Compact objects in General Relativity: singularities, symmetries and stability

V Escola José Plínio Baptista de Cosmologia, 2021

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September 30 - October 5, 2021

Contents

1 Black holes, symmetries and singularities

It's oftentimes remarked that Schwarzschild's solution of Einstein's vacuum field equations,

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \frac{r}{r - r_S} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{1}$$

($r_S > 0$ the Schwarzschild radius) has the following properties:

1. *Symmetries*: it is *static and spherically symmetric.*
2. Has a *coordinate singularity* at $r = r_S$
3. Has a *true/geometric/physical singularity* at $r = 0$.
4. Contains a *black hole (BH) region*

The purpose of is lecture is to explain in detail every concept that appears in italics above.

1.1 Coordinate singularities

In what follows (M, g_{ab}) is a Riemannian or Lorentzian manifold with metric g_{ab}

> Coordinate singularities are singularities in the components, in a given coordinate system, of globally defined smooth tensor fields on M, as we take limits to points *at the boundary of the open set where the coordinates take values.* They do not have any relevance. If M extends beyond this coordinate patch, they can be removed by switching to a valid coordinate patch overlapping at this boundary.

Recall that a manifold M comes equipped with an atlas of coordinate patches, and that the components of a tensor field at the intersection of two coordinate patches are related through:

$$\tilde{T}^{a_1 a_2 \dots}_{b_1 b_2 \dots}\Big|_{\tilde{x}} = T^{c_1 c_2 \dots}_{d_1 .. d_2 \dots}\Big|_{x(\tilde{x})} \left[\frac{\partial \tilde{x}^{a_1}}{\partial x^{c_1}} \frac{\partial \tilde{x}^{a_2}}{\partial x^{c_2}} \dots\right]\Bigg|_{x(\tilde{x})} \left[\frac{\partial x^{d_1}}{\partial \tilde{x}^{b_1}} \frac{\partial x^{d_2}}{\partial \tilde{x}^{b_2}} \dots\right] \tag{2}$$

Since the coordinate change $x \to \tilde{x}$ is –by definition of a manifold– a smooth bijection with a smooth inverse, the Jacobian matrices in (2) are non singular and have smooth entries in the domain of $\tilde{x}$ where $x(\tilde{x})$ is defined, then $\tilde{T}^{a_1 a_2 \dots}_{b_1 b_2 \dots}$ are smooth functions of $\tilde{x}$ if (and only if) the $T^{c_1 c_2 \dots}_{d_1 .. d_2 \dots}$ are smooth functions of x.

Additionally to the atlas of coordinates charts coming with a manifold M of dimension n, we may propose any set of *functionally independent* scalar fields $\phi^1, ..., \phi^n$ as a coordinate system in their open domain $O \subset M$. This will work *as long as the ϕ^j are smooth and functionally independent.* This last condition implies that, when written in terms of a coordinate set $\{x\}$ of M whose patch intersects O, the function $x^a \to \phi^j(x)$ is smooth with smooth inverse. Violating any of these condition introduces singular ϕ^j-components. In particular, $\det(\frac{\partial \phi^j}{\partial x^a}) = 0$ introduces, according to (2), poles in tensors

with lower indices.
Note that the $\{\phi^j\}$ coordinate vector and covector fields are

$$\frac{\partial}{\partial\phi^j} = \frac{\partial x^a}{\partial\phi^j}\frac{\partial}{\partial x^a}, \quad d\phi^j = \frac{\partial\phi^j}{\partial x^a}dx^a \tag{3}$$

become linearly independent at points where the determinant of the Jacobians vanish, and these fields are no longer a basis for vector and covector fields.

1.1.1 Three examples in the Cartesian plane

The Cartesian coordinates (X, Y) are global cordinates of $\mathbb{R}^2$, in terms of which the metric is given by $ds^2 = dX^2 + dY^2$. No further coordinate patch is needed, by we may want to try a few sets of 2-scalar fields as alternative coordinates.

Polar coordinates Polar coordinates (r, ϕ) are defined implicitly by

$$X = r\cos\phi, \;\; Y = r\sin\phi, \;\; \phi \sim \phi + 2\pi, \; r > 0 \tag{4}$$

They are apt as a coordinate patch on the open set $X^2 + Y^2 > 0$ (at $X = Y = 0$, ϕ is not defined). In terms of (r, ϕ) the canonical metric $ds^2 = dx^2 + dy^2$ and its inverse are

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}, \;\; r > 0, \; \phi \sim \phi + 2\pi. \tag{5}$$

The components of these tensors are smooth in the coordinate domain $r > 0$, the singularity $g^{\phi\phi} \to \infty$ as $r \to 0$ is due to the fact that $r = 0$ is a boundary of the coordinate patch, the domain of validity of the polar coordinates.

The involved Jacobian matrices are

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{pmatrix} \tag{6}$$

which has determinant $r = \sqrt{x^2 + y^2}$, and

$$\frac{\partial(r,\phi)}{\partial(x,y)} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{r} & \frac{\cos\phi}{r} \end{pmatrix} = \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \tag{7}$$

which has determinant $1/r = 1/sqrtx^2 + y^2$ They Jacobians are, as expected, singular at $X = Y = 0$ (equivalently, $r = 0$) and responsible for globally defined smooth tensor field to exhibit diverging (r, ϕ) components as $r \to 0$. The divergences are not restricted to the metric inverse but to any tensor field with upper indices. Take, e.g., the globally defined smooth vector field

$$V = 1\frac{\partial}{\partial X} + 0\frac{\partial}{\partial y} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi} \tag{8}$$

then V^ϕ diverges as $r \to 0^+$.

Comment/Exercise: globally defined vector fields can be described with words: e.g., the V field above "has unit norm and points to the right".

i) Describe, in terms of length and direction $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial\phi}$ at different points and as $r \to 0$.

ii) Could i) help explain why V^ϕ "behaves so badly" as $r \to 0$?

Hybrid coordinates Hybrid coordinates (r, x) are given in terms of Cartesian coordinates (X, Y) by

$$(x, r) := (X, \sqrt{X^2 + Y^2}) \tag{9}$$

They have the same value at (X, Y) and $(X, -Y)$ so they can be used as coordinates either in the upper half plane (UHP) or the lower half plane (LHP), but not on the entire plane. The inverses are

$$(X, Y) = (x, \sqrt{r^2 - x^2}),\ r > 0, -r < x < r, \qquad \text{UHP} \tag{10}$$

$$(X, Y) = (x, -\sqrt{r^2 - x^2}),\ r > 0, -r < x < r \qquad \text{LHP} \tag{11}$$

In both cases (that is, as coordinates in the UHP or the LHP) we find

$$g_{ab} = \frac{1}{r^2 - x^2} \begin{pmatrix} r^2 & -rx \\ -rx & r^2 \end{pmatrix}, \quad r > 0, \quad -r < x < r. \tag{12}$$

which becomes singular at points at the boundary of the patch, where $x = \pm r$.

