Black Holes and their Analogues: 100 years of General Relativity.

A workshop in honor of Kirill Bronnikov's 70th birthday.

Ubu - Anchieta, ES, Brazil.
April 13 - 17, 2015

Black hole, a difficult birth...

Jean Eisenstaedt

SYRTE, Observatoire de Paris, CNRS, UPMC ; 61 avenue de l'Observatoire, 75014 Paris, France

Black hole, a difficult birth...

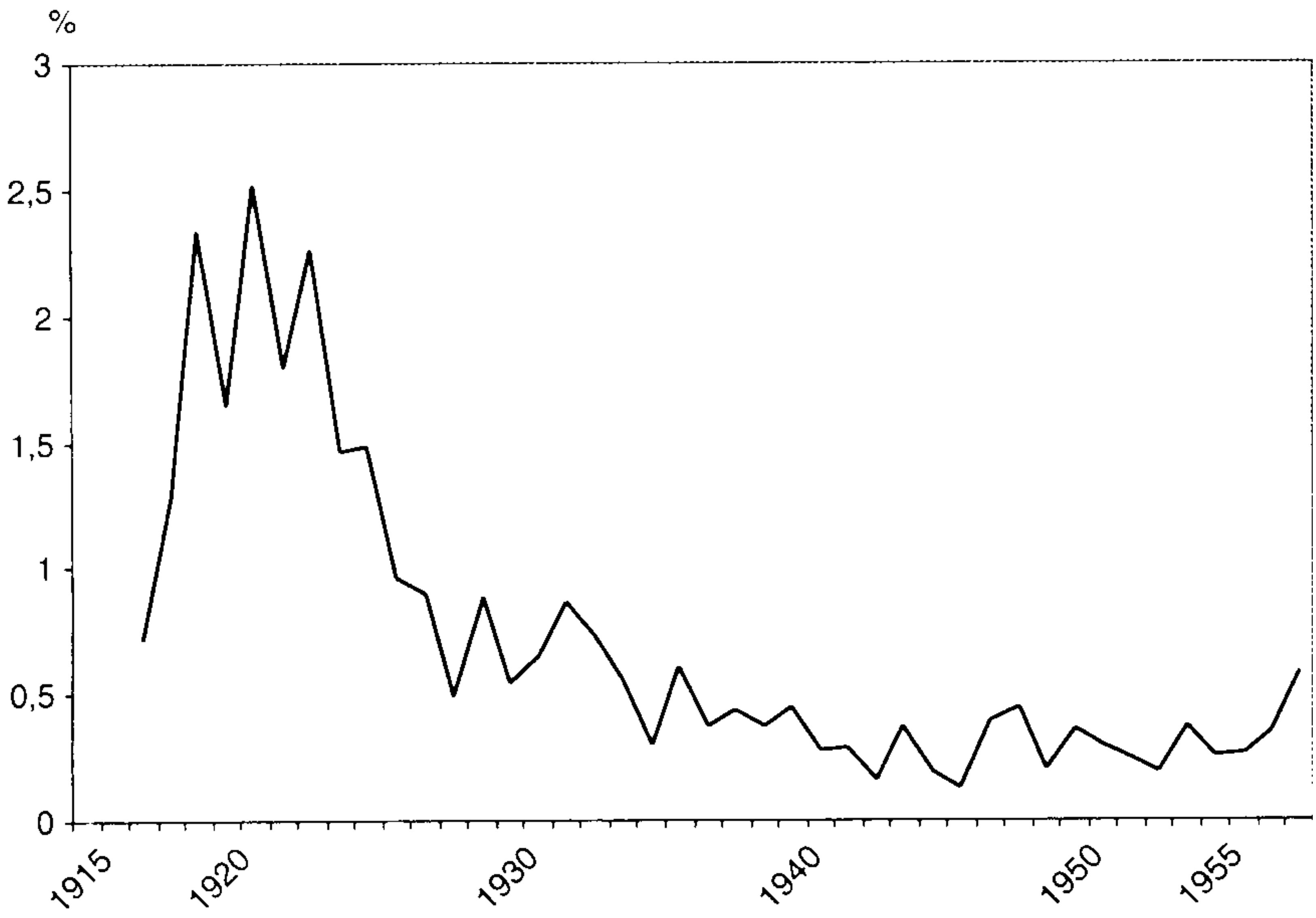
- The Low Water Mark Period
- Schwarzschild’s solution
- The Schwarzschild singularity
- Some accurate remarks and questions...
- The reification of Schwazschild’s singularity
- A fictitious singularity...
- Schwarzschild’s singularity becomes a black-hole horizon...

The Low Water Mark Period

Max Born, 1959

Born, 1955, Bern’s Colloquium

- "I remember that on my honeymoon in 1913 I had in my luggage some reprints of Einstein's papers which absorbed my attention for hours, much to the annoyance of my bride. These papers seemed to me fascinating, but difficult and almost frightening. When I met Einstein in Berlin in 1915 the theory was much improved and crowned by the explanation of the anomaly of the perihelion of Mercury, discovered by Leverrier. I learned it not only from the publications but from numerous discussions with Einstein, - which had the effect that I decided never to attempt any work in this field. The foundation of general relativity appeared to me then, and it still does, the greatest feat of human thinking about Nature, the most amazing combination of philosophical penetration, physical intuition and mathematical skill. But its connections with experience were slender. It appealed to me like a great work of art, to be enjoyed and admired from a distance. “



Number of publications in general relativity
as a percentage of the total number of publication in physics (1915-1955).

A neo-newtonian interpretation.

- "Mechanical laws, according to Einstein's theory, are much more complicated in conception than under the assumption of Newton. However, the motion of celestial bodies under ordinary circumstances differs so little from their Newtonian representation that, for astronomical purposes, relativistic effects may be conveniently treated as first-order perturbations. (Levi-Civita 1937)."

- "it seems that we must give up any hope to approach, in general Relativity, in a rigorous way, the questions we consider usually in classical mechanics."

(Levi-Civita 1950).

— Quel concert de louanges ! Ces dames parlent encore de leurs couturiers.

— Mais non, il s'agit d'Einstein. (*Dessin de* P. PORTELETTE.)

Einstein in Paris: fashionable!

Schwarzschild's solution

Karl Schwarzschild

Schwarzschild's original solution

Schwarzschild's original solution expressed in the coordinates he used: actually: $x_1 = R^3/3$, $x_2 = -\cos\theta$, $x_3 = \phi$, and $x_4 = ct$.

$$ds^2 = (1 - \frac{\alpha}{r})c^2\, dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2\left(d\theta^2 + sin^2\, \theta d\phi^2\right)$$

where $r = \left(R^3 + \alpha^3\right)^{1/3}$; (with $\alpha = \dfrac{2Gm}{c^2}$); or $R = \left[r^3 - \left(\dfrac{2\,Gm}{c^2}\right)^3\right]^{1/3}$.

$$ds^2 = \left(1 - \frac{2Gm}{rc^2}\right)c^2\,dt^2 - \frac{dr^2}{1 - \frac{2Gm}{rc^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Schwarzschild’s solution in Droste’s coordinates.

Defining "a physical distance" δ as:

$$\delta = \int_{\frac{2Gm}{c^2}}^{r} \frac{dr}{\left(1 - \frac{2Gm}{rc^2}\right)^{1/2}}$$

and using t, the Newtonian coordinate time; from his equation of motion:

$$\dot{\delta}^2 = \left(\frac{d\delta}{dt}\right)^2 = \left(1 - \frac{2Gm}{rc^2}\right)\left[1 - \frac{1}{E^2}\left(1 - \frac{2Gm}{rc^2}\right)\right]c^2$$

$$\ddot{\delta} = \frac{d^2\delta}{dt^2} = \frac{Gm}{r^2}\left[1 - \frac{2}{E^2}\left(1 - \frac{2Gm}{rc^2}\right)\right]\left(1 - \frac{2Gm}{rc^2}\right)^{1/2}$$

$\dot{\delta}$ and $\ddot{\delta}$ go to z ero at $r = 2Gm/c^2$ " where the motion stops infinitely smoothly". He will even get a repulsion around the "singularity".

