frac{\partial}{\partial s}\right)^2 =& \frac{\Upsilon}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)^2 + \frac{4amr}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \phi}\right) \\ & - \frac{\Delta}{\Sigma}\left(\frac{\partial}{\partial r}\right)^2 - \frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2 - \frac{\Delta - a^2\sin^2(\theta)}{\Sigma\Delta\sin^2(\theta)}\left(\frac{\partial}{\partial \phi}\right)^2 .\end{aligned} \tag{6}$$

Principal null directions and null tetrad
A preferred null tetrad for the Kerr metric is[Chandrasekhar(1983)]

$$l = \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{a}{\Delta} \frac{\partial}{\partial \phi}, \tag{7}$$

$$n = \frac{r^2 + a^2}{2\Sigma} \frac{\partial}{\partial t} - \frac{\Delta}{2\Sigma} \frac{\partial}{\partial r} + \frac{a}{2\Sigma} \frac{\partial}{\partial \phi}, \tag{8}$$

$$\tilde{m} = \frac{1}{\sqrt{2}(r + ia\cos(\theta))} \left(ia\sin(\theta) \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{i}{\sin(\theta)} \frac{\partial}{\partial \phi} \right); \tag{9}$$

where l and n are the principal null directions.

The curvature tensor
The Riemann tensor is defined in terms of the covariant derivative by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \, v^d = R_{abc}{}^d v^c. \tag{10}$$

Null tetrad and spinors

The relation of the null tetrad with a spinor dyad is given by

$$l^a \Longleftrightarrow o^A o^{A'}, \tag{11}$$

$$m^a \Longleftrightarrow o^A \iota^{A'}, \tag{12}$$

$$\bar{m}^a \Longleftrightarrow \iota^A o^{A'}, \tag{13}$$

$$n^a \Longleftrightarrow \iota^A \iota^{A'}. \tag{14}$$

The Riemann tensor in terms of spinors

The Rieman tensor can[Pirani(1965)] be expressed by

$$\begin{aligned} R_{AA'BB'CC'DD'} = \quad & \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Psi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ & + \epsilon_{AB}\epsilon_{C'D'}\Phi_{CDA'B'} + \epsilon_{A'B'}\epsilon_{CD}\Phi_{ABC'D'} \\ & + 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'}). \end{aligned} \tag{15}$$

Where the Ricci tensor can be expressed by

$$R_{AA'BB'} = 6\Lambda\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}. \tag{16}$$

The Kerr metric V

For the principal null congruence, it is well known that the only nonzero Weyl scalar is

$$\Psi_{2p} \equiv \Psi_{0011} = -\frac{m}{(r + i\,a\cos(\theta))^3}, \tag{17}$$

where we emphasize with the subindex p, that its refers to the principal null tetrad.

Determining the center of mass null congruence for Kerr spacetime I

We are going to construct null functions from a congruence of null geodesics that are hypersurface orthogonal("Double null coordinates for Kerr spacetime", Marcos A. Argañaraz and Osvaldo M. Moreschi, Phys.Rev.D 104, 024049 (2021)) [Argañaraz and Moreschi(2021)].

Null geodesics
Let λ denote an affine parameter and a dot a derivative by λ; then the null geodesic $\ell = (\dot{t} \equiv \frac{dt}{d\lambda}, \dot{r} \equiv \frac{dr}{d\lambda}, \dot{\theta} \equiv \frac{d\theta}{d\lambda}, \dot{\phi} \equiv \frac{d\phi}{d\lambda})$ must satisfy

$$\begin{aligned} 0 =& g_{ab}\ell^a \ell^b = \left(1 - \frac{2mr}{\Sigma}\right)\dot{t}^2 + \frac{4amr\sin^2(\theta)}{\Sigma}\dot{t}\dot{\phi} \\ &- \frac{\Sigma}{\Delta}\dot{r}^2 - \Sigma\dot{\theta}^2 - \left(r^2 + a^2 + \frac{2a^2mr}{\Sigma}\sin^2(\theta)\right)\sin^2(\theta)\dot{\phi}^2; \end{aligned} \tag{18}$$

and the constants of motion

$$E = g_{ab}\ell^a \left(\frac{\partial}{\partial t}\right)^b = \left(1 - \frac{2mr}{\Sigma}\right)\dot{t} + \frac{2amr\sin^2(\theta)}{\Sigma}\dot{\phi} = \dot{t} - \frac{2mr}{\Sigma}(\dot{t} - a\sin^2(\theta)\dot{\phi}), \tag{19}$$

and

$$-L_z = g_{ab}\ell^a \left(\frac{\partial}{\partial \phi}\right)^b = \frac{2amr\sin^2(\theta)}{\Sigma}\dot{t} - \left(r^2 + a^2 + \frac{2a^2mr}{\Sigma}\sin^2(\theta)\right)\sin^2(\theta)\dot{\phi}. \tag{20}$$

For asymptotic studies, the affine parameter will be chosen so that $E = 1$.
There is also another integral of motion given by

$$2\Sigma \ell^a \ell^b l_a n_b = K; \tag{21}$$

where l^a and n^b are the principal null geodesics of the Kerr metric. Note that since ℓ^a is a null vector, one also has:

$$2\Sigma \ell^a \ell^b m_a \bar{m}_b = K. \tag{22}$$

All these conservation equations can be used in order to express the null geodesic equations in the form[Chandrasekhar(1983)]

$$\Sigma \dot{t} = \frac{1}{\Delta}\left[E\left((r^2 + a^2)^2 - \Delta\, a^2 \sin^2(\theta)\right) - 2amrL_z\right] = \frac{1}{\Delta}\left[E\Upsilon - 2amrL_z\right], \tag{23}$$

$$\Sigma \dot{\phi} = \frac{1}{\Delta}\left[2Eamr + (\Sigma - 2mr)\frac{L_z}{\sin^2(\theta)}\right], \tag{24}$$

$$\Sigma \dot{r} = \pm\sqrt{\mathcal{R}}, \tag{25}$$

and

$$\Sigma \dot{\theta} = \pm\sqrt{\Theta(\theta, L_z, K)}; \tag{26}$$

Determining the center of mass null congruence for Kerr spacetime III

where we have used

$$\mathcal{R} \equiv \left[E(r^2 + a^2) - aL_z\right]^2 - K\Delta, \tag{27}$$

and

$$\Theta(\theta, L_z, K) \equiv K - \left[Ea\sin(\theta) - \frac{L_z}{\sin(\theta)}\right]^2. \tag{28}$$

Asymptotic conditions at the center of mass sections

The one form ℓ_a for the general congruence with nonzero angular parameter is

$$\begin{aligned} \ell_a =& g_{ab}\,\ell^b \\ =& Edt_a - \frac{\pm\sqrt{\mathcal{R}}}{\Delta} dr_a - \left(\pm\sqrt{\Theta}\right) d\theta_a - L_z d\phi_a. \end{aligned} \tag{29}$$

At each point of the spacetime, the choice of the constants L_z and K single out a direction in the sphere of directions.

One can ask, how can one choose the 'constants' L_z and K so that *locally* they define a null hypersurface with the properties that we want; that is, although they are constant along each null geodesic, we are free to choose them for each geodesic.

