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Classical and Quantum Cosmological Singularities

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Introduction and Motivations

- Appearance of singularities is one of the most important phenomena in General Relativity and its generalizations and modifications.
- The singularities were first discovered in such simple geometries as those of Friedmann and Schwarzschild and later their general character was established (Penrose, Hawking).
- The investigation of the oscillatory approach to the cosmological singularity (Belinsky, Khalatnikov, Lifshitz) known also as Mixmaster universe (Misner) has opened the way to the birth of a new branch of the mathematical physics chaotic cosmology and hyperbolic Kac-Moody algebras (Damour, Henneaux, Nicolai).

Introduction and Motivations

- One can try to study the opportunity to cross the singularity.
- Very interesting effects arise at the singularities crossing
- Can quantum cosmology eliminate the singularities arising in classical General Relativity?
- What happens with quantum particles at the singularity crossing?
- When is it possible and when it is not possible to describe the transition through the singularity?

  An attempt of a general approach

The Big Brake cosmological singularity, more general soft singularities and their crossing

Description of the tachyon model

The flat Friedmann universe

$$ds^2 = dt^2 - a^2(t)dl^2$$

The tachyon Lagrange density

$$L = -V(T)\sqrt{1-\dot{T}^2}$$

The energy density

$$\rho = \frac{V(T)}{\sqrt{1-\dot{T}^2}}$$

The pressure

$$p = -V(T)\sqrt{1-\dot{T}^2}$$

The Friedmann equation

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \rho$$

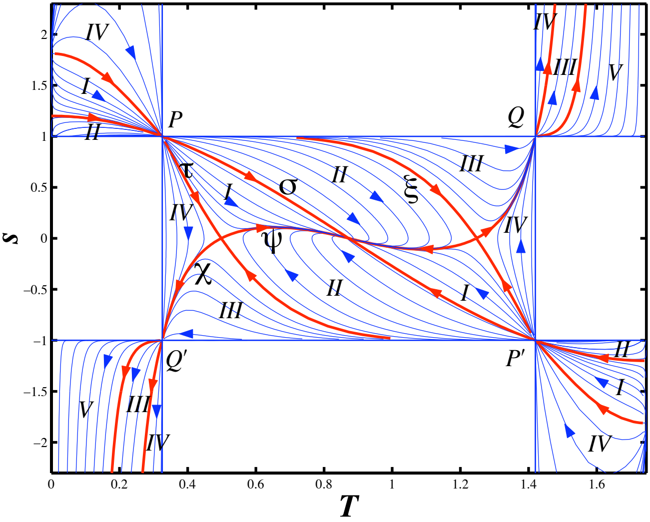
The equation of motion for the tachyon field

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V_{,T}}{V} = 0$$

In our model

$$V(T) = \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\;T\right]}$$
$$\times\sqrt{1-(1+k)\cos^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\;T\right]}\;,$$

where k and $\Lambda > 0$ are the parameters of the model. The case $k > 0$ is more interesting.



Phase portrait of the model for a positive k.

Some trajectories (cosmological evolutions) finish in an infinite de Sitter expansion. In other trajectories the tachyon field transforms into a pseudotachyon field with the Lagrange density, energy density and positive pressure:

$$
\begin{aligned}
L &= W(T)\sqrt{\dot{T}^2 - 1}, \\
\rho &= \frac{W(T)}{\sqrt{\dot{T}^2 - 1}}, \\
p &= W(T)\sqrt{\dot{T}^2 - 1}, \\
W(T) &= \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda\left(1+k\right)}\ T\right]} \\
&\times\sqrt{\left(1+k\right)\cos^2\left[\frac{3}{2}\sqrt{\Lambda\left(1+k\right)}\,T - 1\right]}
\end{aligned}
$$

What happens to the Universe after the transformation of the tachyon into the pseudotachyon ?

It encounters the Big Brake

cosmological singularity.

The Big Brake cosmological singularity and other soft singularities

$$t \to t_{BB} < \infty$$

$$a(t \to t_{BB}) \to a_{BB} < \infty$$

$$\dot{a}(t \to t_{BB}) \to 0$$

$$\ddot{a}(t \to t_{BB}) \to -\infty$$

$$R(t \to t_{BB}) \to +\infty$$

$$\rho(t \to t_{BB}) \to 0$$

$$p(t \to t_{BB}) \to +\infty$$

If $\dot{a}(t_{BB}) \neq 0$ it is a more general soft singularity.