- "a moving particule out of the sphere $r = \alpha$ would never get into this sphere"

- "the particule will never reach the sphere $r = \alpha$".

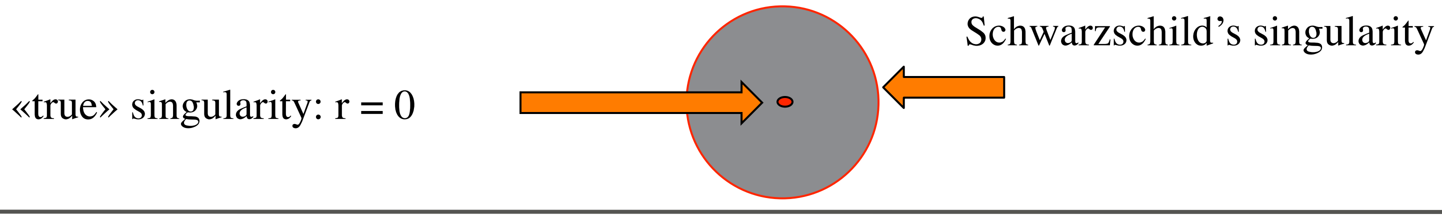
- He "will not consider the space $r < \alpha$"

He will come back to this question in his thesis (1917) but with his radial coordinate only and t the time-coordinate.

Droste on the radial fall (1916, 1917)

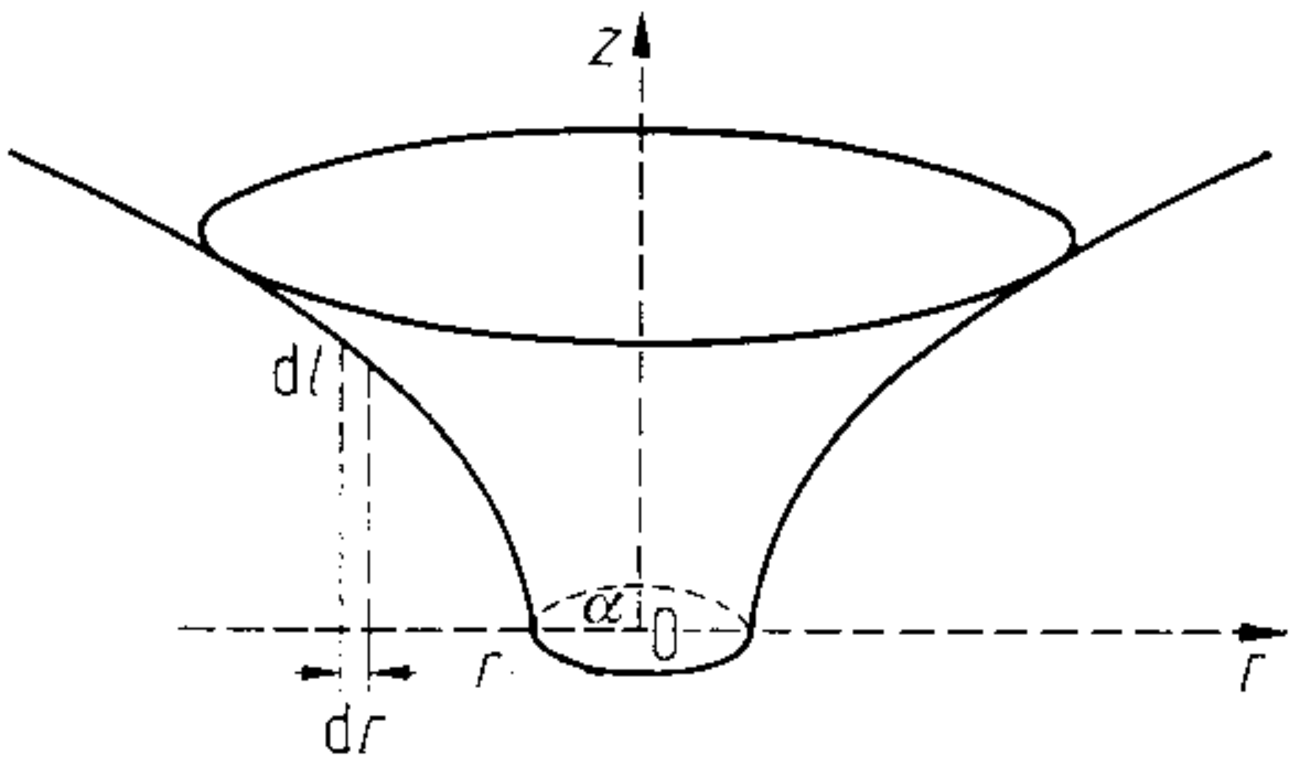
The Schwarzschild singularity

Schwarzschild’s space-time



The structure of Schwarzschild’s solution

oir :



Flamm's representation of Schwarschild's solution 1916 (from Becquerel 1922).

Embedding the spatial part of the Schwarzschild solution in an Euclidean space of four dimensions:

$$ds^2 = \frac{dr^2}{1 - \frac{\alpha}{r}} + r^2 d\theta^2 = dx^2 + dy^2 + dz^2$$

where: $x = r \sin\theta$, $y = r \cos\theta$ and $z = \int_0^r \frac{dr}{\sqrt{\frac{r}{\alpha} - 1}} = 2\sqrt{\alpha(r - \alpha)}$.

Flamm's representation, 1916.

Einstein and Painlevé, Paris 1922

Coordinates and covariance...

1. Schwarzschild's solution in standard polar coordinates:

$$d s^2 = (1 - \frac{\alpha}{r})c^2\, dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right)$$

with $r = \left(R^3 + \alpha^3\right)^{1/3}$.

2. Schwarzschild's solution in Droste's coordinates:

$$d s^2 = (1 - \frac{\alpha}{r})c^2\, dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right)$$

(with $r > \dfrac{2Gm}{rc^2}$).

3. Isotropic system (Droste 1916):

$$ds^2 = \left[\frac{\rho - \frac{mG}{2c^2}}{\rho + \frac{mG}{2c^2}}\right]^2 c^2\, d\,t^2 - \left(1 + \frac{mG}{2\rho c^2}\right)^4 \left\{d\rho^2 + \rho^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right)\right\}.$$

4. (Droste 1916) by means of the transformation $r = \tilde{r} + \alpha$:

$$ds^2 = \frac{c^2\, dt^2}{(1 + \frac{\alpha}{\tilde{r}})} - (1 + \frac{\alpha}{\tilde{r}})d\tilde{r}^2 - \left(\tilde{r} + \alpha\right)^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right).$$

Painlevé, Schwarzschild's solution and coordinates.