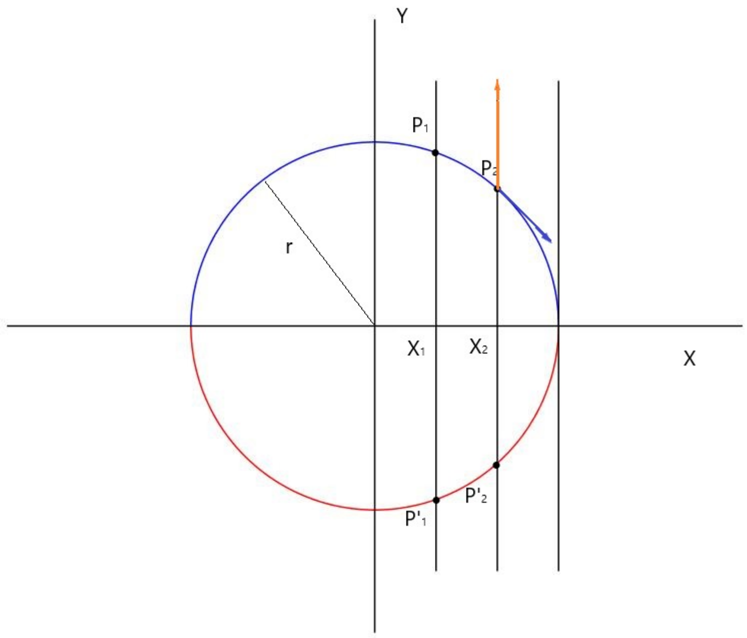


Figure 1: Hybrid (x, r) coordinates can be used in the upper (lower) half plane, a point with coordinates (x, r) lies at the intersection of the vertical line $X = x$ with the half circle of radius r in the upper (lower) plane.

From now on we stick to the case where we use (x, r) coordinates in the UHP.
For the coordinate vector fields we find

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} = 1\,\frac{\partial}{\partial X} - \frac{X}{Y}\frac{\partial}{\partial Y} \\ \frac{\partial}{\partial r} &= \frac{\partial X}{\partial r}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial r}\frac{\partial}{\partial Y} = 0\,\frac{\partial}{\partial X} + \frac{\sqrt{X^2 + Y^2}}{Y}\frac{\partial}{\partial Y} \end{aligned} \tag{13}$$

from where we read $\det \partial(X,Y)/\partial(x,r) = \sqrt{X^2+Y^2}/Y = r/\sqrt{r^2-x^2}$. At point P_2 these vectors are respectively drawn in blue and olive. Note that they become parallel in the patch limit $Y \to 0^+$.
Similarly

$$dx = dX, \quad dr = \frac{X}{\sqrt{X^2+Y^2}}\, dX + \frac{Y}{\sqrt{X^2+Y^2}} \tag{14}$$

become parallel as $Y \to 0^+$. From (14) we read $\det \partial(x,r)/\partial(X,Y) = Y/\sqrt{X^2+Y^2} = \sqrt{r^2-x^2}/r$, as expected.

Note that $\frac{\partial}{\partial r}$ points upwards (not radially) and $\frac{\partial}{\partial x}$ does *not* point to the right). To understand this it is important to keep in mind that the symbol $\frac{\partial}{\partial x}$ by itself means nothing: we need to know what the accompanying coordinates are. In fact, $\partial/\partial x$ means "direction of increase of x *while keeping the other coordinates fixed*": In Figure 1 the coordinate vectors are drawn at P_2: if we keep r fixed then the direction of increase of x is that given by the blue vector, if we keep x fixed and increase r, the resulting direction is that given by the olive vector.

We could borrow the smart notation used in Thermodynamics to indicate what the independent variables are when its is not clear from the context, e.g.,

$$C_V = T\, \frac{\partial S}{\partial T}\bigg)_V, \quad C_p = T\, \frac{\partial S}{\partial T}\bigg)_P, \text{ etc...}$$

In this notation (13) reads

$$\begin{aligned} \frac{\partial}{\partial x}\bigg)_r &= \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} = \frac{\partial}{\partial X}\bigg)_Y - \frac{X}{Y}\,\frac{\partial}{\partial Y}\bigg)_X \\ \frac{\partial}{\partial r}\bigg)_x &= \frac{\partial X}{\partial r}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial r}\frac{\partial}{\partial Y} = 0\,\frac{\partial}{\partial X}\bigg)_Y + \frac{\sqrt{X^2+Y^2}}{Y}\,\frac{\partial}{\partial Y}\bigg)_X \end{aligned} \tag{15}$$

Non smooth scalar fields proposed as coordinates for $\mathbb{R}^2$ Define the scalar fields (here ℓ is a positive constant with dimensions of length)

$$\begin{aligned} u(X,Y) &= X \\ v(X,Y) &= Y + \ell \ln\left(\frac{|X|}{\ell}\right) = \begin{cases} Y + \ell \ln\left(\frac{X}{\ell}\right) & , X > 0 \\ Y + \ell \ln\left(-\frac{x}{\ell}\right) & , X < 0 \\ \text{undefined} & , X = 0 \end{cases} \end{aligned} \tag{16}$$

Which has an inverse in the $u \neq 0$ set :

$$X = u, \quad Y = v - \ell \ln\left(\frac{|u|}{\ell}\right). \tag{17}$$

The set of fields (u, v) satisfy the conditions for a coordinate system on the RHP, also on the LHP, but certainly not on the *entire* plane. The tensor components for the metric and its inverse in (u, v) coordinates look the same in both half planes:

$$g_{ab} = \begin{pmatrix} 1 + \ell^2/u^2 & -\ell/u \\ -\ell/u & 1 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & \ell/u \\ \ell/u & 1 + \ell^2/u^2 \end{pmatrix} \tag{18}$$

Whether we use these coordinates in the RHP or the LHP, the Jacobians will introduce singularities in globally smooth tensor field components at the patch boundary $X = 0$.

1.1.2 Eddington-Finkelstein: a non historical perspective

Suppose the following solution to Einstein's vacuum field equations had been available *before* Schwarzschild's:

$M = \mathbb{R}_v \times \mathbb{R}^+_r \times S^2$, the metric is given by

$$ds^2 = -f(r)dv^2 + 2\,dv\,dr + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \quad f(r) = 1 - \frac{2M}{r}, \quad M > 0. \tag{19}$$

Note that the components of the metric and its inverse are smooth on the full range of the coordinates, $-\infty < v < \infty$, $r > 0$.

