The 2+1 *BH and other 3D weirdos*

PPGCosmo workshop on
Astrophysics, Cosmology and Gravitation
Vitória, Espírito Santo - Brasil
March 05, 2020

Jorge Zanelli
CECs –Valdivia (Chile)
z@cecs.cl

Collaboration with **M. Briceño, C. Martínez, N. Valdés**

In memoriam

Prof. Antonio Brasil Batista

We wouldn’t be here today if it hadn’t been for him.

Outline:

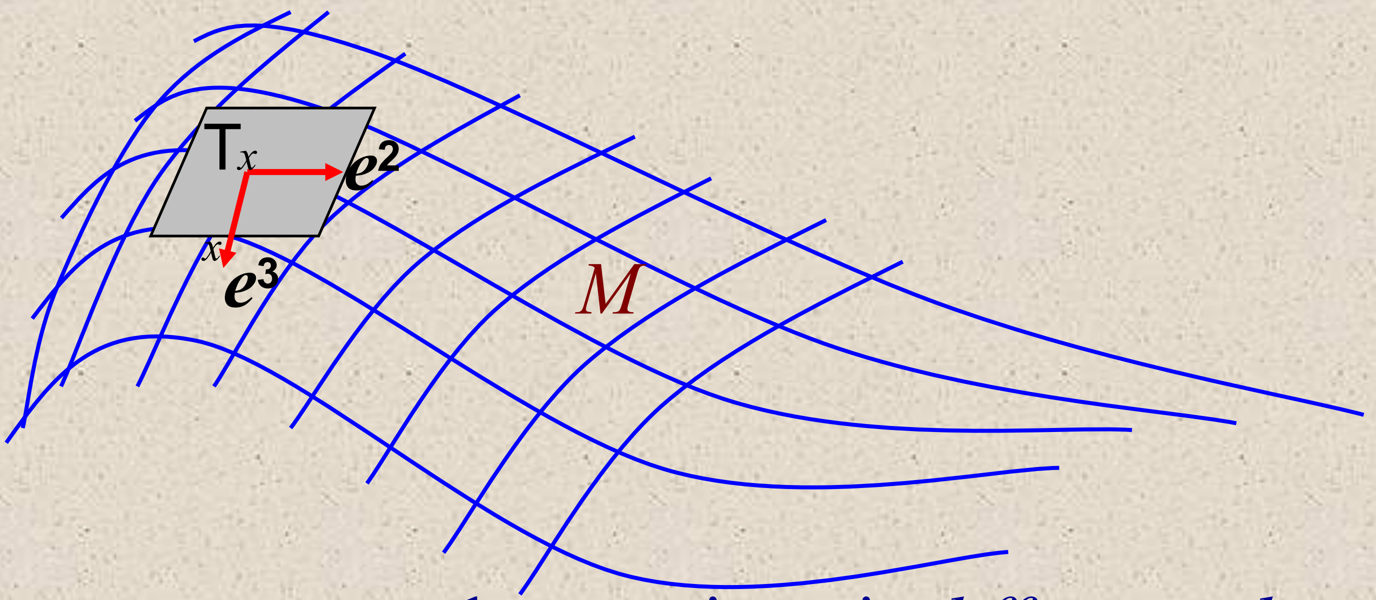
- *GR in* D *dimensions*
- 2+1 *Gravity*
- *The* 2+1 *Black Hole*
- *BHs as identifications*
- *Conical Naked Singularities*
- *Living with NSs*; (*CTCs*?)
- *Summary*

GR in D *dimensions*

Gravitation is the study of the geometry of spacetime

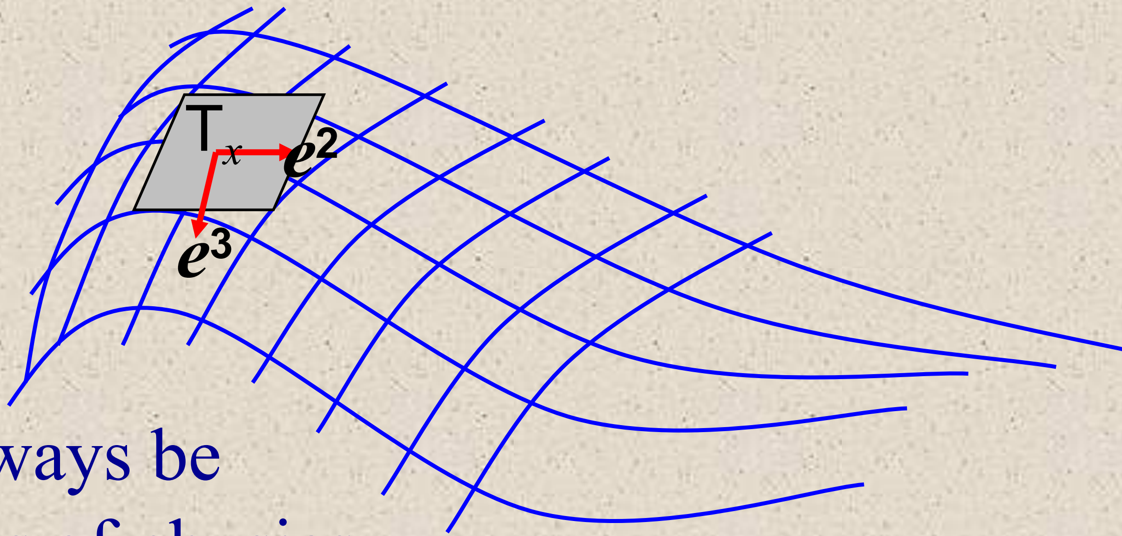
- *D*-dimensional spacetime *M* is a differentiable manifold (up to isolated singularities in sets of zero-measure).

- *M* admits a tangent space T_x at each point.



- An open set around any point x is *diffeomorphic* to an open set in the tangent.