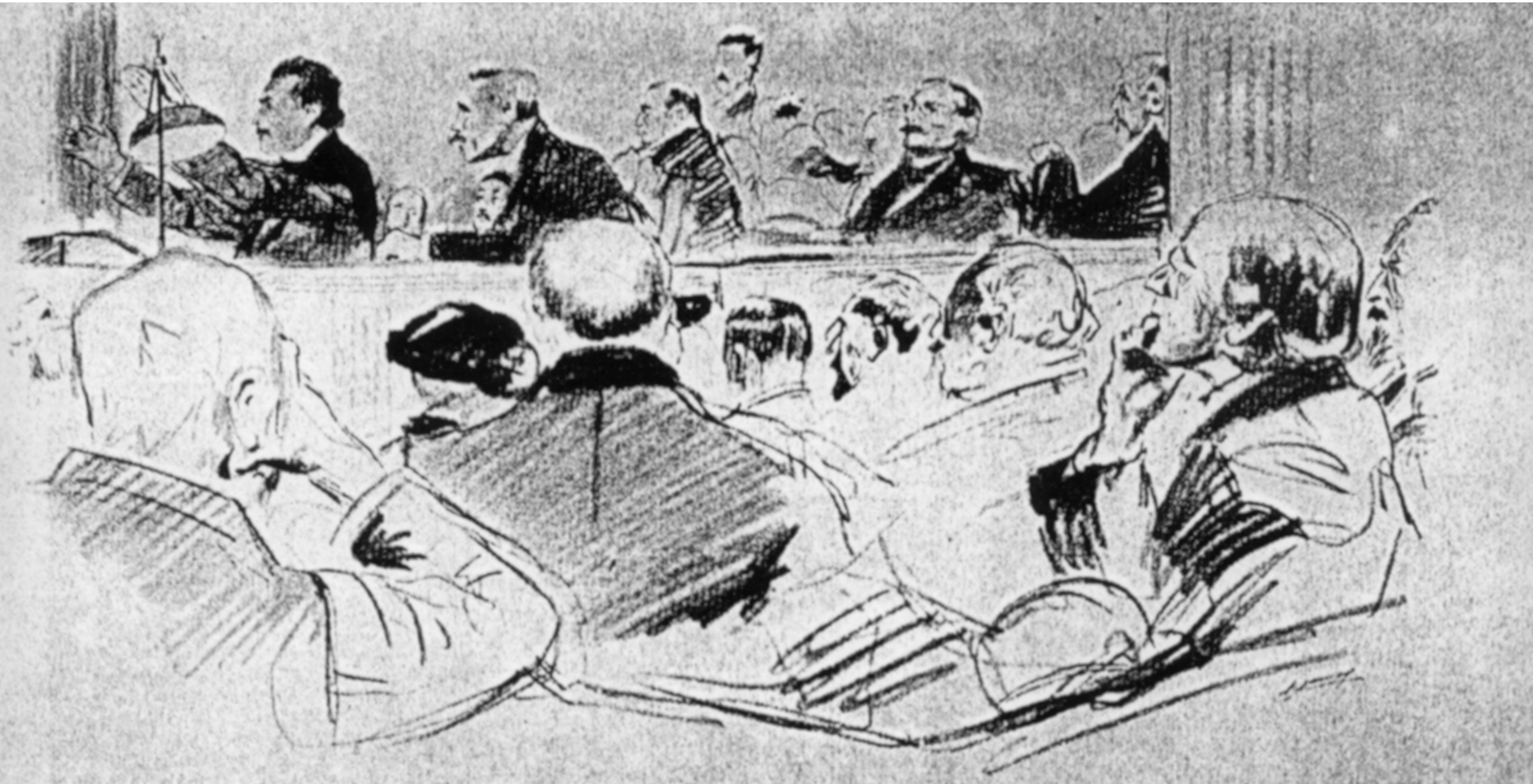
$$ds^2 = (1 - \frac{\alpha}{r})c^2 dt^2 + 2\sqrt{\frac{\alpha}{r}}drcdt - \left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

"It seems to me that the existence of the formula [*] and the possibility of an infinity of others give a clear indication of the hazardous character of such predictions. [...] it's pure imagination to claim that such consequences can be derived from the ds^2." (Painlevé 1921, p.680).

Einstein, Schwarzschild's solution and coordinates.

$$d s^2 = (1 - \frac{\alpha}{r})c^2 \, d t^2 + 2\sqrt{\frac{\alpha}{r}} dr c dt - \left(d r^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right)$$

"When, in the ds^2 of the spherically static solution with central symmetry, you introduce instead of *r* any function of *r* , you do not obtain a *new* solution because the quantity *r* i n itself has no physical meaning. [...] You must always keep in mind that coordinates do not have any physical signification ; which means that they do not represent the result of a measurement ; only the conclusions, accessible by the elimination of coordinates may claim for an objective signification. Furthermore, the metrical interpretation of the quantity *ds* is not "pure imagination" but the deep core of the theory itself." (Einstein to Painlevé, December 7, 1921)



Le grand physicien expose ses théories dans un amphithéâtre du Collège de France, le 31 mars. — Derrière M. Einstein, M. Langevin se tient prêt à lui souffler les mots français qui, quelquefois, ne viennent pas facilement aux lèvres du conférencier.

Collège de France, 1922

Jacques Hadamard

The impossible collapse (from Einstein 1922).

The interior Schwarzschild's solution :

static,

density ρ is constant,

pressure $p = p(r)$.

$$ds^2 = -e^{\lambda} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) + e^{\nu} dt^2$$

The pressure $p = p(r)$ is infinite in $b = \frac{9}{8} \frac{2mG}{c^2}$ before the Schwarzschild's singularity could be reached.

The Eddington line-element, 1924.

$$ds^2 = -dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + c^2dt^2 - \frac{2Gm}{rc^2}\left(cdt - dr\right)^2$$

“We can go on shifting the measuring-rod through its own length time after time, but *dr* is zero; that is to say, we do not reduce *r*. There is a **magic circle** which no measurement can bring us inside. It is not unnatural that we should picture something obstructing our closer approach, and say that a particle of matter is filling up the interior. ”

S. Eddington, 1920

"I felt driven to the conclusion that this was almost a *reductio ad absurdum* of the relativistic degeneracy formula. Various accidents may intervene to save the star, but I want more protection than that. I think there should be **a law of Nature to prevent a star to behave in this absurd way**."

Arthur S. Eddington, 1935

Some accurate remarks and questions...

Droste's on covariance

- "To what formula shall we give the preference, to that of Schwarzschild [1], to [2], [3] or to [4]? It is in fact a matter of personal convenience. But we must remember that the r coordinate **doesn't** represent the measured interval. We are however free to choose any coordinate (**provided that** all points may be reached) but some choice of coordinates may appear more appropriate than another" (Droste, 1916).

Coordinates and covariance...

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with $r = \left(R^3 + \alpha^3\right)^{1/3}$.

2. Schwarzschild's solution in Droste's coordinates:

$$d s^2 = (1 - \frac{\alpha}{r})c^2\, dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right)$$