The guiding idea is that ℓ_a must be an exact differential; but we also demand the congruence to be orthogonal to a sphere at future null infinity, that coincides with the center of mass section.

As a first step, let us consider a surface S_r, defined by (t =constant and r =constant), with tangent vectors $\left(\frac{\partial}{\partial\theta}\right)^b$ and $\left(\frac{\partial}{\partial\phi}\right)^b$. We are interested in the limiting case, where this surface tends to a sphere at future null infinity $S_r \longrightarrow S_\infty$. In our approach, we select the congruence ℓ^a which is orthogonal to S_∞; that is, that

$$\begin{aligned} \lim_{r\longrightarrow\infty} g_{ab}\ell^a \Big(\frac{\partial}{\partial\theta}\Big)^b &= 0, \\ \lim_{r\longrightarrow\infty} g_{ab}\ell^a \Big(\frac{\partial}{\partial\phi}\Big)^b &= 0. \end{aligned} \tag{30}$$

From (29) and (30) we obtain for each geodesic $L_z = 0$ and $K = a^2 \sin(\theta_\infty)^2$, where $\theta_\infty = \theta \mid_{r=\infty}$. These conditions fix the congruence completely, and since it started orthogonal to a topological 2-sphere, it is hypersurface orthogonal.

Local differential condition
For ℓ_a to be an exact differential, one must have

$$
\begin{aligned}
d\ell =& 0 \\
=& -\frac{1}{2(\pm\sqrt{\mathcal{R}})\Delta}\frac{\partial \mathcal{R}}{\partial\theta} d\theta \wedge dr - \frac{1}{2(\pm\sqrt{\Theta})}\frac{\partial\Theta}{\partial r} dr \wedge d\theta \\
& - \frac{\partial L_z}{\partial r} dr \wedge d\phi - \frac{\partial L_z}{\partial\theta} d\theta \wedge d\phi;
\end{aligned} \tag{31}
$$

which immediately implies

$$
\frac{\partial L_z}{\partial r} = \frac{\partial L_z}{\partial\theta} = 0; \tag{32}
$$

and also

$$
\frac{1}{(\pm\sqrt{\mathcal{R}})\Delta}\frac{\partial \mathcal{R}}{\partial\theta} - \frac{1}{(\pm\sqrt{\Theta})}\frac{\partial\Theta}{\partial r} = 0; \tag{33}
$$

which can also be expressed as

$$
\frac{1}{(\pm\sqrt{\mathcal{R}})}\frac{\partial K}{\partial\theta} + \frac{1}{(\pm\sqrt{\Theta})}\frac{\partial K}{\partial r} = 0. \tag{34}
$$

These are the general integrability conditions to build a null function u from the identification $\ell = du$. Note that this condition also demands that E must be constant for the congruence; so there is no loss of generality in assuming our choice $E = 1$ for our construction.

Solving for K and k

Let us now consider the case $\dot{r} > 0$ and try to build a congruence so that ℓ_a coincide with the differential of a null function u, by relaxing K to be a function of r and θ. The sign for $\dot{\theta}$ is chosen by thinking on a spheroid at future null infinity, in the asymptotically flat situation; so that we take (+) for the northern hemisphere and (-) for the southern hemisphere and we will express this by $\pm|_h$. Let us observe that geodesics in this congruence that have one point at the Equator, then are contained at it.
Then, (34) can be expressed as

$$\frac{1}{2\sqrt{(r^2+a^2)^2 - K\Delta}} \frac{\partial K}{\partial \theta} d\theta \wedge dr \pm|_h \frac{1}{2\sqrt{K - \left(a\sin(\theta)\right)^2}} \frac{\partial K}{\partial r} d\theta \wedge dr = 0; \tag{35}$$

which we could also write as

$$\boxed{\sqrt{(r^2+a^2)^2 - K\Delta}\,\frac{\partial K}{\partial r} \pm|_h \sqrt{K - \left(a\sin(\theta)\right)^2}\,\frac{\partial K}{\partial \theta} = 0.} \tag{36}$$

Thus, equation (36) constitute the integrability condition for the one form ℓ_a to be the differential of a null function.
Now that we have relaxed the behavior of the parameter K; let us check if it changes along the original null geodesics; since this will be a problem. Then, the derivative of K with respect to each affine parameter is:

$$\frac{dK}{d\lambda} = \frac{\partial K}{\partial r}\,\dot{r} + \frac{\partial K}{\partial \theta}\,\dot{\theta} = \frac{\sqrt{(r^2+a^2)^2 - K\Delta}}{\Sigma}\,\frac{\partial K}{\partial r} \pm|_h \frac{\sqrt{K - \left(a\sin(\theta)\right)^2}}{\Sigma}\,\frac{\partial K}{\partial \theta} = 0; \quad (37)$$

that is, the imposition of equation (36) guarantees that K is constant along each null geodesic of the congruence.
We are interested in the asymptotic boundary condition

$$\lim_{r\to\infty} K = a^2 \sin(\theta_\infty)^2; \quad (38)$$

where θ_∞ is the value of θ at future null infinity. This is the condition to build a congruence that reaches future null infinity orthogonally to the center of mass sections. Therefore, for a numerical integration we can arrange the equation in the form

$$\boxed{\frac{\partial K}{\partial r} = - \pm|_h \frac{\sqrt{K - \left(a\sin(\theta)\right)^2}}{\sqrt{(r^2+a^2)^2 - K\Delta}}\,\frac{\partial K}{\partial \theta}}; \quad (39)$$

which invites us to define k from the relation

$$K(r,\theta) = a^2 \sin(\theta)^2 + k(r,\theta)^2; \tag{40}$$

and

$$\xi = \frac{1}{r}; \tag{41}$$

so that the equation now is

$$\frac{\partial K}{\partial r} = -2\xi^2 k \frac{\partial k}{\partial \xi} = -\pm|_h \xi^2 \frac{\pm|k|}{\sqrt{(1+\xi^2 a^2)^2 - \xi^4 K \Delta}} (2a^2 \sin(\theta)\cos(\theta) + 2k\frac{\partial k}{\partial \theta}); \tag{42}$$

which is invariant under the interchange $k \to -k$; with the boundary condition

$$\lim_{\xi \to 0} k = 0. \tag{43}$$

This can be solved numerically from $\xi = 0$ to $\xi_+ = \frac{1}{r_+}$, or to larger values.

Determining the center of mass null congruence for Kerr spacetime IX

Let us note that in the northern hemisphere one has the plus sign and also $\sin(\theta)\cos(\theta)$ is positive, so that, after starting at $k = 0$, we can assign the sign of k at will. If we assume $k > 0$ then we have to use the equation

$$\boxed{\frac{\partial k}{\partial \xi} = \frac{a^2 \sin(\theta)\cos(\theta) + k\frac{\partial k}{\partial \theta}}{\sqrt{(1+\xi^2 a^2)^2 - \xi^4 \Delta\big(a^2 \sin(\theta)^2 + k^2\big)}};} \tag{44}$$

and on the contrary if we assume that in the northern hemisphere $k < 0$ then we have to use the oposite sign in the above equation; noting that in principle the sign of k does not intervene in the sign of $\dot{\theta}$. For the sake of simplicity we will take the first choice which assumes $k > 0$, in the northern hemisphere, in a neighborhood of future null infinity. Let us note that with this choice we simply have

$$\pm|_h \sqrt{K - \big(a\sin(\theta)\big)^2} = k. \tag{45}$$

We have solved numerically equation (44), whose graphic representation is given in the next figure.