Crossing the Big Brake singularity and the future of the universe

At the Big Brake singularity the equations for geodesics are regular, because the Christoffel symbols are regular (moreover, they are equal to zero).

Is it possible to cross the Big Brake ?

Let us study the regime of approach to the Big Brake.

On analyzing the equations of motion we find that on approaching the Big Brake singularity the tachyon field behaves as

$$T = T_{BB} + \left(\frac{4}{3W(T_{BB})}\right)^{1/3} (t_{BB} - t)^{1/3}.$$

Its time derivative $s \equiv \dot{T}$ behaves as

$$s = -\left(\frac{4}{81W(T_{BB})}\right)^{1/3} (t_{BB} - t)^{-2/3},$$

the cosmological radius is

$$a = a_{BB} - \frac{3}{4} a_{BB} \left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{4/3},$$

its time derivative is

$$\dot{a} = a_{BB} \left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{1/3}$$

and the Hubble variable is

$$H = \left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{1/3}.$$

All these expressions can be continued in the region where $t > t_{BB}$,which amounts to crossing the Big Brake singularity. Only the expression for s is singular at $t = t_{BB}$ but this singularity is integrable and not dangerous.

Once reaching the Big Brake, it is impossible for the system to stay there because of the infinite deceleration, which eventually leads to a decrease of the scale factor. This is because after the Big Brake crossing the time derivative of the cosmological radius and Hubble variable change their signs. The expansion is then followed by a contraction, culminating in the Big Crunch singularity.

Crossing of the soft singularity in the model with the anti-Chaplygin gas and dust

One of the simplest cosmological models revealing a Big Brake singularity is the model based on the anti-Chaplygin gas with an equation of state

$$p = \frac{A}{\rho}, \quad A > 0$$

Such an equation of state arises in the theory of wiggly strings (B. Carter, 1989, A. Vilenkin, 1990).

$$\rho(a) = \sqrt{\frac{B}{a^6} - A}$$

At $a = a_* = \left(\frac{B}{A}\right)^{1/6}$ the universe encounters the Big Brake singularity.

The anti-Chaplygin gas plus dust

The energy density and the pressure are

$$\rho(a) = \sqrt{\frac{B}{a^6} - A} + \frac{M}{a^3}, \quad p(a) = \frac{A}{\sqrt{\frac{B}{a^6} - A}}.$$

Due to the dust component, the Hubble parameter has a non-zero value at the encounter with the singularity, therefore the dust implies further expansion. With continued expansion however, the energy density and the pressure of the anti-Chaplygin gas would become ill-defined.

Change of the equation of state at soft singularity crossings

The abrupt transition from the expansion to the contraction of the universe does not look natural. There is an alternative/complementary way of resolving the paradox.

One can try to change the equation of state of the anti-Chaplygin gas on passing the soft singularity.

There is some analogy between the transition from an expansion to a contraction of a universe and the perfectly elastic bounce of a ball from a wall in classical mechanics. There is also an abrupt change of the direction of the velocity (momentum).

However, we know that in reality the velocity is changed continuously due to the deformation of the ball and of the wall.

The pressure of the anti-Chaplygin gas

$$p = \frac{A}{\sqrt{\frac{B}{a^6} - A}}$$

tends to $+\infty$ when the universe approaches the soft singularity.

Requiring the expansion to continue into the region $a > a_S$, while changing minimally the equation of state, we assume

$$p = \frac{A}{\sqrt{\left|\frac{B}{a^6} - A\right|}},$$

$$p = \frac{A}{\sqrt{A - \frac{B}{a^6}}}, \text{ for } a > a_S.$$

This implies the energy density

$$\rho = -\sqrt{A - \frac{B}{a^6}}.$$

The anti-Chaplygin gas transforms itself into Chaplygin gas with negative energy density.
The pressure remains positive, expansion continues.
The spacetime geometry remains continuous.
The expansion stops at $a = a_0$, where

$$\frac{M}{a_0^3} - \sqrt{A - \frac{B}{a_0^6}} = 0.$$

Then the contraction of the universe begins.