Introduce the scalar field t (compare to (16))

$$t = v - r - 2M\ln\left|\frac{r}{2M} - 1\right| = \begin{cases} v - r - 2M\ln\left(\frac{r}{2M} - 1\right) & , r > 2M \\ v - r - 2M\ln\left(1 - \frac{r}{2M}\right) & , 0 < r < 2M \end{cases} \tag{20}$$

Imagine that someone had the bad idea of using (t, r, θ, ϕ) as coordinates...
The map $(v, r, \theta, \phi) \to (t, r, \theta, \phi)$ is 1-1 and smooth with smooth inverse only in the $r > 2M$ and $r < 2M$ regions. In any of this regions the metric components in these coordinates will be smooth, but a singularity will be find at the patch boundary $r = 2M$. In fact, in terms of (t, r, θ, ϕ), (19) gives Schwarzschild's solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{21}$$

The choice of (t, r, θ, ϕ) with t given in terms of global coordinates of the Eddington-Finkelstein solution (19) explains the *coordinate* singularity in Schwarzschild's solution at $r = 2M$

1.1.3 Why do we have to deal with coordinate singularities?

According to General Relativity (GR) the spacetime can be modeled as a 4-dimensional manifold M with a Lorentzian metric g_{ab} satisfying Einstein's equations

$$G_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4}T_{ab} \tag{22}$$

A manifold M, as defined, comes equipped with an atlas of coordinate charts that completely covers it. Coordinate transformations in patch intersections are bijective, smooth and have smooth inverse, then no coordinate singularity can slip into tensor components according to (2).
The problem is... Einstein failed to tell us what M is... When we solve a GR problem, we are *never* given *beforehand* the spacetime manifold where to solve Einstein's equations

(look for the metric). Usually, we narrow the search to a workable problem, and this is done by restricting the solution space to metrics with a number of symmetries. As an example, Schwarzschild didn't attempt to solve Einstein's vacuum equation $R_{ab} = 0$, this is an impossible task. He searched for the *spherically symmetric static vacuum region outside a static spherical object.* To do so he proposed a set of coordinates adapted to the spherical symmetry and staticity (more on this below). This led him to a very simple ansatz, which involves only 2 unknown functions of a single variable, $f(r)$ and $h(r)$:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad f(r), h(r) > 0, \tag{23}$$

Plugging this in the vacuum equation $R_{ab} = 0$ gives (after absorbing an irrelevant integration constant into a re-scaling of t) the solution found was $1/h(r) = f(r) = 1-2M/r$, M an integration constant understood (by looking at timelike geodesics in the $r >> M$) region as the mass of the central object. (23) is then valid in the open coordinate set $r > 2M$: it's only there that it gives the static metric Shwarzschild was looking for. Of course, the metric he found is also a valid solution of the vacuum equations for $0 < r < 2M$, but it is non static there. An his solution can certainly not be used on the entire $0 < r$ domain: a pole in the metric component is found at the coordinate boundary limit $r \to 2M$.
Of course, at that time, Schwarzschild was not aware that there is a unique maximal vacuum spherically symmetry spacetime (Kruskal's manifold) of which (19) covers a region. This extends the static solution he was looking for into a dynamical regions that he did not suspect of, and where his coordinate choice is not suitable.

1.2 True singularities

A manifold M is geodesically incomplete if it contains geodesic that do not extend to infinite affine parameter in one direction. A manifold M is singular if it is geodesically complete and cannot be extended to a geodesically complete manifold N.

Example 1: the simplest example is the cone in $(\mathbb{R}^3, dx^2 + dy^2 + dz^2)$, which is defined by the constraint $z = a\sqrt{x^2 + y^2}$, $(x, y) \neq (0, 0)$, a> 0. The induced metric is

$$ds^2 = dx^2 + dy^2 + a^2(xdx + ydy)^2/(x^2 + y^2), \quad (x, y) \neq (0, 0) \tag{24}$$

Alternatively, in cylindrical coordinates, $ds^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2$, the constraint is $z = a\rho$, and the cone metric reads is

$$ds^2 = (1 + a^2)\, d\rho^2 + \rho^2 d\phi^2, \quad \rho > 0 \tag{25}$$

The geodesics $(\rho = \rho_o - s, \phi = \phi_o)$ are incomplete (note that s is an affine parameter: length divided by $\sqrt{1 + a^2}$). The proof that it is not possible to enlarge the cone by "adding the point at $\rho = 0$" so that geodesics could go around the vertex (equivalently, there is no change of coordinates that lifts the singularity at $\rho = 0$ in (25) or at $(x, y) = (0, 0)$ in (24)) is by contradiction: in any Riemannian 2-manifold, a circle of radius R centered at a point p of radius R is the set of endpoints of all geodesic segments of length R emanating from p. If ℓ is the length of such a circle, it holds in general that, for small R

$$\ell(R) = 2\pi R - \frac{\pi}{6}\mathcal{R}(p)R^3 + ... \tag{26}$$

where $\mathcal{R}(p)$ is the Ricci scalar at the circle center p. In particular, the Euclidean relation holds in the small radius limit:

$$\lim_{R\to 0} \frac{\ell(R)}{R} = 2\pi \tag{27}$$

If it were possible to enlarge the cone and add its vertex p, then the set of constant ρ would be a circle of radius $R(\rho) = \sqrt{1+a^2}\rho$ and length $\ell(\rho) = 2\pi\rho$, for which

$$\lim_{R\to 0} \frac{\ell(R)}{R} = \lim_{\rho\to 0} \frac{2\pi\rho}{\sqrt{1+a^2}\rho} = \frac{2\pi}{\sqrt{1+a^2}} \neq 2\pi. \tag{28}$$

which is a contradiction.

Example 2: Any extension of (19), such as Kruskal's, will be incomplete, and therefore singular.
The proof of this statement is again subtle: we show that (19) cannot be extended such that timelike geodesics hitting $r = 0$ could go beyond this point and develop infinity proper time τ.
For radial incoming timelike geodesics in the inner Schwarzschild region

$$\frac{dr}{d\tau} = -\sqrt{-f(r) + E^2} \tag{29}$$

where τ is proper time and E is a constant given, in terms of the initial conditions, by $E = (1 - 2M/r_o)\dot{t}_o$. Note that, since $f < 0$ in the inner region, we can choose initial conditions for timelike radial geodesics such that $E = 0$ ($\dot{t}_o = 0$) while keeping $-f(r)\dot{t}_o^2 + \dot{r}_0^2/f(r_o) = -1$ (a dot means $d/d\tau$). For these timelike geodesics, the amount of proper time from $r = r_o$ to $r = 0$, obtained from (29), is finite:

$$\tau^* = \int_0^{r_o} \frac{dr}{\sqrt{2M/r - 1}} < \int_0^{r_o} dr \sqrt{\frac{r}{2M}} = \frac{\sqrt{2}}{3\sqrt{M}} r_o^{3/2}$$

Assume there is a manifold extension (N, g_{ab}) of (19) beyond $r = 0$, with an overlapping coordinate patch z^a covering a neighborhood of $r = 0$, where our geodesics extends as $z^a(\tau)$, with $\tau = \tau^*$ corresponding to $r = 0$. For $\tau < \tau^*$ we could use Schwarzschild $r < 2M$ solution and coordinates. Now consider the curvature scalar $\Phi := R^{abcd}R_{abcd}$, then $\tau \to \Phi(z(\tau))$ should be a smooth function across the extension. However,

$$\lim_{\tau\to\tau^{*-}} \Phi(z^a(\tau)) = \lim_{\tau\to\tau^{*-}} \Phi(z^a(t(\tau), r(\tau), \theta_o, \phi_o)) = \lim_{r\to 0^+} \frac{6M^2}{r(\tau)^6} = \infty \tag{30}$$

since

$$R^{abcd}R_{abcd} = \frac{6M^2}{r^6} \tag{31}$$

in the inner Schwarzschild region.