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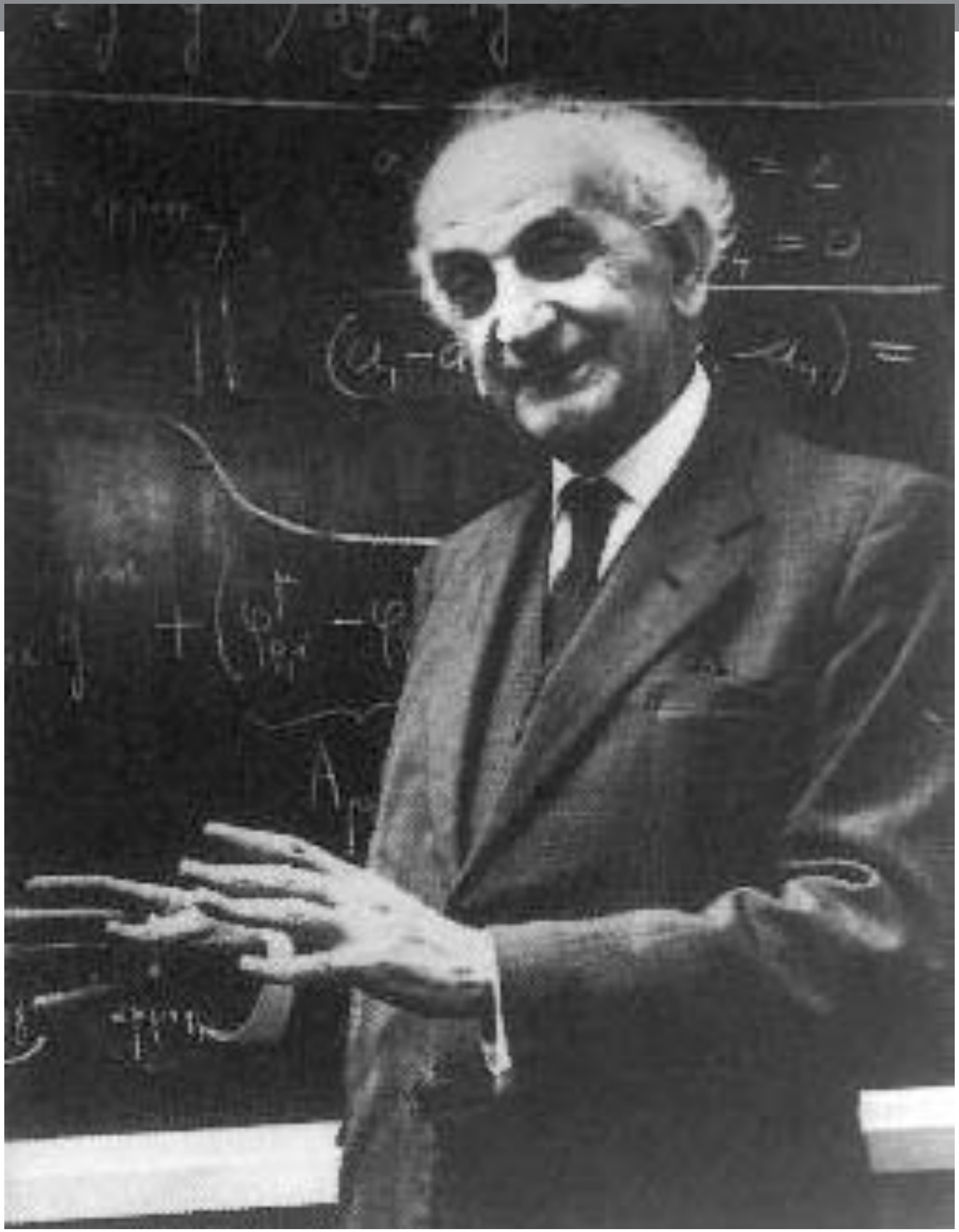
$$ds^2 = \frac{c^2\, dt^2}{(1 + \frac{\alpha}{\tilde{r}})} - (1 + \frac{\alpha}{\tilde{r}})d\tilde{r}^2 - \left(\tilde{r} + \alpha\right)^2\left(d\theta^2 + sin^2\,\theta d\phi^2\right).$$



Cornelius Lanczos

Lanczos' insight, 1922.

With $\bar{r} = r - \frac{\alpha}{2}$, Lanczos got:

$$ds^2 = \left[\frac{\bar{r} - \frac{\alpha}{2}}{\bar{r} + \frac{\alpha}{2}}\right] c^2 dt^2 - \left[\frac{\bar{r} + \frac{\alpha}{2}}{\bar{r} - \frac{\alpha}{2}}\right] d\bar{r}^2 - \left(\bar{r} + \frac{\alpha}{2}\right)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Then, working out the expression for the determinant of the line-element corresponding to the euclidean systems of coordinates associated with the polar system $\bar{r}$, Lanczos obtains: $|g| = \left(1 + \frac{\alpha}{2\bar{r}}\right)^4$. Consequently, at $\bar{r} = 0$ the determinant is **singular**, which was **not** the case at the corresponding point $r = \frac{\alpha}{2}$ of the solution in Droste coordinates."

A question of coordinates...

"This example shows how little one can infer an actual singularity of the field from the singular behavior of the functions g_{mn} since it may be **possible to remove** the latter **by a coordinate transformation.**" (Lanczos 1922, 539)

Lanczos' conclusion, 1922.

Hilbert on regularity (1917)

- "A line element or a gravitational field g_{ij} is *regular* [i.e. non-singular] at a point if it is possible to introduce by a reversible, one-one transformation a coordinate system, such that in this system the corresponding functions g'_{ij} are **regular** at that point, i.e. they are continuous and arbitrarily differentiable at the point and in a neighborhood of the point, and the determinant g' is different from 0."

The reification of Schwazschild’s singularity

Droste thesis, December 1916

$$c^2\left(1-\frac{\alpha}{r}\right)\left(\frac{dt}{ds}\right)^2-\left(1-\frac{\alpha}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2-r^2\left(\frac{d\phi}{ds}\right)^2=1$$