At the moment when the energy density of the Chaplygin gas becomes equal to zero

(again a soft singularity), the Chaplygin gas transforms itself into the anti-Chaplygin gas

and the contraction continues culminating in an encounter with the Big Crunch singularity

$a = 0$.

Analogous effects arise in the model with the tachyon field and dust. The Lagrangian of the Born-Infeld like field changes its form.

Big Bang – Big Crunch crossing ?

- The idea that the Big Bang - Big Crunch singularity can be crossed appears very counterintuitive.
- Some approaches to the description of this crossing were elaborated during the last decade (I. Bars, S.H. Chen, P.J. Steinhardt and N. Turok, C. Wetterich, P. Dominis Prester).
- There is an analogy with the horizon which arises due to a certain choice of the spacetime coordinates: the singularity arises because of some choice of the field parametrization.

- On choosing some convenient field parametrization one can provide a matching between the characteristics of the universe before and after the singularity crossing.
- Analogy to the Kruskal coordinates for the Schwarzschild metric.
- On choosing appropriate combinations of the field variables we can describe the passage through the Big Bang - Big Crunch singularity, but this does not mean that the presence of such a singularity is not essential. Indeed, extended objects cannot survive this passage.

Friedmann cosmology in the presence of a scalar field: Einstein frame versus Jordan frame

$$S = \int d^4x \sqrt{-g} \left[U(\sigma) R - \frac{1}{2} g^{\mu\nu} \sigma_{,\mu} \sigma_{,\nu} + V(\sigma) \right]$$

Conformal coupling

$$U(\sigma) = U_0 - \frac{1}{12} \sigma^2$$

A conformal transformation of the metric

$$g_{\mu\nu} = \frac{U_1}{U}\tilde{g}_{\mu\nu},$$

A new scalar field ϕ:

$$\frac{d\phi}{d\sigma} = \frac{\sqrt{U_1(U + 3U'^2)}}{U} \Rightarrow \phi = \int \frac{\sqrt{U_1(U + 3U'^2)}}{U} d\sigma.$$

$$\phi = \sqrt{3U_1} \ln \left[\frac{\sqrt{12U_0} + \sigma}{\sqrt{12U_0} - \sigma}\right]$$

$$\sigma = \sqrt{12U_0} \tanh \left[\frac{\phi}{\sqrt{12U_1}}\right].$$

The action then becomes the action for a minimally coupled scalar field:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[U_1 R(\tilde{g}) - \frac{1}{2}\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + W(\phi) \right],$$

$$W(\phi) = \frac{U_1^2 V(\sigma(\phi))}{U^2(\sigma(\phi))}.$$

This is called the transformation from the Jordan frame to the Einstein frame.

In a flat Friedmann universe

$$ds^2 = N^2 d\tau^2 - a^2 dl^2,$$

$$d\tilde{s}^2 = \tilde{N}^2 d\tau^2 - \tilde{a}^2 dl^2.$$

$$\tilde{N} = \sqrt{\frac{U}{U_1}} N, \ \tilde{a} = \sqrt{\frac{U}{U_1}} a, \ t = \int \sqrt{\frac{U_1}{U}} d\tilde{t},$$

where t and $\tilde{t}$ are the cosmic time parameters in the Jordan and the Einstein frames.

$$a = \tilde{a}\sqrt{\frac{U_1}{U_0}} \cosh\left(\frac{\phi}{\sqrt{12U_1}}\right).$$

In the vicinity of the singularity in the Einstein frame:

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}} \to 0, \ \text{when} \ \tilde{t} \to 0.$$

However, in the Jordan frame:

$$a \sim \tilde{t}^{\frac{1}{3}} \left(\tilde{t}^{\frac{1}{3}} + \tilde{t}^{-\frac{1}{3}} \right) \to \text{const} \neq 0.$$

Meanwhile, the scalar field σ crosses the value $\pm\sqrt{12U_0}$ and the coupling function U changes its sign.

Thus, the evolution in the Jordan frame is regular, and we can use this fact to describe the crossing of the Big Bang - Big Crunch singularity in the Einstein frame.

If one considers the expansion of the universe from the Big Bang with normal gravity driven by the standard scalar field, the continuation backward in time shows that it was preceded by the contraction towards a Big Crunch singularity in the antigravity regime, driven by a phantom scalar field with a negative kinetic term.