This argument can always be applied in those cases where (M, g_{ab}) exhibits a scalar curvature field, that is, one made from contractions involving the Riemman tensor, volume form, metric and metric inverse, that diverges at points at finite distance / proper time /

null geodesic affine parameter.
Divergences of curvature scalar fields such as

$$R^d{}_{bac} R_d{}^{bac}, \quad R^d_{bac} \epsilon^{ackl} R^e_{fkl} g_{ed} g^{bf}, \tag{32}$$

which should be defined globally in (M, g_{ab}), signal a boundary of the manifold. If this boundary points are at finite affine distance through geodesics, the manifold is incomplete and cannot be enlarged. This is a true singularity.

A key point in the argument above is that, from (2), for a scalar field

$$\tilde{\Phi}(\tilde{x}) = \Phi(x(\tilde{x}))$$

We don't have Jacobian matrices that become singular and introduce or fix singularities: if $\Phi(x)$ is a scalar field that diverges as $x \to x_o$ and we change coordinates to $\tilde{x}(x)$, then $\lim_{x \to \tilde{x}_o} \Phi(\tilde{x}) = \infty$, where $\tilde{x}_o = \tilde{x}(x_o)$. No matter how we label the point, x_o or $\tilde{x}_o$, the field will diverge *at that point*. An example is the divergence of the energy density measured by isotropic observers in FRWL cosmologies as we approach the Big Bang: the divergence will occur, call this point $\tau = 0$ or otherwise! Divergences of scalar fields cannot be "fixed" by coordinate changes.

1.3 Summary on singularities

- **Question:** In a specific coordinate system neither the metric nor the inverse metric have singular components. Are we alright?

  Answer: No, unless all geodesics develop infinite affine parameter in any direction (If this is the case, your coordinate chart is global).

  Example: take

  $$ds^2 = \ell^2 e^{2z} dz^2, -\infty < z < \infty, \tag{33}$$

  where $\ell > 0$ is a constant with dimensions of length. The components of the metric and its inverse are smooth in the entire domain of the coordinate. However, since the distance between points labeled $z_1 < z_2$ is

  $$d(z_1, z_2) = \int_{z_1}^{z_2} \ell e^z dz = \ell e^{z_2} - \ell e^{z_1}$$

  we find that $z = -\infty$ is at a finite distance. To go beyond $z = -\infty$ we need to switch to a geometrically meaningful coordinate, such as the signed distance x to a reference point, say $z = 0$, then

  $$x = \ell(e^z - 1), \quad dx = \ell e^z dz, \quad ds^2 = dx^2 \tag{34}$$

  Originally, this is defined for $x > -\ell$, but $ds^2 = dx^2$ makes sense, and then can be extended, to $-\infty < x < \infty$: this is a geodesically complete extension of (33).

- **Question:** In a specific coordinate system either the metric or the inverse metric has some singular components. Is this a necessarily a problem?

  Answer: It is a problem if the singularity lies at a finite distance (proper time/affine parameter) from points in the open domain where the metric and its inverse are smooth. Otherwise, it is not.

  Example 1: The metric (here $b, c > 0$)

$$ds^2 = \left(1 + \frac{b^2}{(\rho - c)^2}\right) d\rho^2 + \rho^2 \, d\phi^2, \quad \rho > c, \phi \sim \phi + 2\pi \tag{35}$$

has a singularity at $\rho = c$, however the geodesic distance from (ρ, ϕ) to $(\rho \to c^-, \phi)$ is

$$\lim_{\rho^* \to c^-} \int_{\rho^*}^{\rho} \sqrt{1 + \frac{b^2}{(\rho - c)^2}} \, d\rho = \infty$$

It can be shown that (35) is geodesically complete, then (ρ, ϕ) are global coordinates. The singularity of the metric at $\rho = c$ signals the boundary of the patch, which is also the boundary of the manifold.

To gain intuition on this example, note that this is the metric on the surface of revolution $\Sigma_1 \subset \mathbb{R}^3$ defined by the constraint $\rho = a e^{z/b} + c$, $a, b, c > 0$ in cylindrical coordinates (for which the canonical $\mathbb{R}^3$ metric is $ds^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2$). This trumpet-like surface is depicted in Figure 2. Had we eliminated z instead of ρ, we would have found

$$ds^2 = \left(1 + \frac{a^2}{b^2} e^{2z/b}\right) dz^2 + (a e^{z/b} + c)^2 d\phi^2, \quad -\infty < z < \infty, \phi \sim \phi + 2 + \pi \tag{36}$$

Example 2: The surface $\Sigma_2 \subset \mathbb{R}^3$ defined by $\rho = a z^2 + b$, $a, b > 0$: In terms of the coordinates (z, ϕ)

$$ds^2 = (1 + 4a^2 z^2) \, dz^2 + (a z^2 + b)^2 \, d\phi^2, \quad -\infty < z < \infty, \phi \sim \phi + 2\pi \tag{37}$$

Coordinates (ρ, ϕ) can be used either in the $z > 0$ region, or in the $z < 0$ region.

$$ds^2 = \left(1 + \frac{1}{4a(\rho - b)}\right) d\rho^2 + \rho^2 \, d\phi^2, \quad \rho > b, \phi \sim \phi + 2\pi \tag{38}$$

If we didn't know the origin of these metrics and just had been handed them, we could have checked that (37) is geodesically complete (nothing should be done), whereas the singularity in (38) occurs at a finite distance and should be taken care of. The coordinate change $z = \sqrt{(\rho - b)/a}$, the dimension-full positive constant a introduced only for dimensional purposes) in (38) would produce (37). Then lifting the restriction $z > 0$ produces an analytic extension which is geodesically complete. In other words, (37) gives a geodesically complete extension of (38).

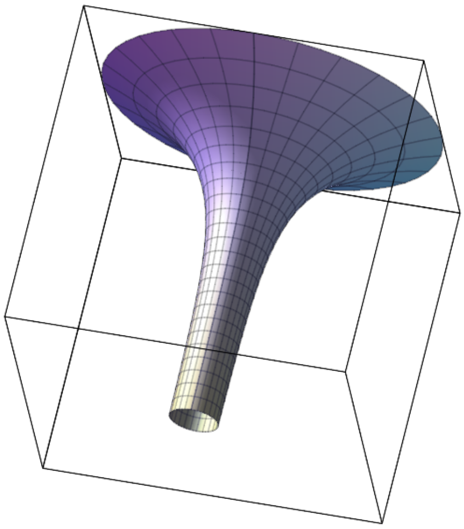


Figure 2: Trumpet like surface Σ_1 in Example 1 above

- **Questions:** are true singularities (more properly: are boundary of incomplete manifolds...) always detected by curvature scalars blowing up?

  Answer: No.

  Example 1: The cone analyzed above.

  Example 2: Coming soon

- **Question:** when singularities are at finite distance and we have no proof that they are true (manifold boundary) singularities, how do we proceed?