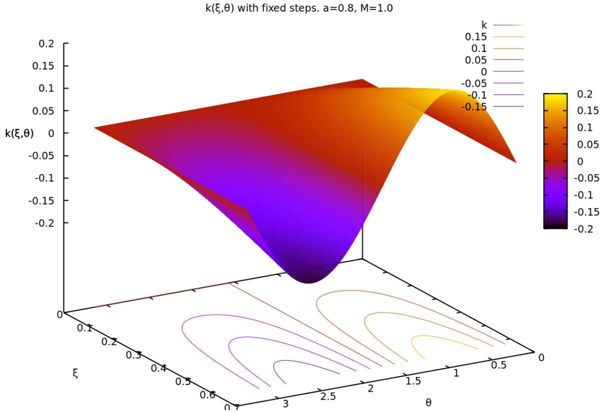


Figure: Numerical solution of k in terms of the coordinates (ξ, θ).

The coordinate (ξ) covers the whole exterior region from $\xi = 0$ to $\xi = \xi_+ = \frac{1}{r_+}$; but one can go through the horizon without difficulties.
We can estimate the residual just from the evaluation of

$$\sqrt{(1+\xi^2 a^2)^2 - \xi^4 K \Delta}\,\frac{\partial k}{\partial \xi} - \left(a^2 \sin(\theta)\cos(\theta) + k\frac{\partial k}{\partial \theta}\right) = 0; \tag{46}$$

which of course is an alternative way to write equation (44); but more adapted to checking the integrability condition. The residual (46) is shown in the next figure, and it can be improved at any higher desired precision.

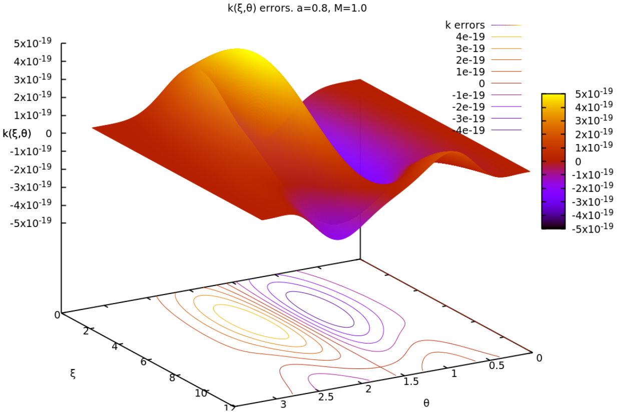


Figure: Numerical solution of k in terms of the coordinates (ξ, θ).

Direct expressions

From the knowledge of k, we can define the null functions u (outgoing) and v (ingoing), as the hypersurfaces generated by the outgoing ℓ_a and ingoing $\mathfrak{n}_a$ null geodesics congruences respectively; namely

$$du = dt - \frac{\sqrt{(r^2 + a^2)^2 - K\Delta}}{\Delta} dr - \pm|_h \sqrt{\Theta(r,\theta)}\, d\theta, \tag{47}$$

$$dv = dt + \frac{\sqrt{(r^2 + a^2)^2 - K\Delta}}{\Delta} dr \pm|_h \sqrt{\Theta(r,\theta)}\, d\theta, \tag{48}$$

where $K = K(r,\theta)$ is a solution of 36, and

$$\Theta(r,\theta) = K(r,\theta) - a^2 \sin^2(\theta). \tag{49}$$

To obtain integrated expressions of u and v, we need to integrate (47) and (48). Because they are very similar, for simplicity we will work with u, but the same process follows for v.

Null functions u and v II

We can integrate du along any curve $\gamma(s')$, which connects an initial point $(t_0, r_0, \theta_0, \phi_0)$ to a final point (t, r, θ, ϕ). Given a curve γ, with $s' \in [s_0, s]$, one can express

$$\begin{aligned} u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) =& t - t_0 - \int_{s_0}^{s} \frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta'')}}{\Delta} \frac{dr'}{ds'} ds' \\ & - \int_{s_0}^{s} \pm|_h \sqrt{\Theta(r'',\theta')} \frac{d\theta'}{ds'} ds'. \end{aligned} \tag{50}$$

As we know, the difference $u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0)$ only depends on the initial and final points. Nevertheless to proceed and find a final computable expression, it is useful to consider an integration path. Because both integrands depends only on two coordinates (r,θ), we have two natural paths to connect initial and final points. One of them is $[(r_0;\theta_0) \to (r;\theta_0) \to (r;\theta)]$, in that case (50) becomes

$$\begin{aligned} u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) =& t - t_0 - \int_{r_0}^{r} \frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta_0)}}{\Delta} dr' \\ & - \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r,\theta')} \, d\theta', \end{aligned} \tag{51}$$

and the other one is $[(r_0;\theta_0) \to (r_0;\theta) \to (r;\theta)]$; in that case (50) becomes

$$\begin{aligned} u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) = t - t_0 - \int_{r_0}^{r} \frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta)}}{\Delta}\, dr' \\ - \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r_0,\theta')}\, d\theta'. \end{aligned} \tag{52}$$

In what follows, we work with both expressions. Because they are very similar, for simplicity we will do it completely only for (51), but the same final result follows from (52). We start with the behavior analysis of the second term integrand at ($\Delta \approx 0$)

$$\begin{aligned} \frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta_0)}}{\Delta} = \frac{(r'^2+a^2)}{\Delta} \\ - \frac{K}{2(r'^2+a^2)} - \frac{K^2\Delta}{8(r'^2+a^2)^3} + \mathcal{O}(\Delta^2), \end{aligned} \tag{53}$$

Null functions u and v IV

from (53) is clear that (51) has a divergent term at ($\Delta \approx 0$). We can isolate such behavior in one simpler term, by simply adding and subtracting $\left(r'^2 + a^2\right)/\Delta$, to obtain

$$
\begin{aligned}
u(t, r, \theta, \phi) - u_0(t_0, r_0, \theta_0, \phi_0) = t - t_0 &- \int_{r_0}^{r} \left(\frac{\left(r'^2 + a^2\right)}{\Delta} \right) dr' \\
&- \int_{r_0}^{r} \left(\frac{\sqrt{\left(r'^2 + a^2\right)^2 - \Delta K(r', \theta_0)}}{\Delta} - \frac{\left(r'^2 + a^2\right)}{\Delta} \right) dr' \\
&- \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r, \theta')}\, d\theta',
\end{aligned}
\tag{54}
$$

where the divergent term can be integrated analytically

$$
\int_{r_0}^{r} \left(\frac{\left(r'^2 + a^2\right)}{\Delta} \right) dr' = \left[r + \frac{r_+^2 + a^2}{r_+ - r_-} \ln(\frac{r}{r_+} - 1) - \frac{r_-^2 + a^2}{r_+ - r_-} \ln(\frac{r}{r_-} - 1) \right] \Big|_{r_0}^{r}; \tag{55}
$$

where r_+ and r_- are the solutions of $\Delta = 0$; namely $r_+ = m + \sqrt{m^2 - a^2}$ and $r_- = m - \sqrt{m^2 - a^2}$.