The possibility of a change of sign of the effective gravitational constant in the model with a conformably coupled scalar field was analyzed in 1981 by A. Starobinsky.
It was shown that in a homogeneous and isotropic universe, one can indeed cross the point where the effective gravitational constant changes sign.
However, the presence of anisotropies changes the situation: these anisotropies grow indefinitely when this constant is equal to zero.
To describe the Big Bang - Big Crunch singularity crossing in anisotropic universes it is necessary to use another methods. We have done it for the Bianchi-I universe.

Quantum cosmology and singularities

Speaking about quantum cosmology and singularities people mean two different things:
Modification of the Friedmann equation.

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \rho_{\rm matter} + \rho_{\rm quantum\ corrections}.$$

Vanishing of the quantum state of the universe.

$$\Psi(\text{geometry} + \text{matter})_{\text{geometry is singular}} = 0.$$

Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = 0.$$

Where is the time ?
What is the probability ?

A time can be defined as a certain function of geometrical variables.
After that the wavefunction describing matter variables satisfies an effective Schrödinger equation.
The singularity is associated with such values of the matter variables when this singularity arises in the classical theory.
Our analysis of some simple models tells that the probability of the arising of soft singularities is not suppressed by the wave function of the universe, while the probability of Big Bang – Big Crunch singularity tends to zero.

Particles, fields and singularities

What happens with particles (in quantum field theoretical sense) when the universes passes through the cosmological singularity ?
The scalar field in the flat Friedmann universe satisfies the Klein-Gordon equation:

$$\Box\phi + V'(\phi) = 0.$$

One can consider a spatially homogeneous solution of this equation ϕ_0, depending only on time t as a classical background.
A small deviation from this background solution can be represented as a sum of Fourier harmonics satisfying linearized equations

$$\ddot{\phi}(\vec{k}, t) + 3\frac{\dot{a}}{a}\dot{\phi}(\vec{k}, t) + \frac{\vec{k}^2}{a^2}\phi(\vec{k}, (t)) + V''(\phi_0(t))\phi(\vec{k}, (t)) = 0.$$

The corresponding quantized field is

$$\hat{\phi}(\vec{x},t) = \int d^3\vec{k}(\hat{a}(\vec{k})u(k,t)e^{i\vec{k}\cdot\vec{x}} + \hat{a}^+(\vec{k})u^*(k,t)e^{-i\vec{k}\cdot\vec{x}}),$$

where the creation and the annihilation operators satisfy the standard commutation relations:

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = \delta(\vec{k} - \vec{k}').$$

The basis functions should be normalized so that the canonical commutation relations between the field ϕ and its canonically conjugate momentum $\hat{\mathcal{P}}$ were satisfied

$$[\hat{\phi}(\vec{x},t), \hat{\mathcal{P}}(\vec{y},t')] = i\delta(\vec{x} - \vec{y}).$$

$$u(k,t)\dot{u}^*(k,t) - u^*(k,t)\dot{u}(k,t) = \frac{i}{(2\pi)^3 a^3(t)}.$$

The linearized Klein-Gordon equation has two independent solutions.

To define a particle it is necessary to have two independent non-singular solutions.

it is a non-trivial requirement in the situations when a singularity or other kind of irregularity of the spacetime geometry occurs.

It is convenient also to construct explicitly the vacuum state for quantum particles as a Gaussian function of the corresponding variable. Let is introduce an operator

$$\hat{f}(\vec{k},t) = (2\pi)^3(\hat{a}(\vec{k})u(k,t) + \hat{a}^+(-\vec{k})u^*(k,t)).$$

Its canonically conjugate momentum is

$$\hat{p}(\vec{k},t) = a^3(t)(2\pi)^3(\hat{a}(\vec{k})\dot{u}(k,t) + \hat{a}^+(-\vec{k})\dot{u}^*(k,t)).$$

We can express the annihilation operator as

$$\hat{a}(\vec{k}) = i\hat{p}(\vec{k},t)u^*(k,t) - ia^3(t)\hat{f}(\vec{k},t)\dot{u}^+(k,t).$$

Representing the operators $\hat{f}$ and $\hat{p}$ as

$$\hat{f} \to f, \quad \hat{p} \to -i\frac{d}{df},$$

one can write down the equation for the corresponding vacuum state in the following form:

$$\left(u^*\frac{d}{df} - ia^3\dot{u}^* f\right)\Psi_0(f) = 0.$$

$$\Psi_0(f) = \frac{1}{\sqrt{|u(k,t)|}}\exp\left(\frac{ia^3(t)\dot{u}^*(k,t)f^2}{2u^*(k,t)}\right).$$

In the case of the Big Bang - Big Crunch singularity, one of the basis functions in the vicinity of the singularity becomes singular and it is impossible to construct a Fock space.