Equivalence Principle



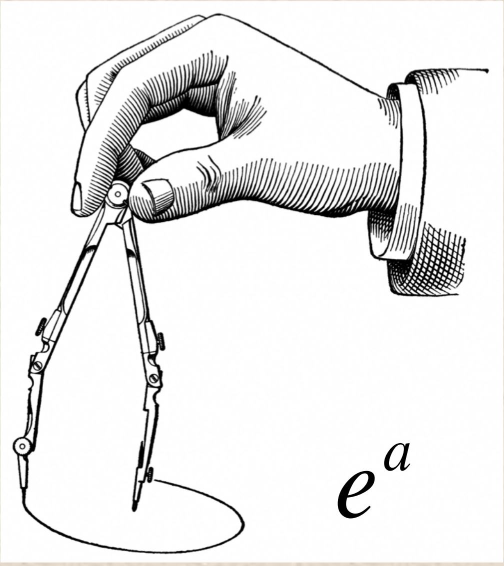
- In a sufficiently small region of spacetime a reference frame can always be found in which the laws of physics are those of *special relativity* (*freely falling observer*).

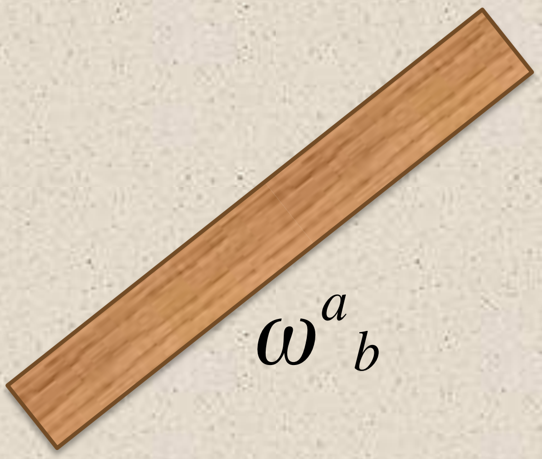
- The laws of physics are invariant under *local Lorentz transformations*.

- General Relativity is a nonabelian gauge theory for the group $SO(3,1)$ in $4D$.

Geometry has two ingredients:

- Metric structure (length/volume, angle, scale) $\longrightarrow e^a$
- Affine structure (parallel transport, congruence) $\longrightarrow \omega^a{}_b$





- These two notions are conceptually and operationally independent. It makes sense to consider them functionally and dynamically independent as well.

Metric Structure

- The diffeomorphism between Minkowski space and an open neighborhood in *M* means that there an invertible relation

$$dz^a = e^a_\mu(x) dx^\mu \equiv e^a(x)$$

(e^a: vielbein, soldering form, local orthonormal frame),

so that

$$ds^2 = \eta_{ab} dz^a dz^b$$

$$= \eta_{ab} e^a_\mu e^b_\nu dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu$$

Metric on *M*:

$$\boxed{g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x)}$$

Affine Structure

- The differential operations appropriate for a dynamical theory compatible with GR should be *Lorentz covariant,*

$$D_\mu u^a = \partial_\mu u^a + \omega^a{}_{\mu b} u^b$$

where u^a is a Lorentz vector and $\omega^a{}_{\mu b}$ is the connection for the Lorentz group:

$$\omega \to \omega' = \Lambda^{-1}(\omega + d)\Lambda$$

- This ensures that under Lorentz transformations, u^a and $D_\mu u^a$ transform in the same manner,

$$u^a \to u'^a = \Lambda^a{}_b u^b,\ \Lambda^a{}_b \in SO(D-1,1)$$

$$Du^a \to (Du^a)' = \Lambda^a{}_b (Du^b)$$

Using only exterior derivatives of e and ω, two new Lorentz tensors are found

$$\left.\begin{aligned} De^a = de^a + \omega^a{}_b \wedge e^b &= \boxed{T^a} \quad \text{Torsion} \\ d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b &= \boxed{R^a{}_b} \quad \text{Curvature} \end{aligned}\right\} \underline{\text{2-forms}}$$

Bianchi identities: $$DR^a{}_b \equiv 0$$

$$DT^a \equiv R^a{}_b \wedge e^b$$

➔ Successive derivatives generate no new functionally independent tensors. The only available ingredients for a Lagrangian are:

$$\underbrace{\eta_{ab}, \quad \varepsilon_{a_1 a_2 \dots a_D}}_{\text{Zero-forms}}, \qquad \underbrace{e^a, \quad \omega^a{}_b}_{\text{one-forms}}, \qquad \underbrace{T^a, \; R^a{}_b}_{\text{two-forms}}$$

The most general gravitational action is

$$\boxed{I[e^a,\omega^a_{\ b}] = \int_{M^D} \underbrace{L(e^a,\omega^a)}_{D\text{-form}}}$$

This requires that under local $SO(3,1)$ transformations,

$$\delta L = d(\text{something}) \ .$$

The Lagrangian must be *quasi*-invariant

If no additional ingredients are included, very few Lorentz invariant actions can be built in each dimension.

2+1 *Gravity*

The most general Lagrangian for 3D gravity is :

$$I[e,\omega] = \int (\varepsilon_{abc}[R^{ab} + \tfrac{\alpha}{3} e^a e^b] e^c + \lambda e^a T_a + \gamma [\omega^a{}_b d\omega^b{}_a + \tfrac{2}{3} \omega^a{}_b \omega^b{}_c \omega^c{}_a])$$

The field equations obtained varying with respect to e^a and $\omega^a{}_b$ are

$$R^{ab} + \Lambda e^a e^b = 0, \qquad T^a + \tau \varepsilon^{abc} e_b e_c = 0$$

➔3 dimensional gravity describes pacetimes of *constant* (Lorentz) *curvature* and *constant torsion*.

In spite of its "triviality", 2+1 gravity admits interesting solutions like rotating black holes, spinning particles and cosmological solutions.

- For $\Lambda=-l^{-2}$ and $T^{a} = 0$ there are spherically symmetric stationary black hole solutions, labeled by 2 constants of integration: *mass* M and *angular momentum* J.