  Answer: see if they can be removed by aiming geodesics at the singularity, and inverting coordinate/s in favor of the affine parameter, and seeing if the resulting expression admits an analytic extension beyond the original domain.

  Example: Schwarzschild to EF is done this way.

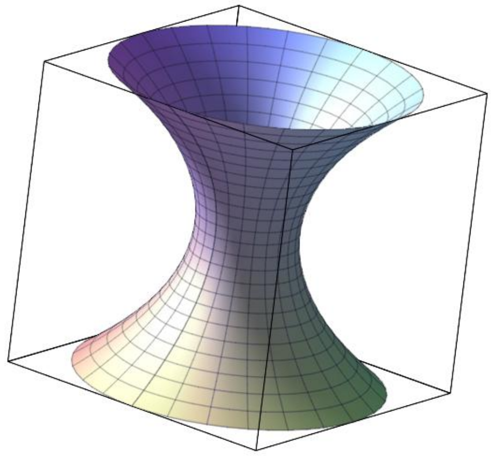


Figure 3: Surface Σ_2 used in Example 2 above.

1.4 Problems on singularities

1. Consider the 2D Riemann metric

$$ds^2 = dx^2 + \frac{b}{x}dy^2, \ \ (b > 0), \ \ -\infty < y < \infty, 0 < x \tag{39}$$

   (a) Is the $x = 0$ set at a finite distance form a generic point $(x, y), x > 0$?

   (b) Can we extend (39) beyond the limit where $x \to 0^+$? Hint: calculate the Ricci scalar.

   (c) What happens at $x = 0$?

   (d) Solve the geodesic problem for this metric. Can we assure that geodesics not hitting $x = 0$ extend infinitely?

2. In the case of (34) we introduced $x = \ell(e^z - 1)$, which has domain is $x > -\ell$. Switching to x gives

$$ds^2 = 1 \ dx^2, \ \ x > -\ell,$$

   and we decided to extend the above expression to all x (analytic extension). Note that we could have equally extended non-analytically as, say

$$ds^2 = \begin{cases} e^{-\frac{1}{(x+\ell)^2}} \ dx^2 & , x < -\ell \\ 1 \ dx^2 & , x \geq -\ell \end{cases}$$

   Which is a C^∞ metric.
   Morale: analytic extensions are unique, smooth extensions are not. Note that this could also be done when extending, e.g., (38): (37) is *not* the only smooth extension.

1.5 A virtuous path: symmetries → ansatz → exact solutions

Einstein's equation is famous for its complexity: it involves 10 unknown functions in 4 variables (the independent components of the metric tensor $g_{ab}(x)$), entangled in a *non-linear* second order PDE. However, as in any physical theory, it is possible to find exact solutions if we assume assume symmetries. *Symmetries* here means *isometries: symmetries of the metric.*

Given a (Lorentzian or Riemannian) manifold M with metric g_{ab}, an isometry is a bijection $\Phi : M \to M$ such that the length of spacelike curves (and proper time of timelike curves in the Lorentzian case), equals that of its image under Φ. Translations, rotations and boots of Minkowski spacetime (as active transformations) are examples of isometries.

Examples: reflections in the plane (discrete), translations and rotations in the plane / space (discrete), boosts in Minkowski spacetime (continuous), P and T transforms in Minkowski spacetime (discrete)

Schwarzschild looked for *static* and *spherically symmetric* solutions. This symmetry assumptions led him to the ansatz:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad f(r), h(r) > 0, \tag{40}$$

This has only 2 unknown metric components which, furthermore, depend on a single variable!

1.6 Vector field flows

Given a vector field k^a and a point of M with coordinates x_o^a, there is a unique solution of the linear system of ordinary equations that give the integral curves of the field:

$$dx^a(s)/ds = k^a(x), \quad x^a(0) = x_o^a \tag{41}$$

Solving this equations *for generic initial condition* x^a (and doing so in every patch coordinate), gives, for each s, a map $\Phi_s : M \to M$ which, in coordinates, reads

$$x^a \to \tilde{x}^a(x, s)$$

This is called the *flow* of x^a along the vector field curves.

The solution is unique and satisfies the *group property*

$$\Phi_s(\Phi_t(x)) = \Phi_{s+t}(x) \tag{42}$$

Example: an incomplete field in $\mathbb{R}^2$

Consider the field $x^2\ \partial/\partial x + 0\ \partial/\partial y$ in $\mathbb{R}^2$. To find the integral curve with initial condition (x, y) we need yo solve the ODE system

$$dx/ds = x^2,\ dy/ds = 0 \quad x(0) = x,\ y(0) = y. \tag{43}$$

The solution is

$$y_s(x,y) = y, \quad x_s(x,y) = \begin{cases} \frac{x}{1-sx} & ,x \neq 0 \\ x & ,x = 0 \end{cases} \tag{44}$$

Note that this solution is only defined for $s \in (-\infty, 1/x)$ if $x > 0$, for $s \in (1/x, \infty)$ if $x < 0$, and for $-\infty < s < \infty$ if $x = 0$. This is an example of an *incomplete vector field*, those for which their integral lines cannot be extended to $-\infty < s < \infty$ for any initial conditions.

Example: a field that generates rotations in $\mathbb{R}^2$

Consider $\mathbb{R}^2$ with Cartesian coordinates, $ds^2 = dx^2 + dy^2$, and the vector field

$$k = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{45}$$

The flow equations are

$$\begin{cases} dx/ds & = -y \\ dy/ds & = x \end{cases}, \qquad x(0) = x, \ \ y(0) = y \tag{46}$$

The solution to the flow equations is

$$\left(\begin{array}{c} x_s(x,y) \\ y_s(x,y) \end{array}\right) = \left(\begin{array}{cc} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right) \tag{47}$$

Note that in this case, the flow is, for every s, an isometry.

1.7 Killing vector fields

We say that k^a is a *Killing vector filed (KVF)* if the flow maps $\Phi_s : M \to M$ are isometries. It can be shown that vector field is Killing if and only if satisfies the equation

$$\nabla_a k_b + \nabla_b k_a = 0 \tag{48}$$

It follows from (48) that linear combinations (with constant coefficients!) of KVF's give KVFs.

Observation (proof left as a problem): if, in a coordinate system $\{x^1, ..., x^n\}$ the metric components do not depend, say, on x^1, then the coordinate vector field $\partial/\partial x^1$ is Killing

In a Riemannian or Lorentzian manifold of dimension n there are at most $n(n+1)/2$ linearly independent KVFs. Minkowski spacetime is an example of a *maximally symmetric spacetime*, as it has a maximum number of linearly independent KVF, which in 4 dimensions is 10. A linearly independent set of these contains: 3 space translations, 1 time translations, 3 rotations and 3 boosts

1.8 Axial symmetry

We say that (M, g_{ab}) has axial symmetry if there is a KVF with closed orbits.