In the case of the Big Rip singularity, when in finite interval of time the universe achieves an infinite volume and infinite time derivative of the scale factor, the Fock space can be constructed for a spectator scalar field, but it does not exist for the phantom scalar field driving the expansion.

In the case of the model with tachyon. field, presented above, we have considered three situations.

The non-singular transformation of the tachyon into pseudo-tachyon. In this case both basis functions are regular and hence the operators of creation and annihilation are well defined.
However, at the moment of the transformation the dispersion of the Gaussian wave function of the vacuum becomes infinite and then becomes finite again.

In the vicinity of the Big Brake singularity it is impossible to define a Fock vacuum.

However, if we add to the universe dust, the character of the soft singularity is slightly changed and then the presence of the Fock vacuum is restored.

Covariant approach to singularities

The crossing of the Big Bang - Big Crunch singularities looks rather counterintuitive.
However, it can be sometimes described by using the reparametrization of fields, including the metric.
One can say that to do this, it is necessary to resort to one of two ideas, or a combination thereof.
One of these ideas is to employ a reparameterization of the field variables which makes the singular geometrical invariant non-singular.

Another idea is to find such a parameterization of the fields, including, naturally, the metric, that gives enough information to describe consistently the crossing of the singularity even if some of the curvature invariants diverge.

The application of these ideas looks in a way as an craftsman work.

Our goal is to develop a general formalism to distinguish "dangerous" and "non-dangerous" singularities, considering the field variable space of the model under consideration.

When the spacetime singularities can be removed by a reparametrization of the field variables?

Our hypothesis: when the geometry of the space of the field variables is non-singular.
The field space $\mathcal{S}$ was developed in order to treat on the same (geometrical) footing both changes of coordinates in the spacetime $\mathcal{M}$ and field redefinitions in the functional approach to quantum field theory.
This approach requires introducing a local metric G in field space $\mathcal{S}$ and computing the associated geometric scalars by defining a covariant derivative which is compatible with G.
G is actually determined by the kinetic part of the action and its dimension depends on the field content of the latter.

After some cumbersome calculations in the functional space, we have shown that the Kretschmann scalar

$$\mathcal{K} = \mathcal{R}_{ABCD}\,\mathcal{R}^{ABCD}$$

is finite in every theory of pure gravity

$$\mathcal{K} = \frac{n}{8}\left(\frac{n^3}{4} + \frac{3n^2}{4} - 1\right),$$

where n is the spacetime dimension.
It can be interpreted as a statement that all the singularities in empty universe can be crossed.

Another hypothesis: quantum effective action and to homotopy group

Let us introduce the functional

$$\psi[\varphi] = e^{i\,\Gamma[\varphi]},$$

where $\Gamma[\varphi]$ is the effective action. We shall call $\psi[\varphi]$ the functional order parameter because ψ plays the analogous role of an order parameter in the theory of phase transitions in ordered media or cosmology.
The field space $\mathcal{M}$ can be thought of as the ordered medium itself, whereas functional singularities correspond to topological defects.
The functional order parameter ψ defines the map

$$\psi : \mathcal{M} \to \mathbb{S}^1,$$

from the field space to the unit circle, the latter playing the role of the order parameter space.

The singularities can be characterized by the fundamental group (first homotopy group).
If this group is trivial the singularity can be removed.
We have checked on the example of some simple systems with removable singularity that the corresponding homotopy group is indeed trivial.

Conclusions

- The appearance of the singularities in the cosmological and other gravitational systems is not drawback of models or theories
- It is their distinguishing feature.
- Rather than avoid singularities, it is better to study how their presence influences the non-singular quantities (just like in quantum field theory).

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