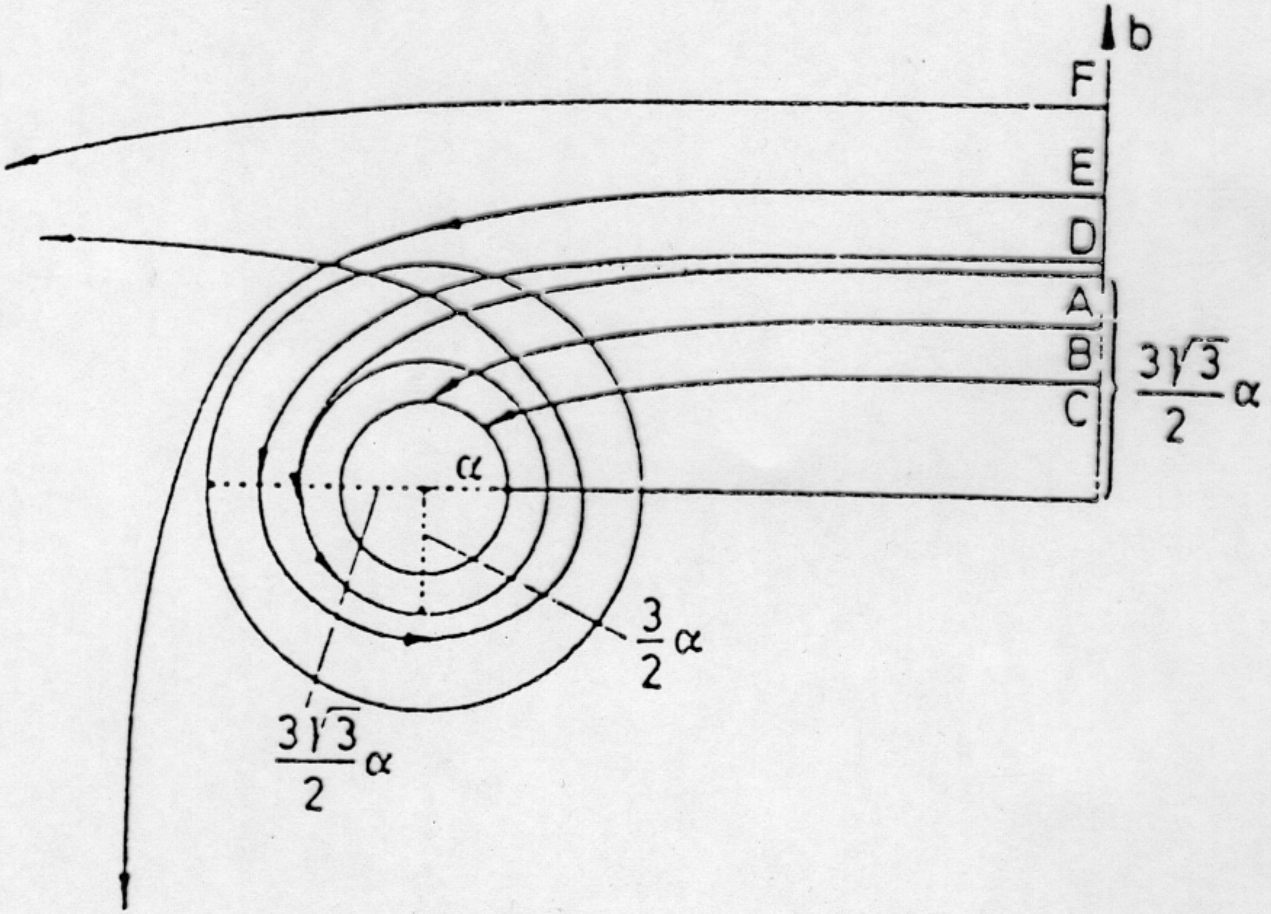
$$r^2\left(\frac{d\phi}{ds}\right)=L$$

$$\left(1-\frac{\alpha}{r}\right)c\left(\frac{dt}{ds}\right)=E$$

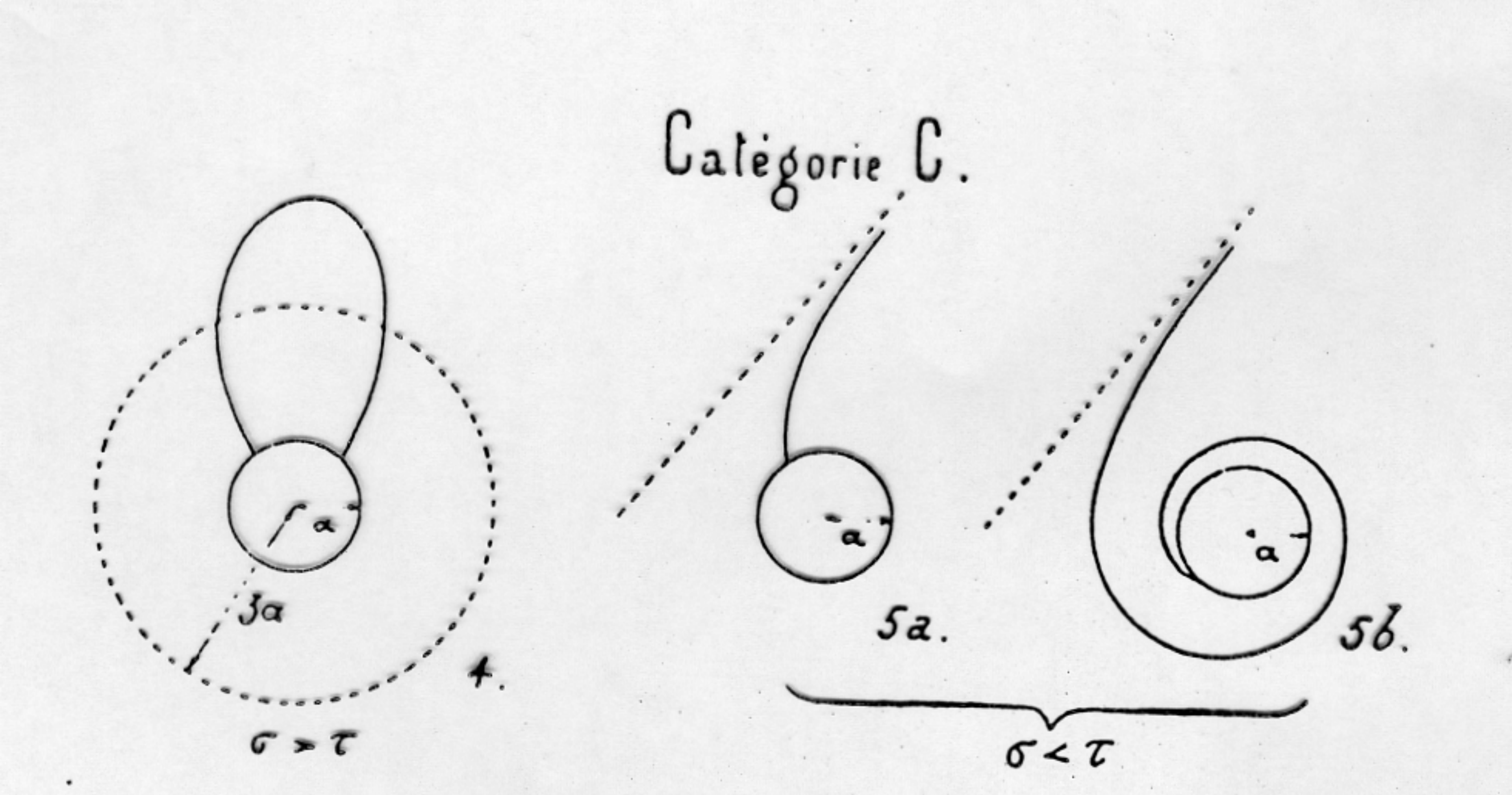
$$\left(\frac{du}{d\phi}\right)^2=\frac{E^2-1}{L^2}+\frac{\alpha}{L^2}u-u^2+\alpha u^3$$

`(where: u = 1/r)`

Trajectories in the Schwarzschild field



Laue 1921: light trajectories



De Jans, 1923, point-masses trajectories

Rabe on the free fall

-
- "No point mass is admissible and the real frontier of a particle cannot in any case be within the singularity » (Rabe 1947).

$$\frac{d^2r}{ds^2} = -\frac{Gm}{r^2c^2}$$

$$\left(\frac{dr}{ds}\right)^2 = \frac{2Gm}{rc^2}$$

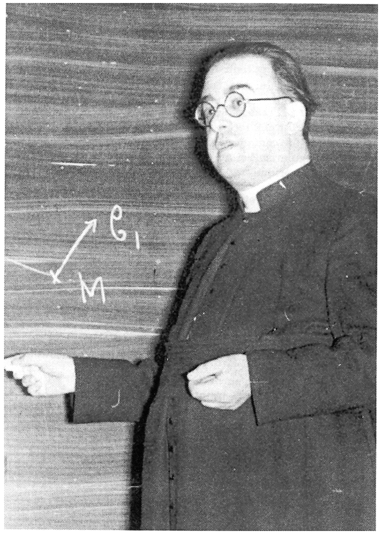
Ratio of gravitational radius to radius

Objet	mG/ac^2
Proton	$1.0\ 10^{-39}$
Metal sphere with radius 1 meter	$3\ 10^{-23}$
Earth	$6.95\ 10^{-10}$
Sun	$2.12\ 10^{-6}$
Certain white dwarfs stars	$2.5\ 10^{-4}$
Galactic nucleus	$3\ 10^{-7}$

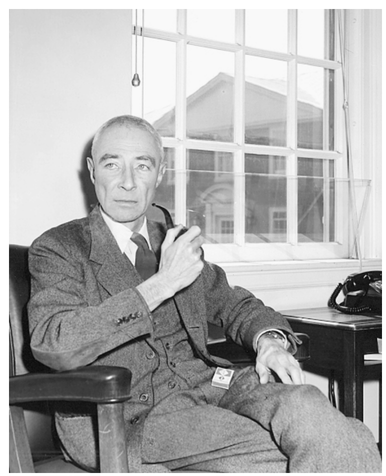
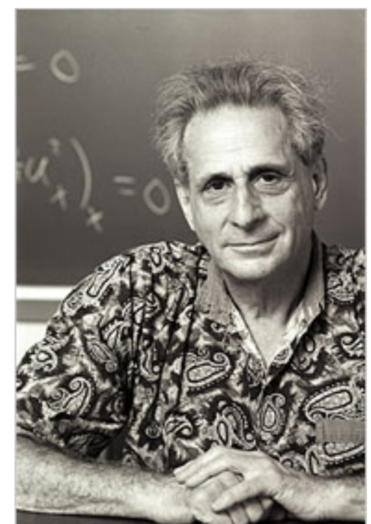
- "There are no known objects with so small a size"

Robertson (in Robertson & Noonan, 1968).

A fictitious singularity...