Example: surfaces in $\mathbb{R}^3$ obtained by imposing a constraint of the form $F(\rho, \phi) = 0$ in cylindrical coordinates. The $\mathbb{R}^3$ metric is $ds^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2$ and we can eliminate either ρ or z from the constraint $F(\rho, z) = 0$ to find the metric on the surface.

Eliminating z from a constraint of the form $z = f(\rho)$ gives

$$ds^2 = \left(1 + f'(\rho)^2\right) d\rho^2 + \rho^2 \, d\phi, \quad \phi \sim \phi + 2\pi, \tag{49}$$

1. $\frac{\partial}{\partial \phi}$ is a KVF with closed orbits: these are $(\rho(s), \phi(s)) = (\rho_o, \phi_o + s)$.
2. The restriction of the metric to an orbit is that of a circle of length $L = 2\pi\rho$: we call $\rho = \frac{L}{2\pi}$ the "length radius"
3. There could be no "center of symmetry" (point left fixed by the KVF flow), since $\rho = 0$ could be out of the domain.
4. Even if there were a center of symmetry, the radius of the KVF flow circles at $\rho = \rho_o$, that is, its geodesic distance to the center of symmetry, *would not be* ρ_o but
$$R(\rho_o) = \int_o^{\rho_o} \sqrt{1 + f'(\rho)^2} \, d\rho > \rho_o$$

Eliminating ρ from a constraint of the form $\rho = q(z)$ gives

$$ds^2 = \left(1 + q'(z)^2\right) dz^2 + q(z)^2 \, d\phi, \quad \phi \sim \phi + 2\pi, \tag{50}$$

1. $\frac{\partial}{\partial \phi}$ is a KVF with closed orbits: these are $(z(s), \phi(s)) = (z_o, \phi_o + s)$.
2. The restriction of the metric to an orbit is that of a circle of perimeter $2\pi q(z_o) = 2\pi\rho(z_o)$
3. There could be no "center of symmetry" (point left fixed by the KVF flow): $q(z) = \rho = 0$ could be out of the domain.
4. Even if there were a center of symmetry, say $q(z_1) = 0$, the radius of the KVF flow circles at $z = z_o$ *would not be* $\rho_o = q(z_o)$ but

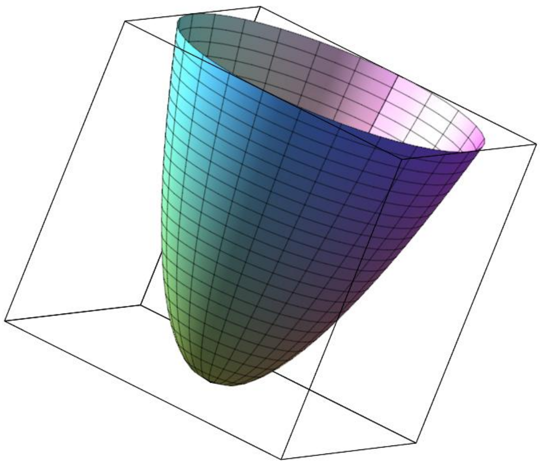
$$R(\rho_o) = \int_{z_1}^{z_o} \sqrt{1 + q'(z)^2} \, dz > q(z_o) - q(z_1) = q(z_o)\rho_o$$

Example: paraboloid

The surface defined by $z = a\rho^2$ (paraboloid): $dz = 2a\rho\, d\rho$ and

$$ds^2 = (1 + 4a^2\rho^2)\, d\rho^2 + \rho^2\, d\phi^2, \quad \rho > 0, \phi \sim \phi + 2\pi \tag{51}$$

Note that this figure is made by piling circles of length $2\pi\rho$, such that the orthogonal distance between two circles with length-radii $\rho_1 < \rho_2$ is bigger that $\rho_2 - \rho_1$

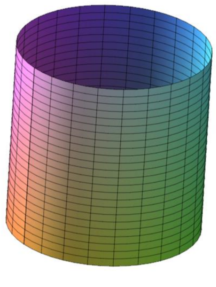


Example: cylinder

This is defined in (ρ, ϕ, z) coordinates by the constraint $\rho = R$. Has line element

$$ds^2 = dz^2 + R^2 d\phi^2$$

There is no center of symmetry (fixed point under the rotational symmetry)
Again, $\sqrt{g_{\phi\phi}} = R = P/(2\pi)$ is the length radius of the orbits of the symmetry group.



Two further examples of axial symmetry are those given in Figure 2 and Figure 3.These were treated in the previous Section.

1.9 Spherical symmetry

- A spherically symmetric spacetime has a metric

$$ds^2 = g_{ab}(x)dx^a dx^b + r^2(x)(d\theta^2 + \sin^2\theta\, d\phi^2), \;\; r(x) > 0. \tag{52}$$

where (x^1, x^2) are coordinates of the *orbit space* manifold with metric $g_{ab}(x)dx^a dx^b$ of signature $(-,+)$

- A spherically symmetric Riemannian manifold has the same form as (52) except that g_{ab} is a positive definite metric on an arbitrary dimension manifold.

These spacetimes / Riemannian manifolds have the KVFs (64), which generate rotations.
The metric induced on the orbits of the rotation group (which are the constant x sets) is $r^2(x)(d\theta^2 + \sin^2\theta\, d\phi^2)$, it has has square root determinant $r^2(x)\sin\theta$, and then the area of the orbits are

$$A = \int_0^{2\pi} d\phi \int_o^{\theta} d\theta\;\sin\theta = 4\pi r^2(x) \tag{53}$$

For this reason $r(x) = \sqrt{A/(4\pi)}$ is called *area radius*, in complete analogy to the perimeter radius. in axial symmetry.

In the 3D Riemannian case

$$ds^2 = g(x)^2 dx^2 + r^2(x)(d\theta^2 + \sin^2\theta\, d\phi^2), \;\; g(x) > 0, r(x)?(\textit{ only know that is positive}) \tag{54}$$

Note that there could be no "center of symmetry" (where $r(x) = 0$). Suppose there is a center of symmetry at $x = x_o$, that is, $r(x_o) = 0$, and that $r(x)$ is invertible near x_o so that $x = q(r)$ and

$$ds^2 = g(q(r))\; q'(r)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \;\; r \geq 0 \tag{55}$$

In general, the area-radius r is *not related* to the geodesic radial distance to the center, which is

$$d = \int_0^r \sqrt{g(q(s))\; q'(s)^2}\; ds \neq 0 \tag{56}$$

Two examples:

1. The 3-cylinder: this is a stack of equal radii spheres (the same way an ordinary cylinder is a stack of equal radii circles):

$$ds^2 = -dz^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2), \;\; x \in \mathbb{R} \tag{57}$$

R is area-radius, there is no center of symmetry, then no “distance to center”.

2. The extreme Reissner-Nordström black hole

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad r > M \tag{58}$$

is spherically symmetric, as well as its t =constant slices, which are spacelike:

$$ds^2_{slice} = \frac{r^2}{(r-M)^2}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad r > M \tag{59}$$

The slice is said to have a *cylindrical end* as $r \to M^+$ and to be asymptotically Euclidean as $r \to \infty$. Can you explain this? Could you analyze the singularity of (59) as $r \to M^+$?