Null functions u and v V

Then, we have an equivalent expression of (51)

$$
\begin{aligned}
u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) = t - r &- \Big(\frac{r_+^2+a^2}{r_+-r_-}\ln(\frac{r}{r_+}-1) - \frac{r_-^2+a^2}{r_+-r_-}\ln(\frac{r}{r_-}-1)\Big) \\
&- \int_{r_0}^{r}\left(\frac{\sqrt{\left(r'^2+a^2\right)^2-\Delta K(r',\theta_0)}}{\Delta} - \frac{\left(r'^2+a^2\right)}{\Delta}\right) dr' \\
&- \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r,\theta')}\, d\theta' + C_0(t_0,r_0),
\end{aligned}
\tag{56}
$$

where C_0 is a constant. The strategy is to elect $u_0(t_0,r_0,\theta_0,\phi_0) = -C_0(t_0,r_0)$, in such a way, that for any initial point $(t_0,r_0,\theta_0,\phi_0)$

$$
\begin{aligned}
u(t,r,\theta,\phi) = t - r &- \Big(\frac{r_+^2+a^2}{r_+-r_-}\ln(\frac{r}{r_+}-1) - \frac{r_-^2+a^2}{r_+-r_-}\ln(\frac{r}{r_-}-1)\Big) \\
&- \int_{r_0}^{r}\left(\frac{\sqrt{\left(r'^2+a^2\right)^2-\Delta K(r',\theta_0)}}{\Delta} - \frac{\left(r'^2+a^2\right)}{\Delta}\right) dr' \\
&- \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r,\theta')}\, d\theta'.
\end{aligned}
\tag{57}
$$

The final step to obtain the coordinate $u(t,r,\theta,\phi)$, is to elect the initial point $(t_0,r_0,\theta_0,\phi_0)$. We are interested to take its value at future null infinity, that is, at $r_0 \to \infty$, $t_0 \to \infty$, $\theta_0 \to \theta_\infty$, $\phi_0 \to \phi_\infty$, where θ_∞ and ϕ_∞ are finite. Note that in (57), there is no dependence on ϕ_0, then we are free to choose ϕ_∞, without any change in u. Also it can be noted that C_0 is finite in this limit. Then, we have

$$
\begin{aligned}
u(t,r,\theta,\phi) = {} & t - r - \Big(\frac{r_+^2 + a^2}{r_+ - r_-}\ln(\frac{r}{r_+} - 1) - \frac{r_-^2 + a^2}{r_+ - r_-}\ln(\frac{r}{r_-} - 1)\Big) \\
& - \int_{r_\infty}^{r} \Bigg(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r',\theta_\infty)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta}\Bigg)\, dr' \\
& - \int_{\theta_\infty}^{\theta} \pm|_h \sqrt{K(r,\theta') - a^2\sin(\theta')^2}\, d\theta'.
\end{aligned} \tag{58}
$$

In the same way, we can start with the second natural integration path (52) and repeat all the steps to obtain

$$
\begin{aligned}
u(t,r,\theta,\phi) = t - r - \Big(&\frac{r_+^2 + a^2}{r_+ - r_-} \ln(\frac{r}{r_+} - 1) - \frac{r_-^2 + a^2}{r_+ - r_-} \ln(\frac{r}{r_-} - 1)\Big) \\
&- \int_{r_\infty}^{r} \Big(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r',\theta)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta} \Big)\, dr' \\
&- \int_{\theta_\infty}^{\theta} \pm|_h \sqrt{K(r_\infty,\theta') - a^2 \sin(\theta')^2}\, d\theta'.
\end{aligned}
\tag{59}
$$

Note that when $a \to 0$, we have $K \to 0$. Then the last two terms in (58) and (59) become zero because each integrand is zero. Also when $a \to 0$, we have $r_- \to 0$, which from (55), we can simply take $a = 0$ and $r_- = 0$. So that in this way we recover the Eddington's outgoing null function for Schwarzschild spacetime

$$
\lim_{a\to 0} u(t,r,\theta,\phi) = t - \Big(r + 2m \ln(\frac{r}{2m} - 1) \Big). \tag{60}
$$

Shorter expressions for u and v

We have already mention that there are two natural integration paths (51) and (52). In this part, we will show how these paths allow us to get shorter expressions for u and v. Remember that the first path was $[(r_0;\theta_0) \to (r;\theta_0) \to (r;\theta)]$, and that we locate the initial point at $r_0 = r_\infty$, $\theta_0 = \theta_\infty$. To obtain a shorter expression we can take advantage of the election freedom and choose $(\theta_0 = \theta_\infty = 0)$. Then we have to consider the property

$$K(r, \theta = 0) = 0, \tag{61}$$

to obtain the shortest expression for u, by replacing (61) in (58) and using that $(r_+^2 + a^2 = 2mr_+)$ and $(r_-^2 + a^2 = 2mr_-)$, we have

$$\begin{aligned} u(t, r, \theta, \phi) = & \, t - r - \Big(\frac{2mr_+}{r_+ - r_-} \ln(\frac{r}{r_+} - 1) - \frac{2mr_-}{r_+ - r_-} \ln(\frac{r}{r_-} - 1) \Big) \\ & - \int_0^\theta \pm|_h \sqrt{K(r, \theta') - a^2 \sin(\theta')^2} \, d\theta'. \end{aligned} \tag{62}$$

The second natural path was $[(r_0;\theta_0) \to (r_0;\theta) \to (r;\theta)]$, with the initial point at $r_0 = r_\infty$, $\theta_0 = \theta_\infty$. To obtain a shorter expression we can take advantage of the election freedom and choose ($r_0 = r_\infty = \infty$). Then we have to consider another property that comes from the boundary condition

$$K(r = \infty, \theta') = a^2 \sin(\theta')^2, \tag{63}$$

to obtain another short expression, by the replacement of (63) in (59)

$$\begin{aligned} u(t,r,\theta,\phi) = & \, t - r - \Big(\frac{2mr_+}{r_+ - r_-} \ln(\frac{r}{r_+} - 1) - \frac{2mr_-}{r_+ - r_-} \ln(\frac{r}{r_-} - 1) \Big) \\ & - \int_\infty^r \Big(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r',\theta)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta} \Big) \, dr'. \end{aligned} \tag{64}$$