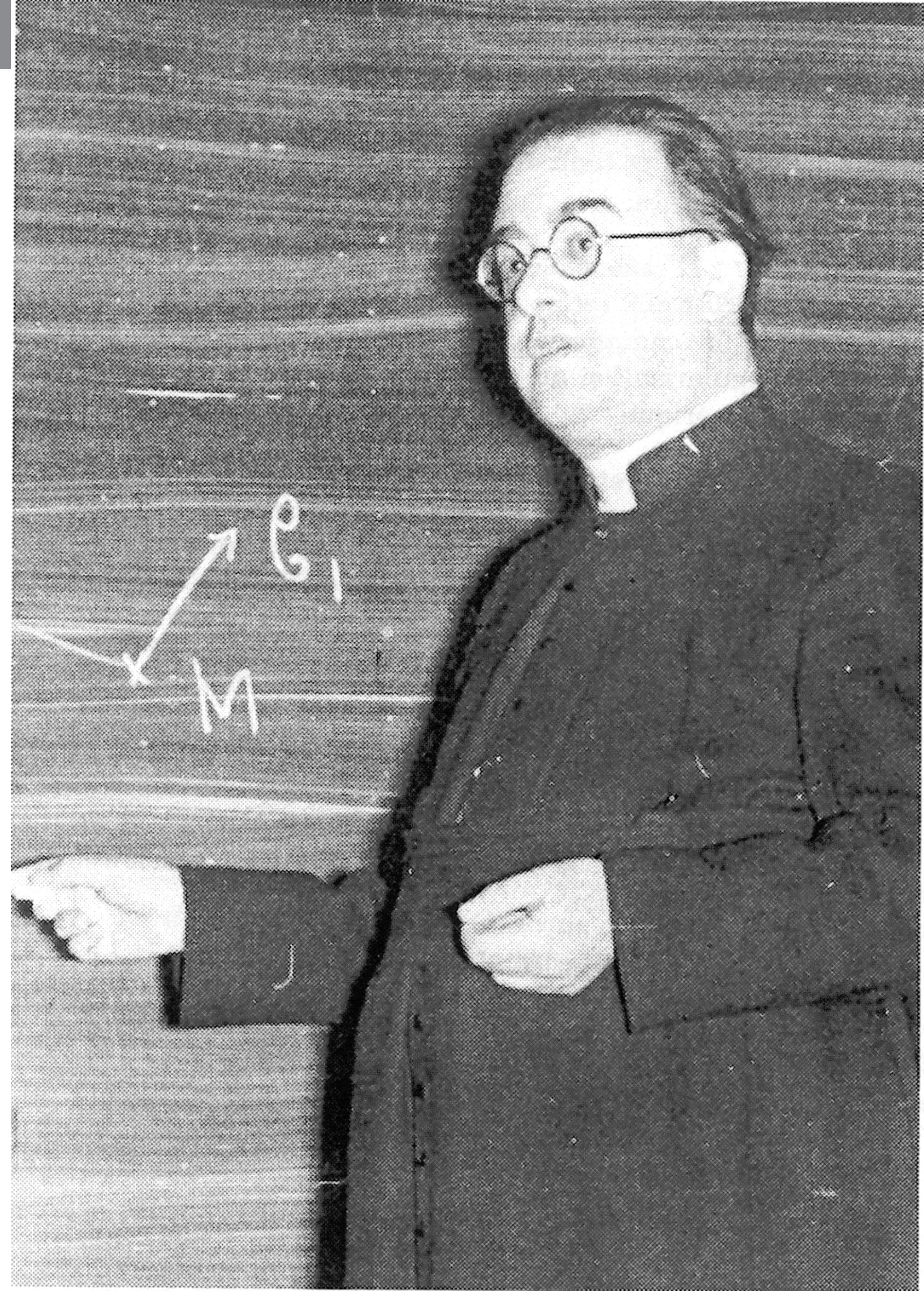



From Lemaître to Kruskal

John L. Synge

Synge, 1950

"Einstein's field equations for an empty region, Rmn = 0, are the relativistic analogue of Laplace's equation [...]. Laplace's equation has played a central part in the mathematical sciences for a century and a half; the physicist has used it again and again in various branches of physics, and the pure mathematician has found in the study of harmonic functions the source of many new ideas, so much so that the theory of potential is now regarded as a substantial mathematical subject. / It is too soon to say whether or not Einstein's equations are destined to play a similar fundamental role. Their study is very difficult for several reasons. Instead of one equation for one unknown, they contain ten equations... for ten unknowns: they are nonlinear (a formidable difficulty); and the fact that they retain their form under general coordinate transformations is an embarrassment rather than an advantage. Our present knowledge of the properties of the solutions of the field equations is very meager. As particular solutions we have the Schwarzschild solution, corresponding to rotational symmetry. But even in these solutions there remain some rather fundamental obscurities which it is desirable to remove." (Synge 1950)



Lemaître, 1959

Friedmann's space

«true» singularity

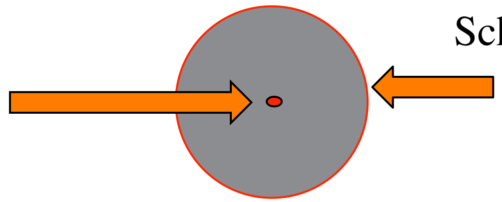
Lemaître from the Friedman to the Schwarzschild « singularity », 1932.

"The equations of the Friedman universe admit [...] solutions in which the radius of the universe goes to zero. This contradicts the generally accepted result that a given mass cannot have a radius smaller than $2Gm/c^2$" (Lemaître 1932, 80).

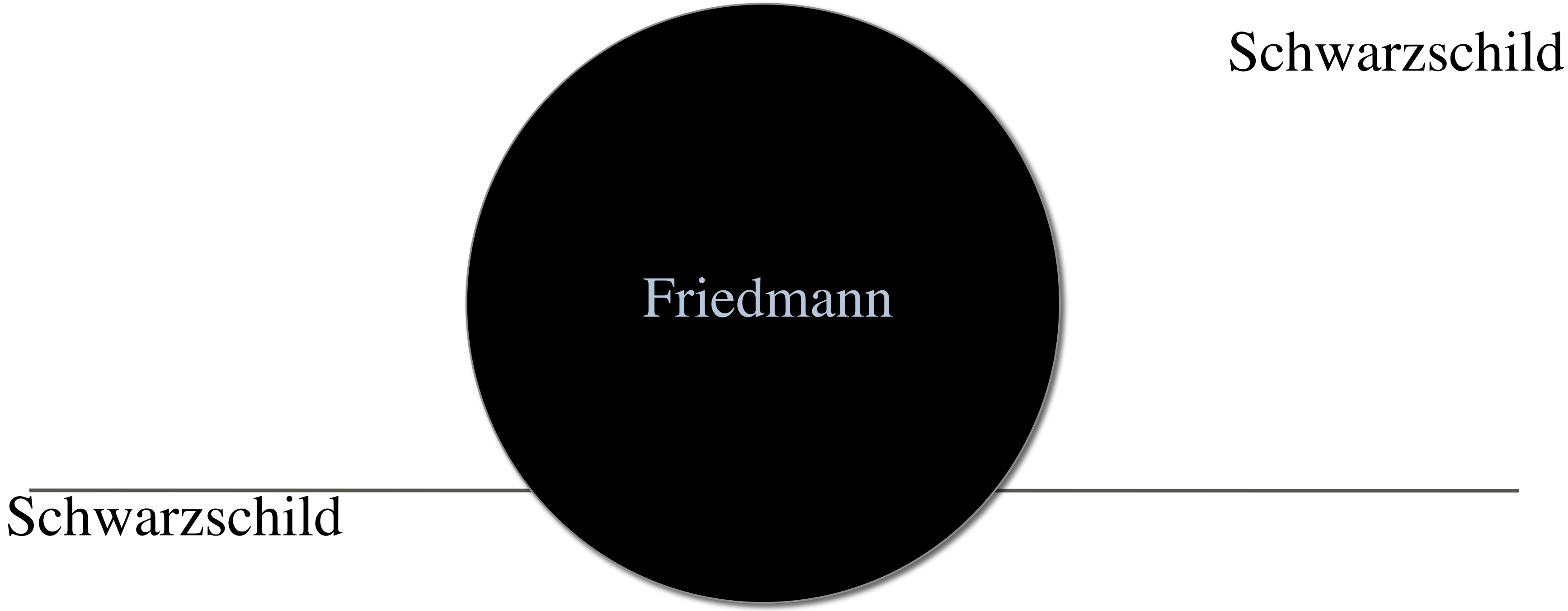
Schwarzschild’s exterior space

Schwarzschild’s singularity

«true» singularity: r = 0



Structure of Schwarzschild’s solution



Structure of Lemaître solution

Lemaître: the null pressure solution

Lemaître solved the field-equations with spherical symmetry, energy density $\rho(\chi, t)$ and **no pressure.** (Usually and wrongly called the Bondi-Tolman solution).