1.10 Static and spherically symmetric spacetimes

A region of a spacetime is *stationary* if it has a timelike KVF.

A region of a spacetime is *static* if it has a timelike KVF which is orthogonal to hypersurfaces.

Metric ansatz to take advantage of staticity: Put arbitrary coordinates $\{x^1, x^2, x^3\}$ on an hypersurface Σ_o orthogonal to the timelike KVF. Suppose the $+++$ metric on Σ_o is

$$ds^2_{\Sigma_o} = h_{ij}(x)\, dx^i\, dx^j$$

Use coordinates $\{x^1, x^2, x^3\}$ for the point $\Phi_t(x^1, x^2, x^3)$ in spacetime given by the flow by the timelike KVF with initial condition the point in Σ_o with coordinates (x^1, x^2, x^3). Note that Σ_o is the level set $t = 0$ and that

$$ds^2 = -f(x)dt^2 + h_{ij}(x)\, dx^i\, dx^j \tag{60}$$

The timelike KVF is $\frac{\partial}{\partial t}$

Staticity + spherical symmetry gives Schwarzschild ansatz:

$$ds^2 = -f(r)dt^2 + \underbrace{h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}_{h(x)_{ij}dx^i dx^j}, \quad f(r), h(r) > 0, \tag{61}$$

The Schwarzschild case Applying the construction leading to (60) in the exterior EF spacetime $r > 2M$, which is static, one understands what happens when the staticity ansatz breaks:

The exterior region of the *EF* spacetime is spherically symmetric *and* static. Schwarzschild took a hypersurface there, orthogonal to the static KVF and let it flow, but this flow never stays in the $r > 2M$ region. To understand why, recall the coordinate change for (v, r, θ, ϕ) to (t, r, θ, ϕ):

$$\begin{pmatrix} v \\ r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} t + r + 2M \ln\left(\frac{r}{2M} - 1\right) \\ r \\ \theta \\ \phi \end{pmatrix} \tag{62}$$

Thus, the static KVF is:

$$\left.\frac{\partial}{\partial t}\right)_{r,\theta,\phi} = \left.\frac{\partial}{\partial v}\right)_{r,\theta,\phi}$$

and the $t = 0$ surface in the static region corresponds to

$$v = r + 2M \ln\left(\frac{r}{2M} - 1\right), r > 2M$$

The flow from this surface by the timelike KVF is

$$(v = r{+}2M \ln\left(\frac{r}{2M} - 1\right), r, \theta, \phi) \to (v = s{+}r{+}2M \ln\left(\frac{r}{2M} - 1\right), r, \theta, \phi), r > 2M, -\infty < s < \infty. \tag{63}$$

This fills in the entire region $r > 2M$, but never reaches the BH horizon.

1.11 Problems on symmetries

3. Prove that (44) is the solution of (43), and that it satisfies the group property: $x_s(x_t(x, y), y_t(x, y) = x_{s+t}(x, y)$, $y_s(x_t(x, y), y_t(x, y) = y_{s+t}(x, y)$.
   Same for (46) and (47) Hint for this case: note that, if $z = x + iy$, then $dz/dz = iz$.

4. i) Prove that $P_x = \frac{\partial}{\partial x}$, $P_y = \frac{\partial}{\partial y}$, $P_z = \frac{\partial}{\partial z}$, are KVF that generate translations in $(\mathbb{R}^3, dx^2 + dy^2 + dz^2)$, and $L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $L_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$ are KVF that generate rotations.

   ii) Switching to spherical coordinates in $\mathbb{R}^3$, (r, θ, ϕ), rewrite L_x, L_y, L_z. You should get

$$\begin{aligned} L_x &= -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ L_y &= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \\ L_z &= \frac{\partial}{\partial\phi} \end{aligned} \tag{64}$$

   iii) In Minkowski spacetime $(\mathbb{R}^4, -dt^2{+}dx^2{+}dy^2{+}dz^2)$ we have, besides translations and rotations, time translations and boosts. Find the (4) KVFs that generate these.

5. Given a Lorentzian or Riemannian manifold (M, g_{ab}) of dimension n, prove that, if in a given coordinate system the components of the metric do not depend on, say, x^1, then $\frac{\partial}{\partial x^1}$ is a KVF. Do it in two ways: a) Show that the length [proper time] of a spacelike [timelike] curve $(x^1(t), x^2(t), ..., x^n(t))$ equals that of the curve $(x^1(t) + s, x^2(t), ..., x^n(t))$ (which is the flow of the original curve under the vector field $\frac{\partial}{\partial x^1}$), and also by showing that this vector field satisfies the Killing equation (48).

6. Analyze the extreme Reissner-Nordström spacelike slice (59).

   i) is $r = M$ at a finite distance? Should we care about the singularity at $r = M$?

   ii) The Ricci scalar of this 3-manifold is found to be

   $$\mathcal{R} = \frac{2M^2}{r^4} \tag{65}$$

   Should we care anbout this expression having a pole at $r = 0$?

   iii) Can you explain why $\mathcal{R} \to 0$ as $r \to \infty$ and $\mathcal{R} \to 2/M^2$ as $r \to M$?

   iv) Is this 3-dimensional slice geodesically complete?

1.12 The black hole region

To illustrate how tricky loose statements like a region of spacetime that not even light rays can abandon, consider the interior $\mathcal{C}_p$ of the future light cone of a point p in Minkowski spacetime $\mathbb{M}$. No points in $\mathcal{C}_p$ can be connected by a future causal curve to points in $\mathbb{M} - \mathcal{C}_p$ (the complement of $\mathcal{C}_p$ in $\mathcal{M}$).

This does not mean that Minkowski spacetime contains a BH region: the impossibility of reaching points outside $\mathcal{C}_p$ is simply a re-statement of Einstein's postulate that nothing travels faster than light. In fact, future causal (timelike or null) curves originating within $\mathcal{C}_p$ can extend to arbitrary proper time/affine parameter and reach places arbitrarily far away.

When we talk of a black hole, we are considering a spacetime that contains a region that "far away" approaches the region in Minkowski spacetime where future null geodesics go ("future null infinity" in the jargon).

More precisely, starting from Minkowski in spherical coordinates,

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) && (66)\\ &= -dv^2 + 2dv\, dr + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) && (67)\end{aligned}$$

the region where future null geodesics go corresponds to $v \equiv t + r \to \infty$ while $u \equiv t - r$ is kept finite.

In *Eddingstein* spacetime

$$ds^2 = -(1 - 2M/r)\, dv^2 + 2dv\, dr + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$\simeq -dv^2 + 2dv\, dr + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2) \text{ for } r >> M \quad (68)$$

future null infinity corresponds to $(v, r) = (u_o + 2s, s), s \to \infty$ Although future null lines originating in the $r > 2M$ region can reach this region of arbitrarily large r values, those originating in the *Black Hole region* $r < 2M$ cannot, they are confined to the BH region.