In an analogous way, we can start with (48) to obtain the corresponding expressions for $v(t,r,\theta,\phi)$. The expressions are

$$\begin{aligned} v(t,r,\theta,\phi) = & \, t + r + \Big(\frac{2mr_+}{r_+ - r_-} \ln(\frac{r}{r_+} - 1) - \frac{2mr_-}{r_+ - r_-} \ln(\frac{r}{r_-} - 1) \Big) \\ & + \int_0^\theta \pm|_h \sqrt{K(r,\theta') - a^2 \sin(\theta')^2} \, d\theta', \end{aligned} \tag{65}$$

and

$$v(t,r,\theta,\phi) = t + r + \Big(\frac{2mr_+}{r_+ - r_-} \ln(\frac{r}{r_+} - 1) - \frac{2mr_-}{r_+ - r_-} \ln(\frac{r}{r_-} - 1) \Big) \\ + \int_\infty^r \Big(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r',\theta)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta} \Big) \, dr'. \tag{66}$$

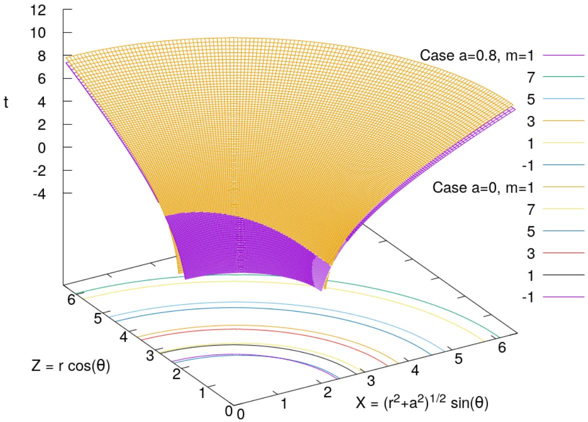
Then, we can compute u with (62) or (64) and v with (65) or (66) to obtain the same result.

Plot of u

In the next figure it is shown the Kerr center of mass null surface, when $u = 0$, with angular parameter values $a = 0.8$ and $a = 0$; to also include the comparison with the Schwarzschild case. Note that such comparison is only about the functional dependence of $u(t,r,\theta)$ with respect to coordinates (t,r,θ).

We have computed numerically both expressions (62) and (64). We corroborate they have the same value up to machine rounding error.

Null functions u and v XI



Center of mass null surface ($u = 0$) for Kerr and Schwarzschild spacetimes.

Note that (47) and (48) can be written as

$$du = dt - dr_s, \tag{67}$$

$$dv = dt + dr_s; \tag{68}$$

where

$$dr_s \equiv \frac{\sqrt{\mathcal{R}}}{\Delta}\, dr + k\, d\theta. \tag{69}$$

The intersection of both null coordinates families u and v, that is, ($du = dv = 0$), define a family of 2-dimensional spacelike surfaces S_{r_s}, where the function $r_s = (v-u)/2$ is constant. In terms of Boyer-Lindquist coordinates, such surface ($dr_s = 0$), is given by $r(\theta)$ which satisfies

$$\frac{dr}{d\theta} = -\frac{k\,\Delta}{\sqrt{\mathcal{R}}}. \tag{70}$$

Note that r_s can be interpreted as the Kerr extension of the Schwarzschild's tortoise coordinate. In particular, we can express the function r_s in the exterior region as

$$\boxed{\begin{aligned} r_s(r,\theta) =& r + \frac{2mr_+}{r_+ - r_-}\ln(\frac{r}{r_+} - 1) - \frac{2mr_-}{r_+ - r_-}\ln(\frac{r}{r_-} - 1) \\ &+ \int_0^\theta k(r,\theta')\, d\theta' \end{aligned}}. \tag{71}$$

It is easy to show that the induced metric over the 2-dimensional spacelike surface family S_{r_s}, is given by

$$ds^2 = -\frac{\Upsilon\,\Sigma}{\mathcal{R}}\, d\theta^2 - \frac{\Upsilon}{\Sigma}\sin^2(\theta)\, d\phi^2; \tag{72}$$

which can be obtained directly from (80) below, by simply taking ($du = dv = 0$). Note that in Boyer-Lindquist coordinates at each surface S_{r_s}, equation (70) is satisfied; which also implies that at the horizons the surface S_{r_s} is given by $r = r_+, r_-$, as one could expect. But it should be remarked that (70) still makes sense even at the horizons where r_s diverges; that is the family S_{r_s} have a smooth extension at the horizons, even though the behavior of r_s.

A convenient geometric description of such surface family, can be done in terms of their Gaussian and Extrinsic curvatures, as described in the GHP[Geroch et al.(1973)Geroch, Held, and Penrose] formalism. Then, one has

$$C_{Gaussian} = \left(\bar{Q}_{GHP} + Q_{GHP}\right), \tag{73}$$

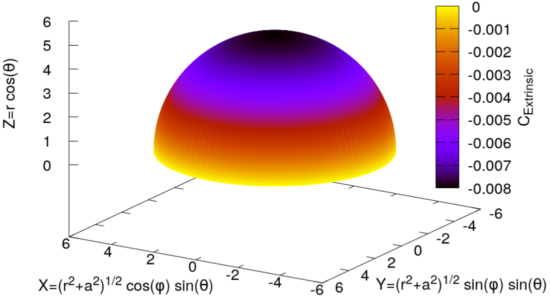
$$C_{Extrinsic} = i\left(\bar{Q}_{GHP} - Q_{GHP}\right), \tag{74}$$

which are given in terms of the complex curvature scalar Q_{GHP}, given by

$$Q_{GHP} = \sigma\sigma' - \rho\rho' - \Psi_2 + \Lambda + \Phi_{11}, \tag{75}$$

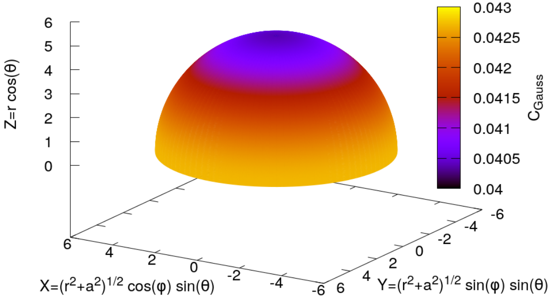
Surface family S_{r_s} III

where $\sigma, \sigma', \rho, \rho'$ are the spin coefficients in the GHP formalism; and in the Kerr spacetime one has $\Lambda = \Phi_{11} = 0$.
Next figures show the numerical calculation of these quantities, where the smooth nature of them can be inferred.



Extrinsic Curvature of S_{r_s}. Using parameters $a = 0.8$, $m = 1.0$. Case $r(\theta) \approx 3r_+$.

Surface family S_{r_s} IV



Gaussian Curvature of S_{r_s}. Using parameters $a = 0.8$, $m = 1.0$. Case $r(\theta) \approx 3r_+$.

It is important to mention that in the limit case where $a = 0$ (Schwarschild), one has

$$\lim_{a\to 0} C_{Gauss} = \frac{1}{r^2}, \tag{76}$$

$$\lim_{a\to 0} C_{Extrinsic} = 0; \tag{77}$$

where the two dimensional spacelike surfaces $(dr_s = 0)$, in Schwarzschild's case are given by $(dr = 0)$.

