Naked singularities and the Quasi-normal modes

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Introduction

Schwarzschild black hole

- The paradigm of a black hole is given by the Schwarzschild solution in General Relativity:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2.$$

- An important remark: this solution is stable.

Introduction

Schwarzschild black hole

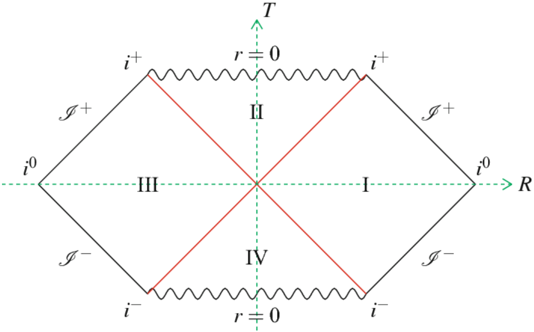
- The corresponding naked singularity solution can be constructed by considering a negative mass:

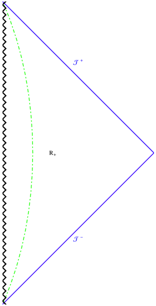
$$ds^2 = \left(1 + \frac{2G|M|}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2G|M|}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2.$$

- Only recently it has been stablished the instability of this solution (Gleiser&Dotti, 2006).

Introduction

Black hole and naked singularity





Introduction

Schwarzschild black hole

- The study of the stability of naked singularities is frequently delicate for many reasons.
- Among them
  1. A naked singularity spacetime is not globally hyperbolic;
  2. Connect to the previous problem, the boundary conditions are not always evident;
  3. And yet, it must be verified if the operator connected with the perturbed equation is self-adjoint assuring
     - a unitary evolution and
     - the validity of the spectral theorem (which allows a Fourier decomposition for example).

Quasi Normal Modes - QNM

The set up

- Perturbing the Schwarzschild space-time, we introduce the fluctuations around the previous solution:

$$g_{\mu\nu} = g^B_{\mu\nu} + h_{\mu\nu},$$

where $g^B_{\mu\nu}$ is the previous *background solution* and $h_{\mu\nu}$ is a small fluctuation around it.

Quasi Normal Modes - QNM

The set up

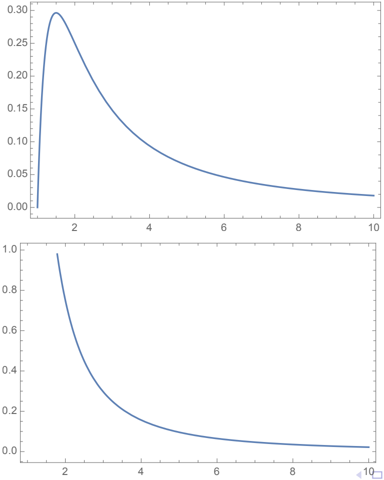
- The perturbed equations can be divided into polar and axial modes.
- Let us consider, for simplicity, only the the axial modes, given by

$$\frac{d^2\phi}{dr_*^2} + f(r)\left\{\omega^2 - V(r)\right\}\phi = 0, \tag{1}$$

$$\begin{aligned} r_* &= r + 2GM \ln\left\{\frac{2GM}{r} - 1\right\}, &(2)\\ f(r) &= 1 - \frac{2GM}{r}, &(3)\\ V(r) &= f(r)\left\{\frac{l(l+1)}{r^2} + \frac{\beta M}{r^3}\right\}, \quad \beta = 1, 0, -3. &(4) \end{aligned}$$

Quasi Normal Modes - QNM

The potential



Quasi Normal Modes - QNM

The frequencies

- Roughly speaking, the QNM is characterised by a complex frequency:

$$\omega = \omega_0 - i\omega_i,$$
$$\omega_0, \omega_i > 0.$$

- They corresponds to damped oscillations.

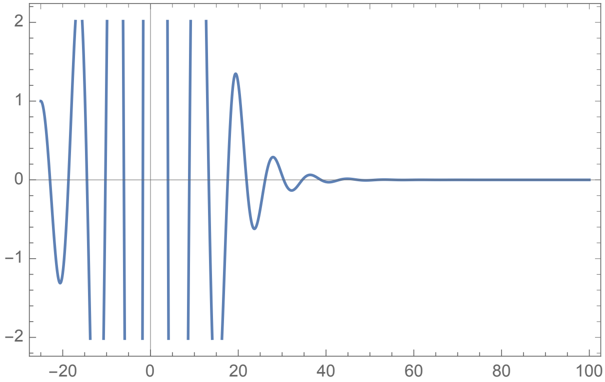
Quasi Normal Modes - QNM

The boundary conditions

- The Black Hole perturbations are characterised by specific boundary conditions:
  - On the horizon, only ingoing modes;
  - At the spatial infinity, only outgoing modes.