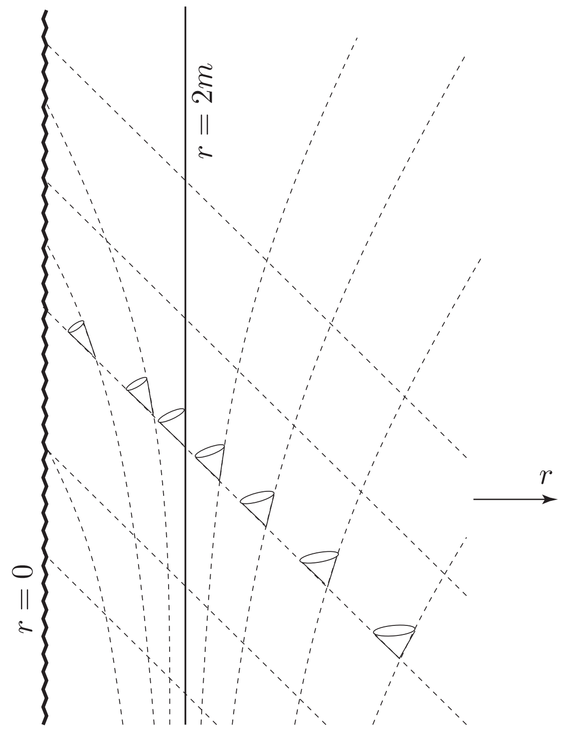


Figure 4: Eddington-Finkelstein solution.
Dotted straight lines are $v =$ constant curves (incoming radial null geodesics).
Dotted curved lines are the "outgoing" null radial geodesics defined by $dv/dr = 2/f(r)$.
Plotted half cones are future pointing.
(From *Exact Space-Times in Einstein's General Relativity* by Podolsky and Griffiths, CUP, 2009)

1.13 The BH region of a collapsed star

The exterior region of a spherical collapse, the vacuum region outside the surface of the collapsing star must be, by the uniqueness of the spherical symmetric GR vacuum, a piece of the EF extension of Schwarzschild solution. This should match smoothly the interior, non vacuum spherically symmetric solution. As a consequence, a dust particle *at the surface* traces a radial in-falling future timelike curve both from the viewpoint of the vacuum exterior metric and the inner metric, and the corresponding piece of EF left outside will be a part of the exact collapse solution. This is the shaded area depicted in the figure below, this is the part of the EF spacetime that is physically relevant. The area that is not shaded gets replaced by the (unknown, matter model dependent) non-vacuum metric. Even without knowing the details of the non-vacuum part of the spacetime diagram, we can ascertain that there is a BH, ending at a spacelike singularity. This is a *real* (as real as spherical, non rotating collapse) BH.

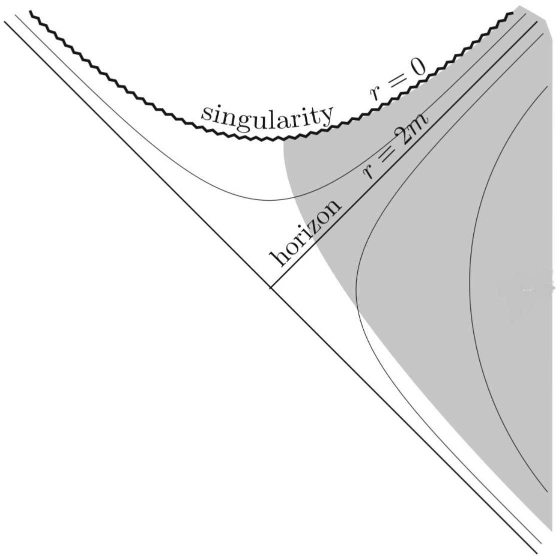


Figure 5: Eddington-Finkelstein solution: diagonal (upper left to lower right) line corresponds to $v = -\infty$, all the other plotted curves are $r = constant$. Curves at 45 degrees are null.
The shaded area models the exterior of the collapsing body. For the complete spacetime of a collapsed star, the non-shaded area should be replaced with the inner, non-vacuum, collapsing matter.
(From *Exact Space-Times in Einstein's General Relativity* by Podolsky and Griffiths, CUP, 2009)

1.14 The Kerr spacetime

$$ds^2 = -\frac{(\Delta_r - a^2\sin^2\theta)}{\Sigma}\mathrm{d}t^2 + \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \frac{4aMr\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi + \Sigma\mathrm{d}\theta^2 + \frac{\sin^2\theta[(r^2+a^2)^2 - a^2\sin^2\theta\Delta_r]}{\Sigma}\mathrm{d}\phi^2, \quad (69)$$

where

$$\Delta_r = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad M > a > 0$$
$$t \in \mathbb{R}, \quad r > m + \sqrt{M^2 - a^2}, \quad 0 < \theta < \pi, \quad \phi \sim \phi + 2\pi$$

- Metric elements do not depend on (t, ϕ), then $\partial/\partial t$ and $\partial/\partial\phi$ are KVFs

- If $a = 0$ reduces to Schwarzschild (i.e., Kerr's solution generalizes Schwarschid's)
- For large r approaches Minkowski's: $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
- In the region $r > M + \sqrt{M^2 - a^2\cos^2\theta}$, the KVF $\partial/\partial t$ is timelike: thus Kerr's solution is axially symmetric (KVF is $\frac{\partial}{\partial\phi}$) and stationary (KVF is $\partial/\partial t$) –but not static– in this region.
- For large r it models the exterior of a rotating central object.
- It has *coordinate singularities* where $\Delta_r = 0$:

  $$r_\pm = M \pm \sqrt{M^2 - a^2}$$

  The following coordinate change removes the $r = r\pm$ coordinate singularities: $(t, r, \theta, \phi) \to (v(t, r), r, \theta, \chi(t, \phi))$ with $v = v(t, r)$ and $\chi = chi(\phi, r)$ defined by

  $$\frac{\partial v}{\partial t} = 1, \quad \frac{\partial v}{\partial r} = \frac{r^2 - a^2}{\Delta_r}, \quad \frac{\partial \chi}{\partial \phi} = 1, \quad \frac{\partial \chi}{\partial r} = \frac{a}{\Delta_r}$$

- It has a true singularity (spacetime boundary points) at $r = 0, \theta = \pi/2$ (that is, where $\Sigma = 0$)

  $$R^{ab}{}_{cd}\, \epsilon^{cdef}\, R_{efab} = \frac{M^2 r\, a\cos\theta(3a^2\cos^2\theta - r^2)(a^2\cos^2\theta - 3r^2)}{(r^2 + a^2\cos^2\theta)^6}$$

- The region $r < r_+$ is a BH

The following Theorem follows from results of Carter [?], Robinson [?], Hawking and Wald:

Theorem: If (M, g) is an asymptotically-flat stationary vacuum spacetime containing a BH, then it is a member of the two-parameter Kerr family (the parameters are the mass M and angular momentum per unit mass $a = J/M$).

This result makes the Kerr solution the most important exact solution in GR. If the BHs we are currently observing in a direct way are modeled by GR, then their outside horizon geometry is nearly Kerr's, if matter around can be treated perturbatively.