- 2+1 BHs have the same constant negative curvature for all values of M and J:

$$R^{ab} = -l^{-2}\, e^{a}\, e^{b}\,,\ or\ R^{ab}{}_{\mu\nu} = -l^{-2}[\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}-\delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu}].$$

(In what follows we will use l=1)

The 2+1 black hole (1992)

Static case

$$ds^2 = -(r^2 - M)dt^2 + \frac{dr^2}{(r^2 - M)} + r^2 d\varphi^2$$

$$t \in \mathbb{R}, \quad r \in \mathbb{R}^+, 0 \leq \varphi \leq 2\pi \sim 0.$$

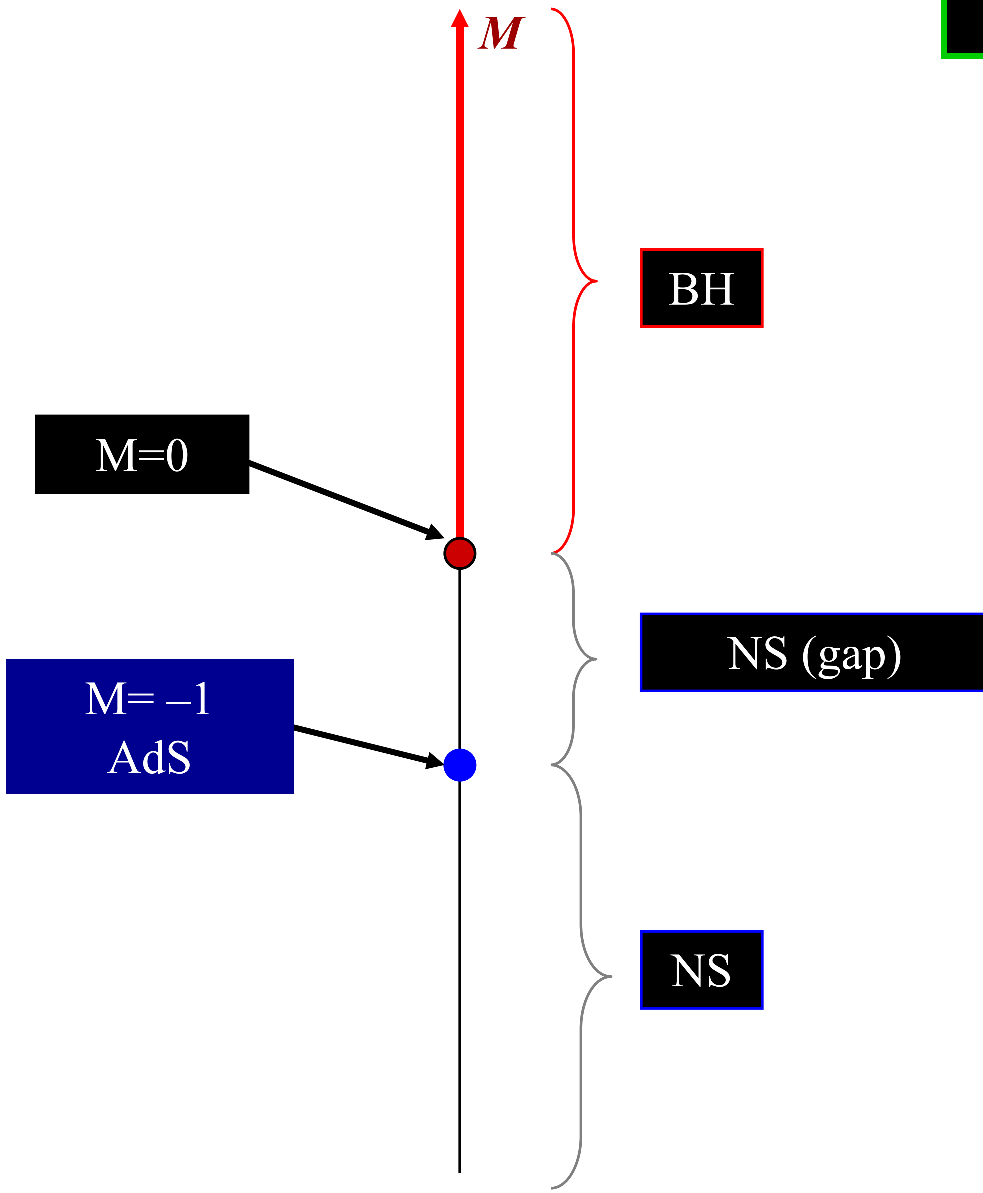
$M \geq 0$ → BH; horizon at $r_+ = M^{1/2}$

$M = -1$ → AdS spacetime ($\Lambda = -1$)

$M<0$ → No horizon: *Naked singularity*

2+1 BH spectrum

(*J*=0)