In the co-moving coordinate system he chose, the line element reads:

$$ds^2 = c^2\, d\tau^2 - a^2\, d\chi^2 - r^2\, (\, d\theta^2 + \sin^2\theta\ \, d\phi^2\,)\ ,$$

where c, a, and r are functions of χ and t. Writing his dust solution in the exterior case and using the **non-static** transformation of coordinates:

$$r^{\frac{3}{2}} = \sqrt{\frac{G\,m}{\lambda\, c^2}}\, \sinh\left(\sqrt{\frac{3\,\lambda\, c^2}{4}}\, (\, \tau - \chi\,) \right)\ ,$$

Lemaître obtains the Schwarzschild line element in his own coordinates:

$$ds^2 = c^2\, d\tau^2 - (\, \frac{\lambda\, c^2}{3}\, r^2 + \frac{2Gm}{r}\,)\, d\chi^2 - r^2\, (\, d\theta^2 + \sin^2\theta\ d\phi^2\,).$$

By setting $\lambda = 0$ and using the coordinate transformation:

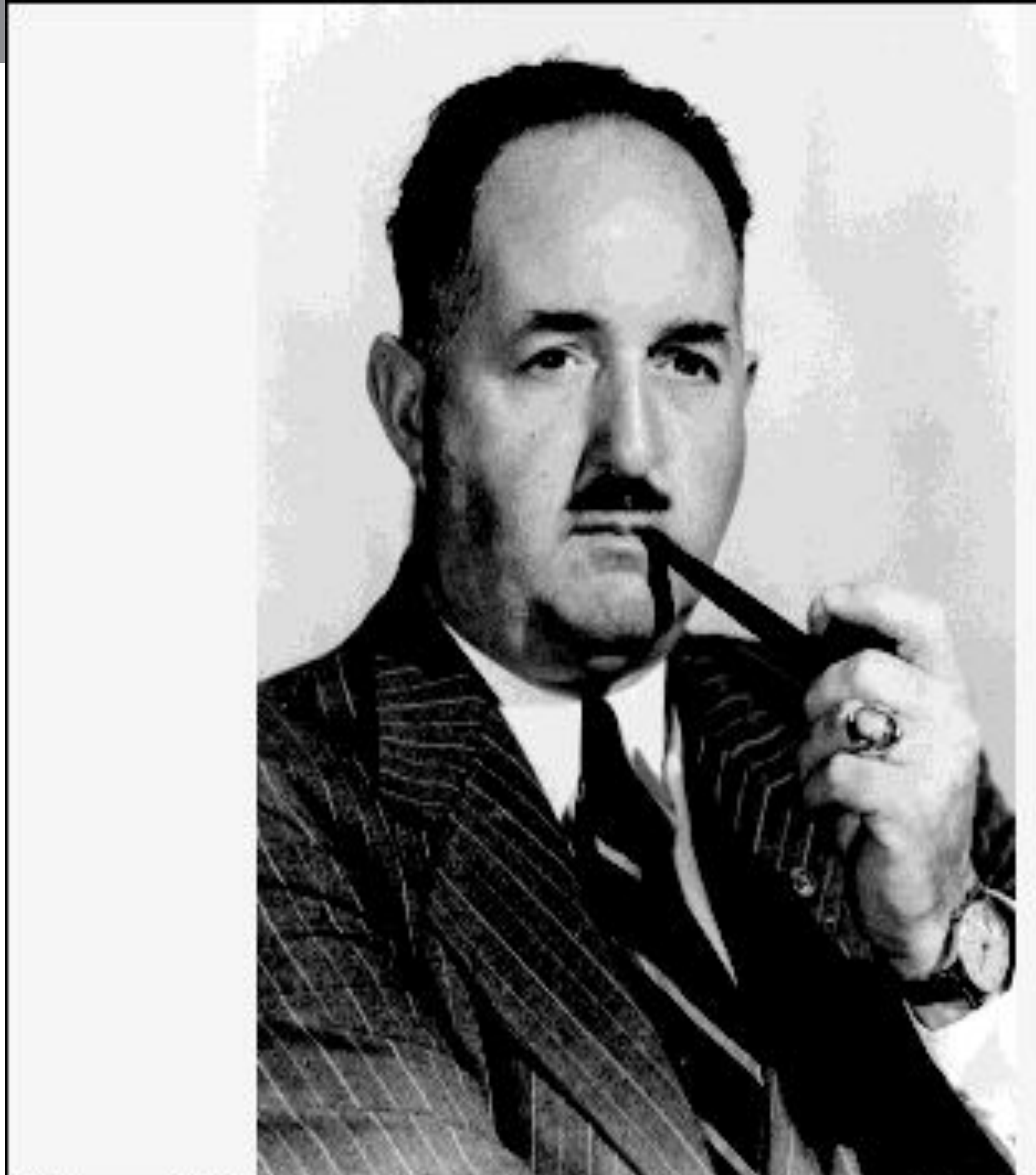
$$\chi = \frac{2}{3}\, \rho^{\frac{3}{2}}$$

One get Robertson's form of the Schwarzschild line-element:

$$ds^2 = c^2 d\tau^2 - \frac{\rho}{r}\, d\rho^2 - r^2\, (\, d\theta^2 + \sin^2\theta\ d\phi^2\,)$$

Lemaître on the Schwarzschild "singularity"

"The singularity of the Schwarzschild field then is a fictitious singularity, analogous to the one appearing on the horizon of the center in the original form of the De Sitter universe." (Lemaître 1932).