In what follows we will start from Kerr's metric in Boyer-Lindquist coordinates (1), and then we will make a coordinate transformation to the *center of mass null coordinates*

Coordinates $\{u, v, \theta, \phi\}$
From (47) and (48) we can write

$$dt = \frac{dv + du}{2}, \tag{78}$$
$$dr = \left[\frac{(dv - du)}{2} - k\, d\theta \right] \frac{\Delta}{\sqrt{\mathcal{R}}}. \tag{79}$$

Then we can replace (78) and (79) in (1) to obtain

$$\begin{aligned} ds^2 =& \frac{1}{4}\left(1 - \frac{2mr}{\Sigma} - \frac{\Sigma\Delta}{\mathcal{R}}\right)(du^2 + dv^2) + \frac{1}{2}\left(1 - \frac{2mr}{\Sigma} + \frac{\Sigma\Delta}{\mathcal{R}}\right) du\, dv \\ &+ dv\left(\frac{2amr\sin^2(\theta)}{\Sigma}\, d\phi + \frac{\Sigma\Delta}{\mathcal{R}} k\, d\theta\right) \\ &+ du\left(\frac{2amr\sin^2(\theta)}{\Sigma}\, d\phi - \frac{\Sigma\Delta}{\mathcal{R}} k\, d\theta\right) - \frac{\Upsilon\,\Sigma}{\mathcal{R}} d\theta^2 - \frac{\Upsilon}{\Sigma}\sin^2(\theta)\, d\phi^2. \end{aligned} \tag{80}$$

Kerr metric in double null coordinates II

We can also obtain the contravariant expression of (80) if we start with the inverse metric line element in Boyer-Lindquist coordinates. After a long calculation one obtains

$$
\begin{aligned}
\left(\frac{\partial}{\partial s}\right)^2 = & 4\,\frac{\Upsilon}{\Sigma\Delta}\left(\frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial v}\right) - \frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2 \\
& + 2\left(\frac{\partial}{\partial u}\right)\left[\frac{2amr}{\Sigma\Delta}\left(\frac{\partial}{\partial \phi}\right) + \frac{k}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)\right] \\
& + 2\left(\frac{\partial}{\partial v}\right)\left[\frac{2amr}{\Sigma\Delta}\left(\frac{\partial}{\partial \phi}\right) - \frac{k}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)\right] \\
& - \frac{\Delta - a^2\sin^2(\theta)}{\Sigma\Delta\sin^2(\theta)}\left(\frac{\partial}{\partial \phi}\right)^2 .
\end{aligned}
\tag{81}
$$

It is easy to verify that $\ell_a = (du)_a$ and $\mathfrak{n}_a = (dv)_a$ are null covectors, which is consistent with our definition.

New angular coordinate φ

Let's start analyzing the most general null geodesic, where the angular coordinate ϕ changes according to (24). It is clear that

$$\lim_{\Delta \to 0} \frac{d\phi}{d\lambda} \to \infty; \tag{82}$$

where $\Delta = (r - r_+)(r - r_-)$; that is, $\Delta \to 0$, when $r \to r_+$ or $r \to r_-$.
We want to define a new angular coordinate without that bad behavior at $\Delta = 0$.
We choose in this work the usual definition

$$d\varphi_{\pm_{oi}} = d\phi - \pm_{oi} \frac{a}{\Delta} dr; \tag{83}$$

which has an integral expression given by

$$\varphi_{\pm_{oi}} = \phi - \pm_{oi} \frac{a}{2\sqrt{m^2 - a^2}} \ln \left| \frac{r - r_+}{r - r_-} \right|. \tag{84}$$

Depending on the election of $\pm_{oi}$, this is well behaved for the outgoing null congruence ($\pm_{oi} = +$) and for the ingoing one ($\pm_{oi} = -$).

Next, it will be useful to express the differential of (83) in terms of *center of mass null coordinates* u, v; which from (79) can be expressed as

$$d\varphi_{\pm_{oi}} = d\phi - \pm_{oi} \frac{a}{\sqrt{\mathcal{R}}} \left[\frac{(dv - du)}{2} - k\, d\theta \right]. \tag{85}$$

Extended null functions U, V

A natural geometric way to study the behavior of a function (U) across a hypersurface, is to study its behavior along geodesics that cross this hypersurface; since its dependence on the affine parameters is geometric and independent of coordinate choices. Therefore we begin by studying the asymptotic behavior of the null coordinate u along null geodesics that reach the future horizon H_f. Since it is immaterial which are those geodesics, we choose the null geodesics that are contained in the congruence $v =$const.; that is, an ingoing one for which one has to take ($\pm_{oi} = -$) and ($\dot{\theta} = - \pm_h \sqrt{\Theta} = -k$). From the first order expressions of the geodesic equations, and $dv = 0$, that allows to express $dt = -\frac{\sqrt{\mathcal{R}}}{\Delta}\dot{r} - \; k\,\dot{\theta}$, one obtains

$$\dot{u}|_{v=\mathbf{const.}} = -\,2\frac{\sqrt{\mathcal{R}}}{\Delta}\dot{r} - 2\,k\,\dot{\theta} = 2\left[\frac{\mathcal{R}}{\Sigma\Delta} + \frac{k^2}{\Sigma}\right]. \tag{86}$$

Defining the null function U by

$$U = -\exp(-\kappa u), \tag{87}$$

one has the behavior

$$\dot{U} = -\,\kappa U \dot{u} = -2\kappa U \,\left[\frac{\mathcal{R}}{\Sigma\Delta} + \frac{k^2}{\Sigma}\right]; \tag{88}$$

which indicates a divergent behavior of the first term in brackets. One must take $\kappa = \kappa_+$ in order to have a smooth behavior of U as a function of the affine parameter λ; where

$$\kappa_+ = \frac{(r_+ - r_-)}{2(r_+^2 + a^2)} = \frac{\sqrt{m^2 - a^2}}{2mr_+}; \tag{89}$$

which is customary referred to as the surface gravity of the black hole.
Also, at the future horizon, one has

$$U \propto \Delta; \tag{90}$$

Kerr metric in double null coordinates VI

where the proportionality factors are smooth functions on the horizon. In order to have a double null system that is smooth across the outer past event horizon we also define the null function V in a similar way; so that we have

$$U = -\exp(-\kappa_+ u), \tag{91}$$

and

$$V = \exp(\kappa_+ v). \tag{92}$$

Let us remark that we have been studying the asymptotic behavior approaching the horizon from the outside region. In the inner region, $U > 0$ and one would use the relation

$$U = \exp(\kappa_+ u_{\text{inner}}); \tag{93}$$

where u_{inner} is the analogous inner version of the null coordinate u in the outer region.