Rotating black hole

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\varphi)^2$$

$t \in \mathbb{R}, \quad r \in \mathbb{R}^+, 0 \leq \varphi \leq 2\pi \sim 0.$

$$f^2 = -M + r^2 + \frac{J^2}{4r^2}$$

$$N = -\frac{J}{2r^2}$$

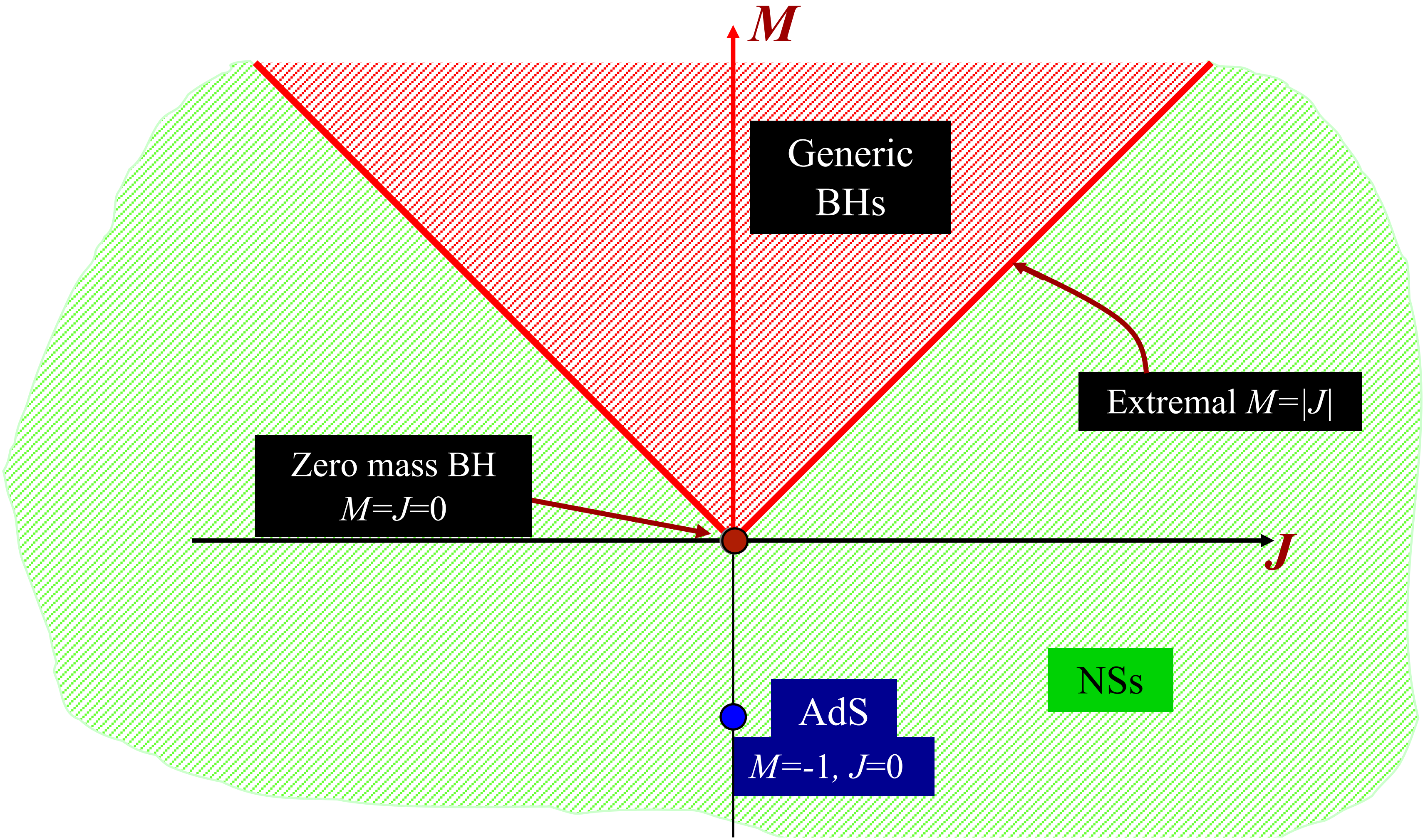
$$r_\pm^2 = \tfrac{1}{2}\left(M \pm \sqrt{M^2 - J^2}\right) \in \mathbf{R} \qquad \Leftrightarrow M \geq |J|$$

$M \geq |J| > 0$ $\Rightarrow$ BH; horizons $r_\pm \geq 0$

$M = -1, J = 0$ $\Rightarrow$ AdS

$M < |J|, \neq -1$ $\Rightarrow$ Naked singularities

2+1 *Black Hole Spectrum*



Every point in this plane represents a *constant negative curvature geometry, locally indistinguishable from AdS_3.*

BHs as identifications

- AdS_{2+1} can be defined as the pseudoesphere

$$\boxed{-(x^0)^2-(x^1)^2+(x^2)^2+(x^3)^2=-1}$$

This geometry has 6 Killing vectors:

$$k=\tfrac{1}{2}k^{ab}(x_a\partial_b-x_b\partial_a)=\tfrac{1}{2}k^{ab}J_{ab}\in so(2,2)$$

- Identifications (quotienting) by a K-vector does not change the local geometry.

- 2+1 *BHs are locally isometric to* AdS_3. *Can they be obtained by identifications in AdS_3?*

Of course !

Black hole as identifications

$$AdS_{2+1}: \boxed{-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -1}$$

- Generic BH: $M > |J| \neq 0;\ \ 0 < r_- < r_+$

$$r_\pm = (2^{-1/2})\sqrt{M \pm (M^2 - J^2)^{1/2}}$$

$$\boxed{\xi(r_+, r_-) \ = \ r_+ J_{12} - r_- J_{03} \quad \sim \partial_\varphi}$$

These are non-compact elements of $so(2,2)$

No fixed points $\Longrightarrow$ *No conical singularities*

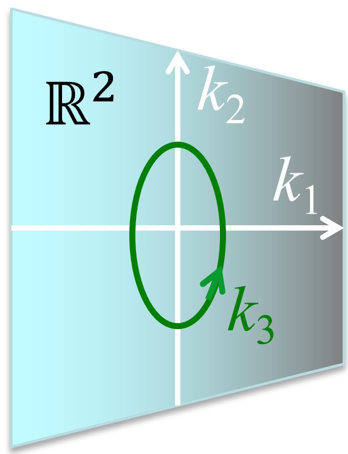
$\xi \cdot \xi = g_{rr} = r^2 > 0$ $\Longrightarrow$ *No closed timelike curves*

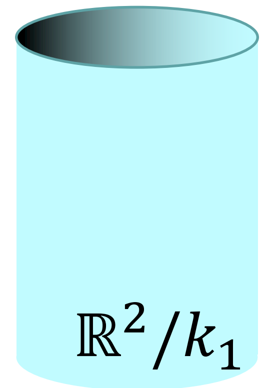
Conical Naked Singularities

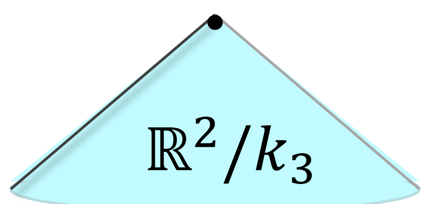
O.Mišković, J.Z. (2009)