Quasi Normal Modes - QNM

The wave form



Calculating the frequencies

Calculating the frequencies

- One fundamental task is to computing the frequencies associated with the potential of a black hole.
- There are numerical, perturbative and semi-analytical methods.

Calculating the frequencies

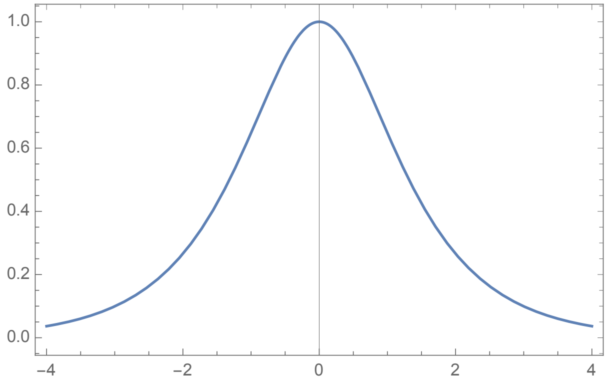
Approximative method

- The potential of the black hole perturbations can be approximated by

$$V(x) = \frac{1}{\cosh^2 x}.$$

Calculating the frequency

The potential



Calculating the frequencies

A theorem

- The problem to face is essentially a scattering problem.
- There is a theorem by Heisenberg and Kramer that such problem can be reduced to that of computing the bound states of the inverse potential.

The Pöschl-Teller problem

The Schrödinger equation

- The energy eigenstate in quantum mechanics is given by the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}+\biggl\{-E+V(x)\biggr\}\Psi=0.$$

- Since we are interested in bounded states, with the Pöschl-Teller potential, let us consider,

$$\begin{aligned} E &= -\Omega^2, \\ V &= -\frac{\bar{V}_0}{\cosh^2 x}, \quad \bar{V}_0 > 0. \end{aligned}$$

The Pöschl-Teller problem

Solving the equation

- By defining,

$$
\begin{aligned}
y &= \tanh x, \\
\Psi &= (1 - y^2)^{\frac{\Omega}{2}} f(x),
\end{aligned}
$$

we end up with the equation,

$$
(1 - y^2)\ddot{f} - 2y(1 + \Omega)\dot{f} + \left\{ \bar{V}_0 - \Omega - \Omega^2 \right\} f = 0.
$$

The Pöschl-Teller problem

Solving the equation

- The equation can be solved using the Fröbenius method:

$$f(x) = \sum_{n=0}^{\infty} a_n y^n . +,$$

- The resulting series is divergent.
- To obtain square-integrable functions, the series must terminate. This implies

$$\Omega = -m - \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\bar{V}_0}.$$

- The resulting functions are the Legendre polynomials.

The Pöschl-Teller problem

Relating to the Black Hole problem

- The black hole perturbed equation can be written as

$$\frac{d^2\Psi}{dr_*^2} + \left\{\omega^2 - V(r_*, \alpha)\right\}\Psi = 0.$$

- The potential can be approximated by ,

$$V(r_*, \alpha) = \frac{V_0}{\cosh^2(\alpha r_*)}.$$

- The parameters V_0 and α are conneced with the maximum of the potential and the curvature (second derivative) of the potential at the maximum of the barrier, respectively.

The Pöschl-Teller problem

Relating to the Black Hole problem

- The approximative BH perturbation equation can be recasted into a Pöschl-Teller problem with the definitions,

$$\begin{aligned} r_* &= -ix, \\ \alpha &= i\alpha', \\ V_0 &= -\frac{\bar{V}_0}{\alpha^2}, \\ \omega &= \frac{\Omega}{i\alpha}. \end{aligned}$$

The Pöschl-Teller problem

Relating to the Black Hole problem

- In this way, an approximation expression can be obtained for the frequencies of the QNM:

$$\omega = \pm\sqrt{V_0 - \frac{\alpha^2}{4}} - i\left(m + \frac{1}{2}\right)\alpha.$$

- This approximation is quite good for high l.

Naked singularities

Test fields

- This analysis can be repeated for naked singularities?
- Such study has been done, using some specific methodologies, by many authors including A. Saa and C. Chirenti.