Coordinates $\{U, V, \theta, \varphi\}$
From (91) and (92) we have $du = -\frac{1}{\kappa_+}\frac{dU}{U}$ and $dv = \frac{1}{\kappa_+}\frac{dV}{V}$. from which one obtains

$$\begin{aligned}
ds^2 =& \frac{1}{4}\left(1 - \frac{2mr}{\Sigma} - \frac{\Upsilon a^2 \sin^2(\theta) + \Sigma^2 \Delta}{\Sigma \mathcal{R}} - \frac{\pm_{oi} 4mra^2 \sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right) \frac{1}{\kappa^2 U^2}\, dU^2 + \\
& \frac{1}{4}\left(1 - \frac{2mr}{\Sigma} - \frac{\Upsilon a^2 \sin^2(\theta) + \Sigma^2 \Delta}{\Sigma \mathcal{R}} \pm_{oi} \frac{4mra^2 \sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right) \frac{1}{\kappa^2 V^2}\, dV^2 \\
& - \frac{1}{2}\left(1 - \frac{2mr}{\Sigma} + \frac{\Upsilon a^2 \sin^2(\theta) + \Sigma^2 \Delta}{\Sigma \mathcal{R}}\right) \frac{1}{\kappa^2 UV}\, dU\, dV \\
& + \left[\frac{\left(\Upsilon a^2 \sin^2(\theta) + \Delta\Sigma^2\right)}{\Sigma\mathcal{R}} \pm_{oi} \frac{2mra^2 \sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right] \frac{k}{\kappa U}\, dU d\theta \\
& + \left[\frac{\left(\Upsilon a^2 \sin^2(\theta) + \Delta\Sigma^2\right)}{\Sigma\mathcal{R}} - \frac{\pm_{oi} 2mra^2 \sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right] \frac{k}{\kappa V}\, dV d\theta \\
& - \left(\frac{2amr \sin^2(\theta)}{\Sigma} \pm_{oi} \frac{\Upsilon a \sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right) \frac{1}{\kappa U}\, dU\, d\varphi + \left(\frac{2amr \sin^2(\theta)}{\Sigma} - \pm_{oi} \frac{\Upsilon a \sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right) \frac{1}{\kappa V}\, dV\, d\varphi \\
& - \left[\Sigma + \frac{k^2 \left(\Upsilon a^2 \sin^2(\theta) + \Sigma^2 \Delta\right)}{\Sigma\mathcal{R}}\right] d\theta^2 \pm_{oi} \frac{2\Upsilon a \sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}} k\, d\theta\, d\varphi - \frac{\Upsilon}{\Sigma}\sin^2(\theta)\, d\varphi^2,
\end{aligned} \tag{94}$$

where one has to consider $\kappa = \kappa_+$.

To compute the contravariant expression, we can start from (81) to make a coordinate transformation. After a long calculation, one obtains

$$\begin{aligned}\left(\frac{\partial}{\partial s}\right)^2 =& -4\,\kappa^2\,\frac{\Upsilon}{\Sigma\Delta}\,U\,V\left(\frac{\partial}{\partial U}\right)\left(\frac{\partial}{\partial V}\right)\\ &-\frac{2\kappa k}{\Sigma}\left[U\left(\frac{\partial}{\partial U}\right)+V\left(\frac{\partial}{\partial V}\right)\right]\left(\frac{\partial}{\partial \theta}\right)\\ &-\frac{2\,\kappa\,a\,U}{\Sigma\Delta}\left(2mr-\pm_{oi}\sqrt{R}\right)\left(\frac{\partial}{\partial U}\right)\left(\frac{\partial}{\partial \varphi}\right)\\ &+\frac{2\,\kappa\,a\,V}{\Sigma\Delta}\left(2mr\pm_{oi}\sqrt{R}\right)\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial \varphi}\right)\\ &-\frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2-\frac{1}{\Sigma\sin^2(\theta)}\left(\frac{\partial}{\partial \varphi}\right)^2,\end{aligned} \tag{95}$$

where one has to consider $\kappa = \kappa_+$. The determinant is given by

$$g = -\frac{\Delta^2\Sigma^2\sin(\theta)^2}{4\kappa^4\mathcal{R}U^2V^2}. \tag{96}$$

Other useful algebraic relation.
The product between U(91) and V(92) is

$$UV\mid_{\kappa=\kappa_+}=\frac{-\,e^{2\kappa_+\left(r+\int_0^\theta k(r,\theta')d\theta'\right)}\left(r_-\right)^{\frac{r_-}{r_+}}}{\left(r_+\right)\left(r-r_-\right)^{\frac{r_++r_-}{r_+}}}\,\Delta. \tag{97}$$

Note the θ dependence of this expression; which generalizes the usual Kruskal one.

The causal structure of Kerr spacetime

One of the great benefits of having constructed a double null coordinate system in Kerr spacetime, is that one can now completely describe its causal structure. Recall that up to now, the causal structure was well understood at the axis, namely for $\theta = 0$ or $\theta = \pi$; as it is depicted in diagram (a) of figure 1 in [Carter(1966)], or in figure 27 of page 312 in [Chandrasekhar(1983)]. However, our construction allows us to extend the validity of those causal diagrams, where now each point represents the intersection of the null coordinates. The intersections of the null hypersurface $u =$const. with the null hypersurface $v =$const., determines a particular topological sphere with $r_s =$constant. Using the coordinates U and V that allow us to cross the future horizon $H = H_f$, one has the same property; so that we can give general validity to the causal diagrams bases on these type of null coordinates. In other words, we can construct diagrams as the one shown in the next figure, where each point represents a topological sphere which it could be at one of the horizons with $r = r_+$ or r_-, or the sphere has $r_s =$constant.

Global causal structure II

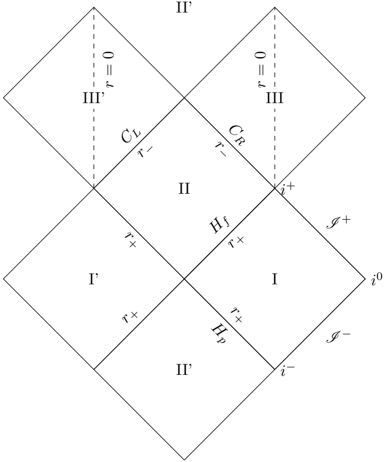


Figure: General causal diagram of Kerr spacetime. The dashed lines indicates the presence of the singularity.

Global causal structure III

Once one has the appropriate definition of the extended coordinates, the causal diagram is straight forward to construct; following the ideas already presented in [Carter(1966)]. We can take Carter's functions $\psi = U_{\text{compact}} + V_{\text{compact}}$ and $\xi = V_{\text{compact}} - U_{\text{compact}}$, where in region I we can define

$$U_{\text{compact}} = \arctan(U), \tag{98}$$

$$V_{\text{compact}} = \arctan(V). \tag{99}$$

To cross each horizon one has to use the appropriate null coordinates. The coordinate U has been constructed to be smooth across the future horizon H_f. We next describe the null coordinates which are smooth at the Cauchy horizons C_R and C_L.