Identifications by Killing vectors respect the *local geometry*:

k_1, k_2, k_3: isometries







k_1 leaves no *fixed* points
→ no singularities

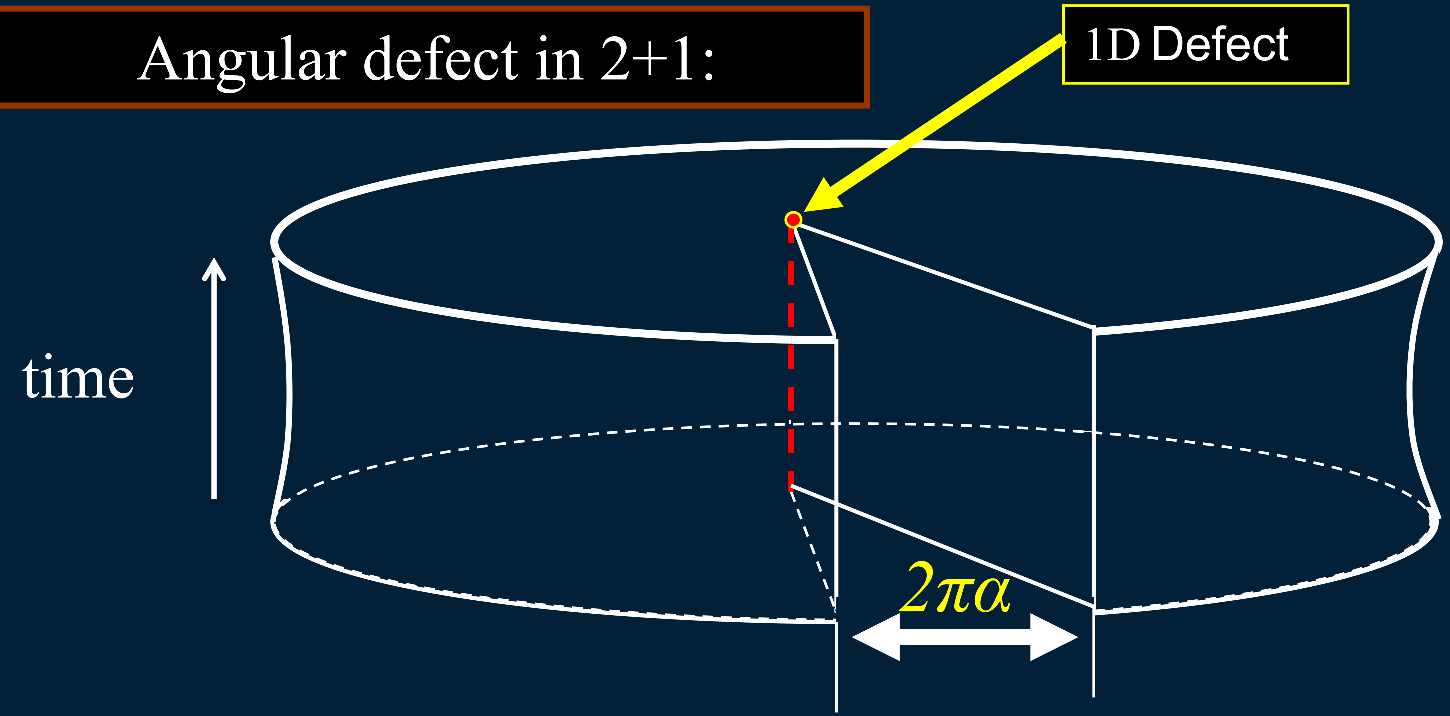
$r=0$ fixed point of k_3
→ Conical Naked Singularity

Conical defects in 2+1 dim. are localized, static/stationary, spherically symmetric geometries.

At large distances point particles, black holes and conical singularities cannot be distinguised.

Unlike black holes, a conical singularity is not surrounded by an event horizon ⟹ Naked Singularity (orbifold)

Angular defect in 2+1:



Identification in the x^1-x^2 plane generates a conical singularity along the set of fixed points of the Killing vector:

$$\xi = -2\pi\alpha \, (x_2\partial_3 - x_3\,\partial_2)$$
$$= -2\pi\alpha \, \partial_\varphi$$

Angular defect $\Leftrightarrow$ δ-like curvature singularity at $r = 0$

The conical geometry looks like a BH:

$$ds^2 = -(r^2 - M)dt^2 + (r^2 - M)^{-1}dr^2 + r^2 \mathrm{d}\varphi^2$$

But here "the mass" is negative, $M = -(1-\alpha)^2$, related to the deficit angle, $\Delta = 2\pi\alpha = 2\pi[1-(-M)^{1/2}]$.

Exceptional cases:

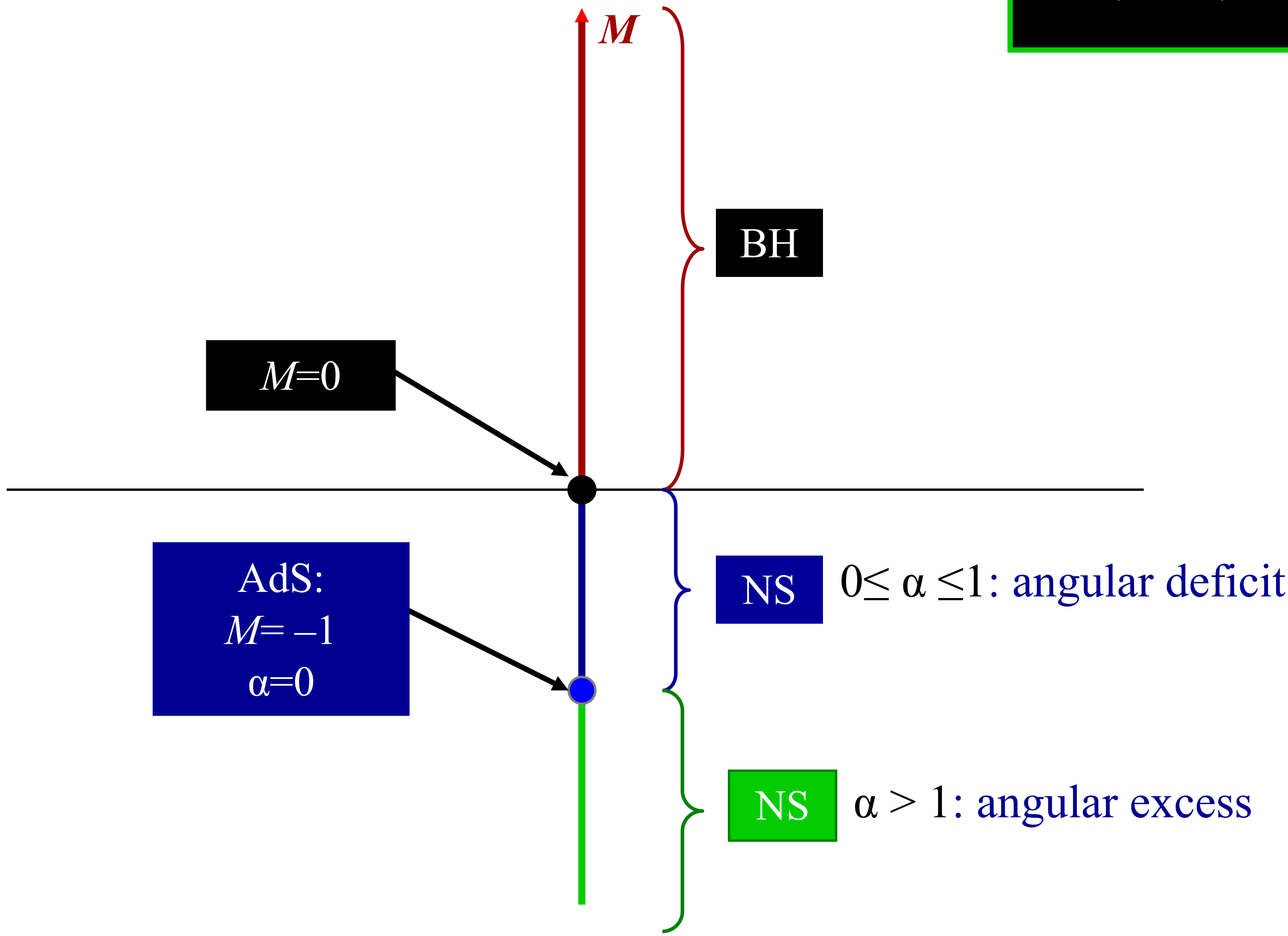
$\alpha = 0,\ M = -1$ → anti-de Sitter; no angular deficit

$\alpha = 1,\ M = 0$ → zero-mass bh; maximum deficit

For $M < 0$ these naked singularities are rather harmless point particles.

2+1 *BH-NS spectrum*

(*J*=0)



Like BHs, conical singularities can also be boosted…

Rotating 2+1 Naked Conical singularity

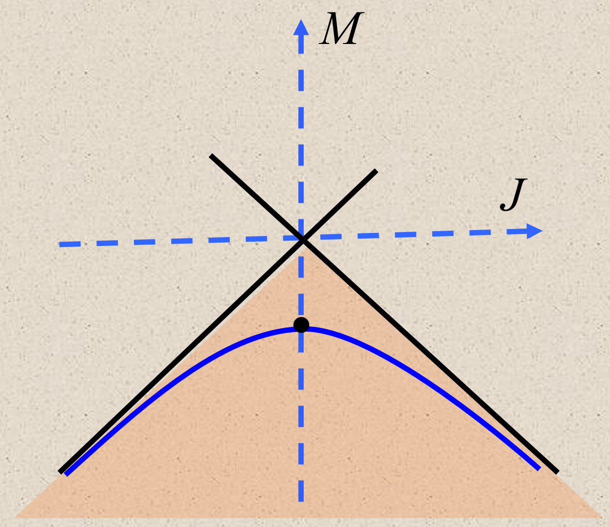
$0 \neq -|J| > M$

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\varphi)^2$$

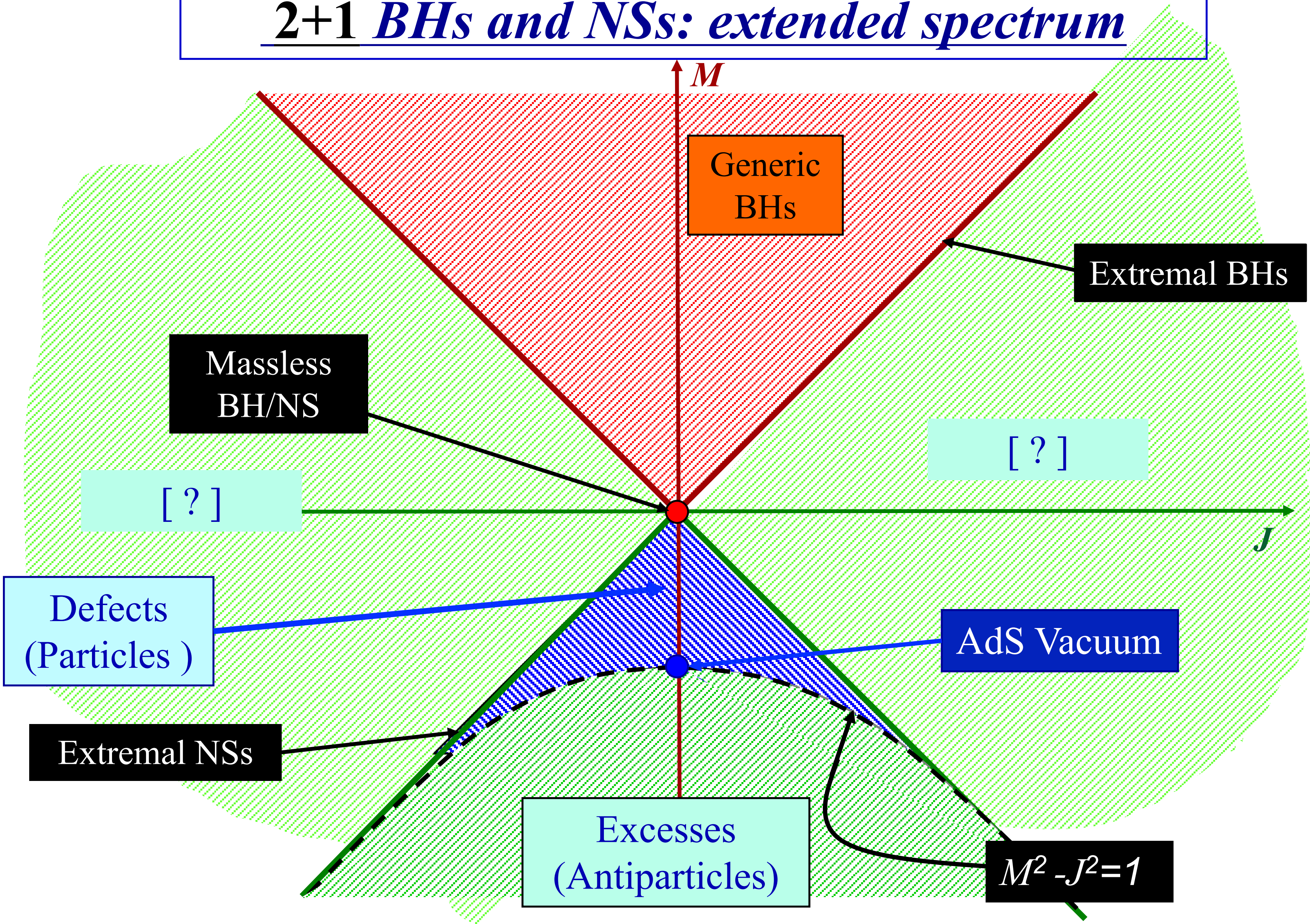
$$f^2 = r^2 - M + \frac{J^2}{4r^2} > 0$$