Naked singularities

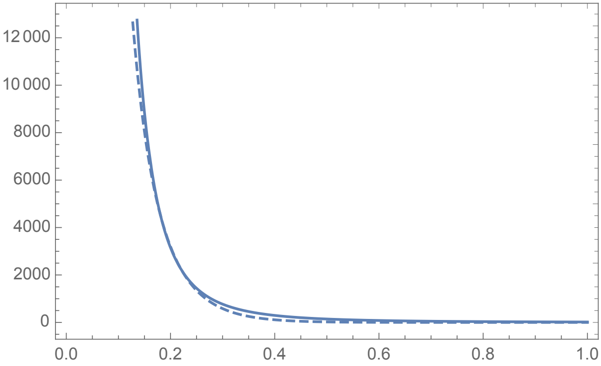
The potential

- However, the shape of the potential (except perhaps for the scalar mode) can be approximated by the expression

$$V(x) = \frac{V_0}{\sinh^2 \alpha x}.$$

Naked singularities

The potential



Naked singularities

The Schödinger equation

- The corresponding equation leading to *bound sates*, would be,

$$-\frac{d^2\Psi}{d^2}+\left\{-\Omega^2-\frac{\bar{V}_0}{\sinh^2\alpha' x}\right\}\Psi=0.$$

- The correspondence with the gravitational perturbations is done, as before, by setting,

$$r_* = -ix, \quad \alpha' = i\alpha, V_0 = -\bar{V}_0.$$

Naked singularities

The evolution equation

- Using the variable transformations,

$$u = \tanh x, \quad z = u^2,$$

the resulting equation is,

$$z(1-z)\frac{d^2\Psi}{dz^2} + \left\{\frac{1}{2} - \frac{3}{2}z\right\}\frac{d\Psi}{dz} + \left\{\frac{\bar{V}_0 - (\Omega^2 + V_0)z}{4z(1-z)}\right\}\Psi = 0.$$

Naked singularities

The evolution equation

- The following redefinition is done:

$$\Psi = z^p(1-z)^q\lambda(z).$$

Naked singularities

The evolution equation

- If,

$$
\begin{aligned}
p &= \frac{1}{4}\left\{1 \pm \sqrt{1 - 4\bar{V}_0}\right\}, \\
q &= \pm\frac{\Omega}{2},
\end{aligned}
$$

we end up with a hypergeometric equation:

$$
z(1-z)\lambda'' + \left\{c - (a+b+1)z\right\}\lambda' - ab\lambda = 0.
$$

Naked singularities

Solution

- The solution is,

$$\Psi = z^p(1-z)^q\Big\{A\,{}_2F_1(a,b,c;z)$$

$$+B\,z^{(1-c)}{}_2F_1(a-c+1,b-c+1,2-c;z)\Big\},$$

with A, B constants.

Naked singularities

Solution

- The parameters a, b, c obey the relations,

$$\begin{aligned} a &= p + q, \\ b &= p + q + \frac{1}{2}, \\ c &= 2p + \frac{1}{2}. \end{aligned}$$

Naked singularities

Convergence

- But, we must verify the convergence of the solution.

Naked singularities

Convergence

- From now on, let us suppose, for simplicity, $A \neq 0$ and $B = 0$.

Naked singularities

Convergence

- In order to make sense, the solutions found previously must be square integrable.
- One way to assure this is to impose that,

$$\begin{aligned} \Psi(0) &= 0. \\ \Psi(1) &= 0. \end{aligned}$$

- The first condition can be obtained if we impose $p > 0$.

Naked singularities

Convergence

- One way to assure the second condition is to impose,

$$ {}_2F_1(a, b, c; 1) = 0. $$

Naked singularities

Convergence

- But,

$$ {}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = 0. $$

- This implies either,

$$ \begin{aligned} c - a &= -n, \quad n \in \mathcal{I}, \\ c - b &= -m, \quad m \in \mathcal{I}. \end{aligned} $$

Naked singularities

Convergence

- The first possibility implies,

$$\omega = \pm 2(n+p)\alpha\, i.$$

- The second possibility implies,

$$\omega = \pm[2(m+p)+1]\alpha i.$$

Naked singularities

Stability

- The frequencies are pure imaginary, and can be of both signs.
- They can be both positive or negative.
- Since,

$$\Psi(r,t) = \Psi(r)e^{-i\omega t},$$

a pure imaginary frequency (positive) implies a perturbation that grows indefinitely with time.

Naked singularities

Stability

- The configuration is unstable!

Conclusions

- An adaptation of the Pöschl-Teller method for computing the perturbations of naked singularities reveals the absence of QNM and the presence of instabilities.
- The original quantum mechanical problem is clearly unstable: the potential is not bounded below.
- The use of such method is consistent in this case?

Brasil, §09/10/1938 †01/03/2020