The $(\tilde{U}, \tilde{V})$ coordinates
Using an analog study, as presented before, and defining

$$\kappa_- = \frac{(r_+ - r_-)}{2\sqrt{\mathcal{R}(r_-)}} = \frac{(r_+ - r_-)}{2(r_-^2 + a^2)} = \frac{\sqrt{m^2 - a^2}}{2mr_-}, \tag{100}$$

which we call κ_- because of its analogy with κ_+; the null functions that are regular at the Cauchy horizons are

$$\tilde{V} = -\exp(-\kappa_- v_{\text{inner}}); \tag{101}$$

which is regular at the Cauchy horizon C_R, and

$$\tilde{U} = -\exp(-\kappa_- u_{\rm inner}); \tag{102}$$

which is regular at the Cauchy horizon C_L.
The behavior of u_{inner}, v_{inner} near $r = r_-$ is

$$\lim_{r \to r_-, v = v_0} u_{inner} = \infty, \tag{103}$$

$$\lim_{r \to r_-, u = u_0} v_{inner} = \infty. \tag{104}$$

which is consistent with our definitions (101), (102).
It is important to remark, that with these definitions we obtain the same metric functional expression as (94) and (95), with the only difference that in region II, one has to choose $(\kappa = \kappa_-)$, $(\pm_{oi} = -)$ and take an opposite sign in $g_{U_{\bf inner}\theta}$, $g_{U_{\bf inner}\varphi}$, $g_{V_{\bf inner}\theta}$ and $g_{V_{\bf inner}\varphi}$.
Note that to cover regions I and II with null functions that are smooth across H_f and C_R, one can use the set $(U, \tilde{V})$; which we plan to employ in a future numerical work.

Global causal structure V

It is worthwhile to remark that one can obtain the integral expressions, following an analogous deduction process, as previously explained for the outer region, but now with an initial point $(r_0, \theta_0) = (r_+, 0)$, namely

$$u_{inner} = -t_{inner} + r_s, \tag{105}$$

$$v_{inner} = t_{inner} + r_s; \tag{106}$$

where from the definitions of κ_+ and κ_-, one can express r_s in region II as

$$\begin{aligned} r_s(r,\theta) = &\, r + \frac{1}{2\kappa_+}\ln(1 - \frac{r}{r_+}) - \frac{1}{2\kappa_-}\ln(\frac{r}{r_-} - 1) \\ &+ \int_0^\theta k(r,\theta')\, d\theta'. \end{aligned} \tag{107}$$

Then, one can also reproduce a similar previous algebraic study, which in this case is

$$\begin{aligned} \tilde{U}\tilde{V} =& e^{-2\kappa_- r_s} = e^{-2\kappa_-\left(r + \int_0^\theta k(r,\theta')d\theta'\right)} \frac{(\frac{r}{r_-} - 1)}{(1 - \frac{r}{r_+})^{\frac{\kappa_-}{\kappa_+}}} \\ =& \frac{e^{-2\kappa_-\left(r + \int_0^\theta k(r,\theta')d\theta'\right)} (r_+)^{\frac{\kappa_-}{\kappa_+}}}{(r_-)(r_+ - r)^{\frac{\kappa_-}{\kappa_+}}(r - r_+)} \Delta = - \frac{e^{-2\kappa_-\left(r + \int_0^\theta k(r,\theta')d\theta'\right)} (r_+)^{\frac{r_+}{r_-}}}{(r_-)(r_+ - r)^{\frac{r_+ + r_-}{r_-}}} \Delta; \end{aligned} \tag{108}$$

where we have used that $\frac{\kappa_-}{\kappa_+} = \frac{r_+}{r_-}$. Note that $\tilde{U}\tilde{V} \geqslant 0$ in region II and is well behaved across C_R.
Finally, the natural definition of compact extended functions in region II are

$$\tilde{U}_{\text{compact}} = \arctan(\tilde{U}), \tag{109}$$

$$\tilde{V}_{\text{compact}} = \arctan(\tilde{V}); \tag{110}$$

and one can extend the definitions of ψ and ξ accordingly.

Region II and III: null tetrad and r_s family
In a similar way as we did previously; we can build an adapted null tetrad for region II. As before, we take

$$(\ell_{\text{inner}})_a = (du_{\text{inner}})_a \,, \tag{111}$$

$$(n_{\text{inner}})_a = -\frac{1}{2}\frac{\Sigma\Delta}{\Upsilon}(dv_{\text{inner}})_a \,, \tag{112}$$

$$(m_{\text{inner}})_a = i\,\frac{\sqrt{2}amr\sin(\theta)}{\sqrt{\Upsilon\,\Sigma}}dt_a + k\sqrt{\frac{\Sigma}{2\Upsilon}}dr_a \tag{113}$$

$$-\sqrt{\frac{\Sigma\,\mathcal{R}}{2\Upsilon}}d\theta_a - i\sqrt{\frac{\Upsilon}{2\Sigma}}\sin(\theta)d\phi_a;$$

where m^a_{inner} is tangent to the surfaces r_s=constant, and the usual null tetrad metric conditions are satisfied.
In region III, each one-form dt_a and dr_a, recovers the same causal character of region I, but with the difference that r and r_s decreases to the right. Therefore, we can use a similar definition

$$du_{\text{III}} = dt_{\text{III}} + dr_s, \tag{114}$$
$$dv_{\text{III}} = dt_{\text{III}} - dr_s. \tag{115}$$

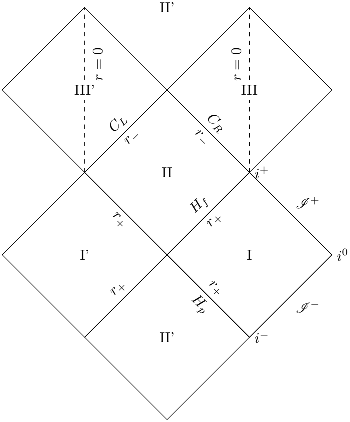
Then to build the null tetrad we take $\ell_{\text{III}} = du_{\text{III}}$, $n_{\text{III}} = \frac{\Sigma\Delta}{2\Upsilon} dv_{\text{III}}$ and we complete the tetrad in an analogous way, as we did previously; namely, m^a_{III} is tangent to the surfaces r_s=constant, and the usual null tetrad metric conditions are satisfied. In this case, for each surface r_s =constant, we also obtain the same functional expressions of Extrinsic and Gaussian curvature.
It is useful to remark that in order to obtain the metric in the $\{u_{\text{III}}, v_{\text{III}}, \theta, \phi\}$ coordinate system one has to replace in (80): $(du \to dv_{\text{III}})$ and $(dv \to du_{\text{III}})$.
Note also that the functional expressions of $\tilde{U}$ and $\tilde{V}$ are good coordinates for region III.

Final comments I

We have presented a smooth double null foliation of Kerr spacetime which allows for a direct interpretation of its global causal structure

- Constructions of null foliations by other authors, have problems on the axis of symmetry.
- Smooth null coordinates adapted for the outer horizons were presented for region I.
- We have also introduced smooth null coordinates for region II adapted to the Cauchy horizons.
- Smooth null coordinates for region III were also constructed.
- The fact that we were able to construct a genuine conformal diagram of region III, does not mean that its intrinsic problems have disappear. Still, equatorial geodesics can reach the singularity at $(r = 0, \theta = \frac{\pi}{2})$. And, region III is acausal, since there exist close timelike curves in it.
- Conformal diagrams based on (U, V) type coordinates, are a useful tool to study the characteristic problem of wave equations. Since data on a wedge $\{U, V\}$ would have a clear domain of dependence.

In any case, our construction has allowed us to create a conformal diagram that it can be applied globally.



General global causal diagram of Kerr spacetime; which could be extended to upper and lower regions.

The dashed lines indicates the presence of the singularity.

Thank You!

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