$$N = -\frac{J}{2r^2}$$

No horizons



2+1 *BHs and NSs: extended spectrum*



Identifications, fixed points and NSs

Conical singularities as identifications

$$AdS_{2+1}: \quad -(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -1$$

- Generic spinning cone, $M<-|J|\neq 0$

$$\boxed{\xi(\beta_+,\beta_-) = \beta_+ J_{23} + \beta_- J_{01} \quad \sim \partial_\varphi}$$

$$\beta_\pm = \sqrt{-M+J} \pm \sqrt{-M-J}$$

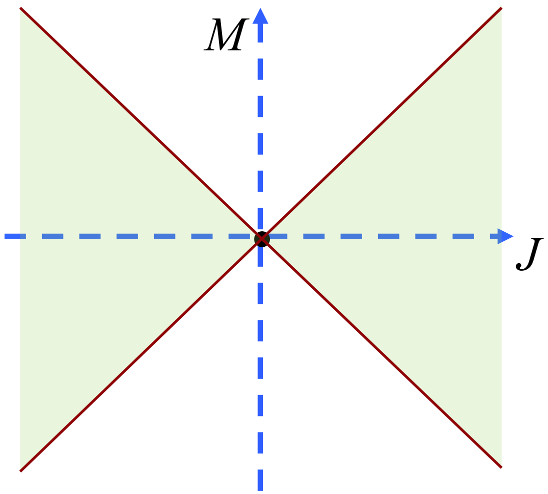
These are compact elements of SO(2,2)

- $r = 0$ fixed point ⟹ *Conical Naked Singularity*
- $\xi\cdot\xi > 0$ ⟹ *No CTCs*

What about the *overspinning* naked singularities $|J| > M > -|J|$?

These geometries are also obtained by identifications:

$$\xi = a\underbrace{(J_{01} + J_{23})}_{\text{Rotation}} + b\underbrace{(J_{03} + J_{12})}_{\text{Boost}} \sim \partial_\varphi$$



where

$$\lambda_\pm = a \pm ib$$
$$= \frac{1}{2}\left[(-M + |J|)^{\frac{1}{2}} \pm (-M - |J|)^{\frac{1}{2}}\right]$$

These are noncompact elements of SO(2,2)

- $r = 0$ fixed point ⟹ *Conical Naked Singularity*
- $\xi \cdot \xi > 0$? ⟹ *CTCs* ?

Living with NSs

It has been claimed that these geometries have CTCs:

arXiv:1204.3288[hep-th] [Barnich, Gomberoff & Hernández, PRD (21012)]:

"… if we make the transformation

$$r'^2 = \frac{16G^2J^2}{\alpha^4} - r^2/\alpha^2\ ,$$

which produces the line element

$$ds^2 = \left(-\alpha\, dt^2 + \frac{4GJ}{\alpha}\right)^2 + dr^2 - \alpha^2 r'^2 d\varphi^2.$$

Therefore, inside this region, the direction ∂φ is always time-like, generating closed time-like curves. It is a bounded time machine."

arXiv:1801.07064 [hep-th] [Compère & Fiorucci; Springer Lectures (2019)]

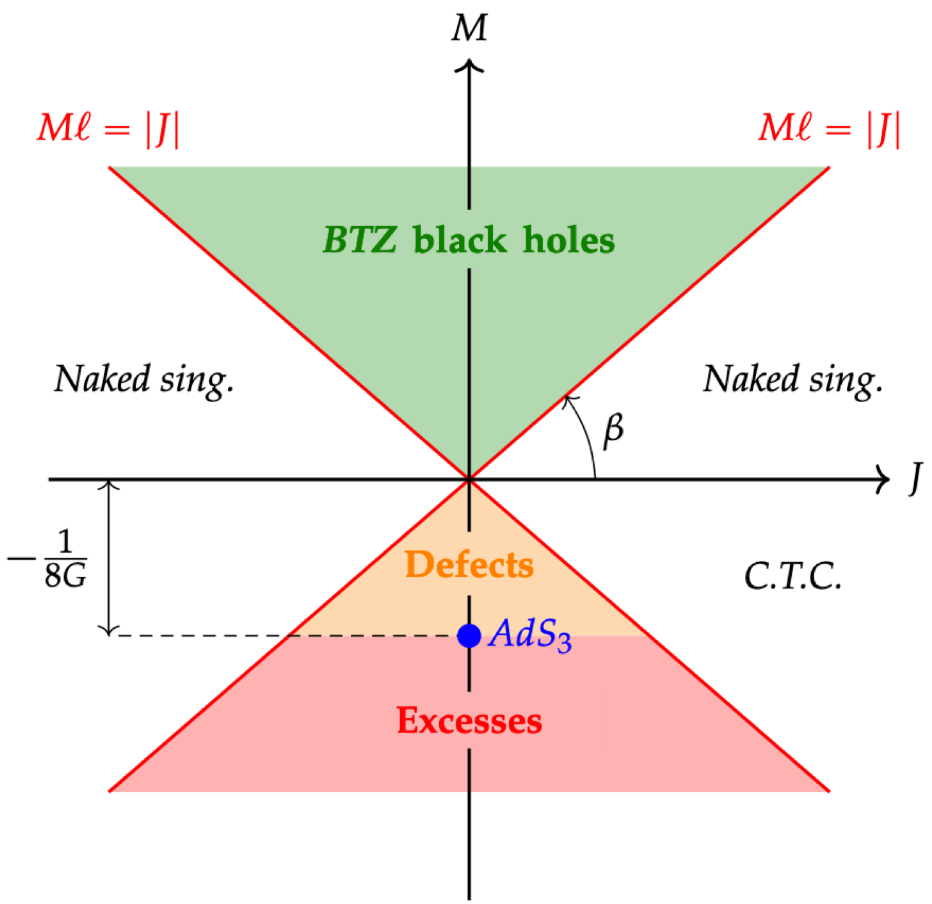


Figure 2.7: *Solutions described by the BTZ metric. The slope of extremal lines is* $\tan\beta = \ell^{-1}$.

Figure adapted with permission from [33].Copyrighted by the American Physical Society.

arXiv:1902.01583 [hep-th] [Casals, Fabbri, Martinez & Zanelli PRD (2019)]

QUANTUM-CORRECTED ROTATING BLACK HOLES AND… PHYS. REV. D **99**, 104023 (2019)

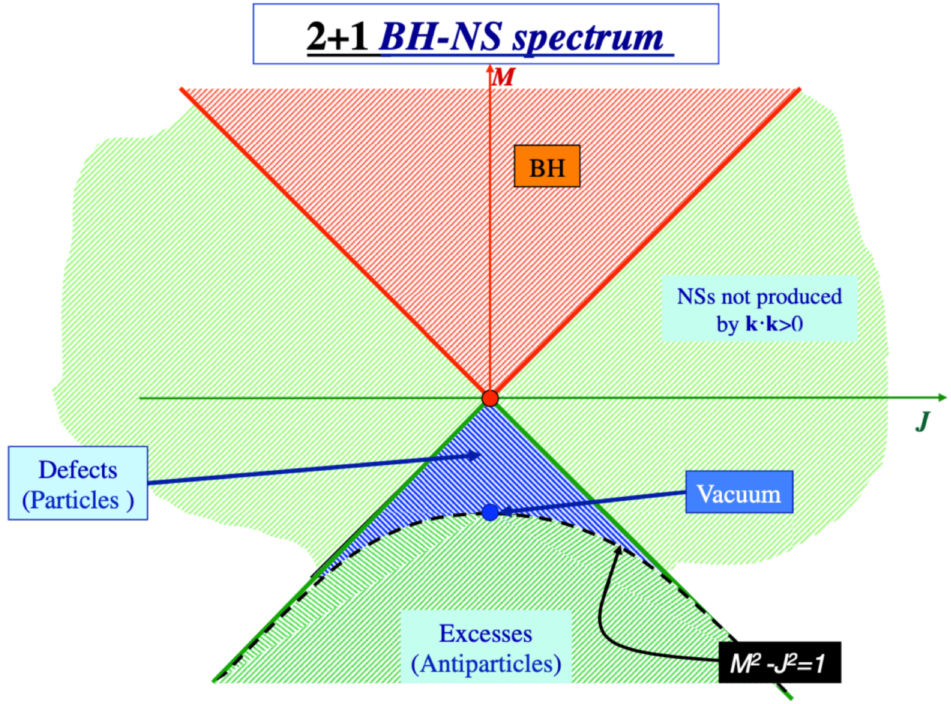


FIG. 2. Schematic representation of BTZ black hole and naked singularity states for different values of M and J.

But this is not true: the geometry has no CTCs as long as $r^2 > 0$

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\varphi)^2$$

Singularities represent a breakdown of spacetime where the notion of geometry fails and the "normal" physical laws no longer apply:

- *Causality*
- *Energy conservation*
- *Stability*
- *Predictability*

Can be violated

Under reasonable generic initial conditions in GR, singularities are inevitable (Penrose-Hawking theorems).

Dressed singularities (surrounded by a horizon) are not problematic, while NSs are *dangerous*.

Cosmic Censorhip (CC) Conjecture:

NSs cannot exist in nature (Roger Penrose -1968)

- CC seems to be true, but there is no proof of it.
- Christodoulu: Generic collapsing matter can produce NSs.
- This, however, requires fine tuning of initial conditions.

➔ The need of "*fine tuning*" suggests that *NS*s may be perturbatively unstable.

Cosmic Censorhip (CC) Conjecture

(Roger Penrose -1968)

"NSs cannot exist in nature"

On the other hand...

- H-P theorems imply that singularities in spherical collapse are stable against non-spherical perturbations.
- Spherical black holes also require fine tuning, but are stable under non-spherical perturbations

➔ Classical stability doesn't seem to rule out NSs

➔ Can quantum mechanics prevent NSs as it prevents the collapse of the electron to $r = 0$ in the H atom?

Intractable problem in 3+1D. However, we recently showed that quantum effects generate a horizon around a NS.

arXiv:1902.01583 [Casals, Fabbri, Martinez &Z, PRD (2019)]

Summary

- Black hole-like metrics for all values of (M, J) are obtained by identifications in AdS_3.
- All of these geometries have a singularity at $r = 0$, which arises from cutting away the region with CTCs.
- In all of these geometries the curvature has a δ-like singularity at the origin.
- In the cases of conical NSs ($M < -|J|$) it has ben shown that cosmic censorship results from quantum effects.
- Like the Covid-19 epidemic, NSs may be scary and complicate life but are much more harmless than expected.

Obrigado, professor Brasil