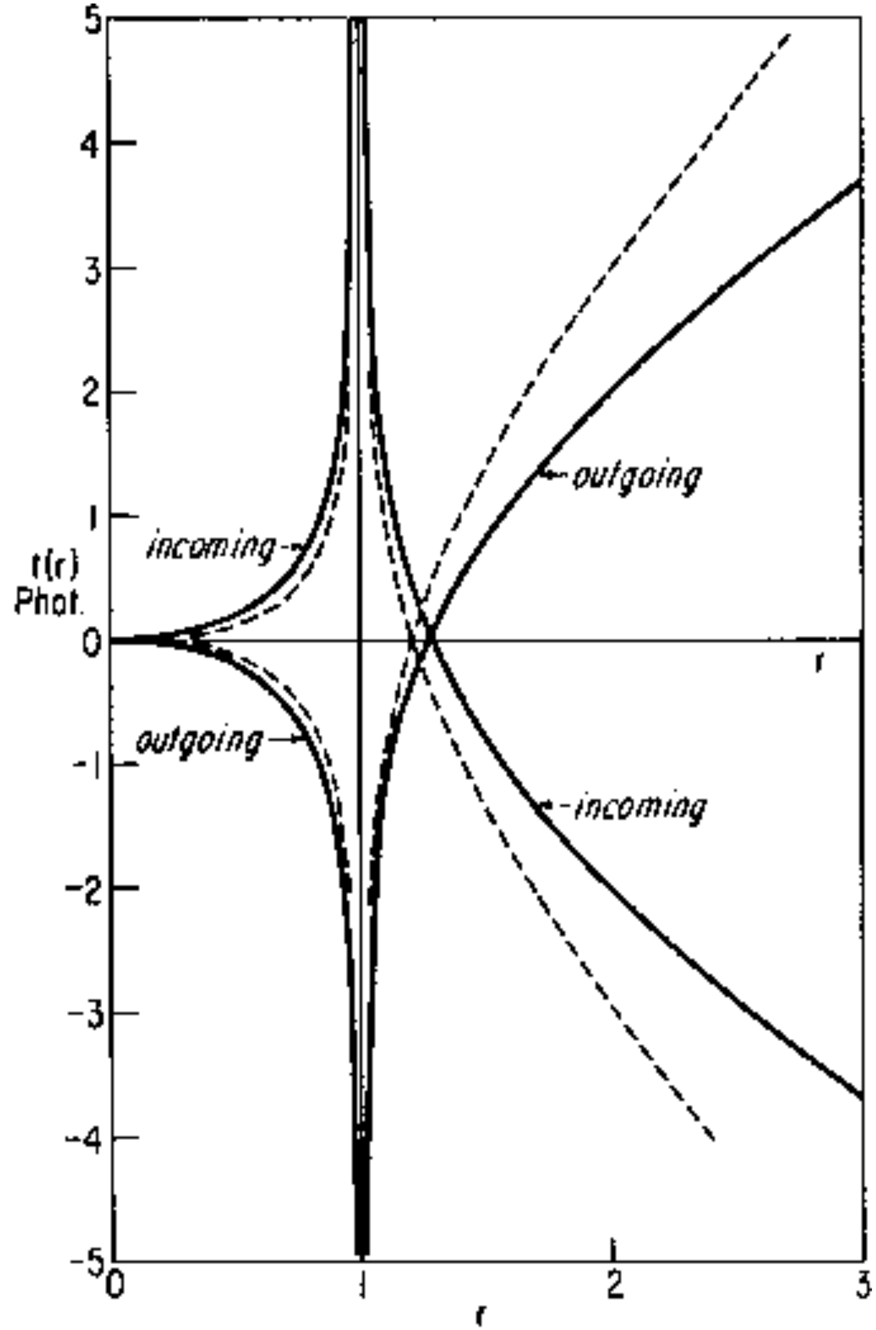


Howard Robertson

Robertson and Schwarzschild's "singularity"

"But there remains the nagging question of the Schwarzschild singularity. Suppose that there existed an object which was smaller than its Schwarzschild radius 2m. There are no known objects with so small a size (see Fig. 4), but **suppose that somehow a star or a galaxy had collapsed to a size smaller than its Schwarzschild radius.** The curious properties of the metric for such a case will be examined" (Robertson in Robertson & Noonan 1968, 237).

"The "incoming" particle begins at the origin at t = 0 and moves outward to meet itself coming in from the outside at t = ∞, r = 2m. The "outgoing" particle separates at t = - ∞ r = 2m, and one of itself moves inward to the origin while the other moves outward." (Roberston and Noonan 1968, 247).



Robertson through the “singularity”

"The observer never sees the particle reach r = 2m, although the particle passes r = 2m and **reaches r = 0 in a finite proper time!** [. . .] the light from the particle is redshited more and more; as the particle approaches r = 2m, z approaches ∞."

Robertson at the end of the thirties

(in Robertson & Noonan 1968).

Oppenheimer and Snyder on collapse

- "When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse...the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles...The total time of collapse for an observer comoving with the stellar matter is finite [...] an external observer sees the star asymptotically shrinking to its gravitational radius." (Oppenheimer and Snyder 1939, 455)

Oppenheimer and Snyder : gravitationnal collapse

- "When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse [...] the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles [...] The total time of collapse for an observer comoving with the stellar matter is finite [...] an external observer sees the star asymptotically shrinking to its gravitational radius." (Oppenheimer and Snyder 1939, 455)

Einstein on gravitationnal collapse?

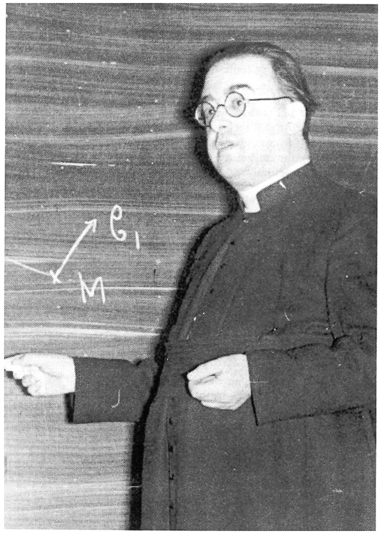
- "This investigation arose out of discussions the author conducted with Professor H. Robertson and with Drs. V. Bargmann and Bergmann on the mathematical and physical significance of the Schwarzschild singularity. The problem quite naturally leads to the question, answered by this paper in the negative, as to whether physical models are capable of exhibiting such a singularity" (Einstein 1939).

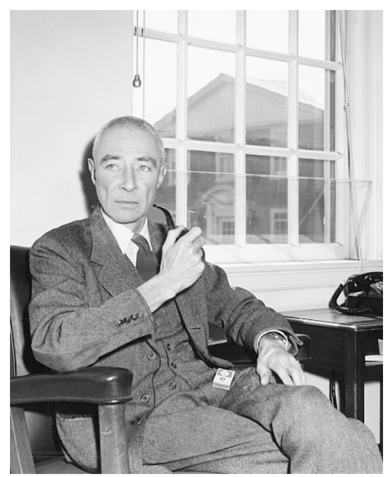
Einstein and the impossible gravitationnel collapse

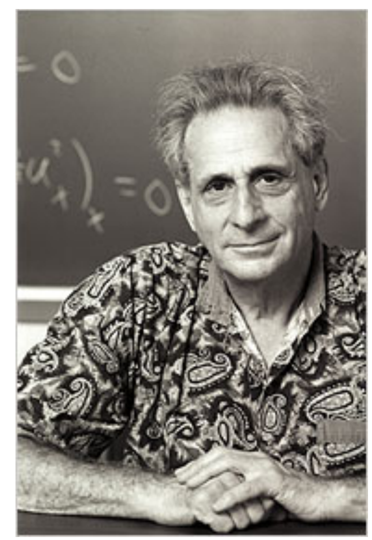
- "The essential result of this investigation is a clear understanding as to why the “Schwarzschild singularity” do not exist in reality. Although the theory given here treats only clusters whose particles move along circular paths it does not seem to be subject to reasonable doubt that more general cases will have analogous results. The “Schwarzschild singularity” does not appear for the reason that matter cannot be concentrated arbitrarily. And this is due to the fact that otherwise the constituting particles would reach the velocity of light ». (Einstein 1939, 936).

Bergmann and the gravitationnal collapse

- (Bergmann 1942). "Robertson has shown that, if a Schwarzschild field could be realized, a test body which falls freely toward the center would take only a finite proper time to cross the "Schwarzschild" singularity, even though the coordinate time is infinite; and he has concluded that at least part of the singular character of the surface $r = 2m$ must be attributed to the choice of the coordinate system."







From Lemaître to Kruskal

Richard Tolman

Tolman, 1934

- "In view of these uncertainties in observational knowledge, much of our actual work must necessarily consist in a study of cosmological *models*, constructed in accordance with the theory of relativity, but not necessarily agreeing in all particulars with the real universe. Indeed we shall feel justified in studying some models, which are known to differ from the real universe in important ways, **provided the results can illuminate our thinking** by indicating the kind of phenomena that might occur without controverting established theory. With the help of such studies, however, we shall certainly make progress in understanding the behaviour of nature on the largest possible scale, and this presents a task as interesting as the human mind can set, and provides a goal as noble as the human spirit can conceive." (Tolman 1934, 332-333).

Schwarzschild’s singularity becomes a black-hole horizon...

John A. Wheeler

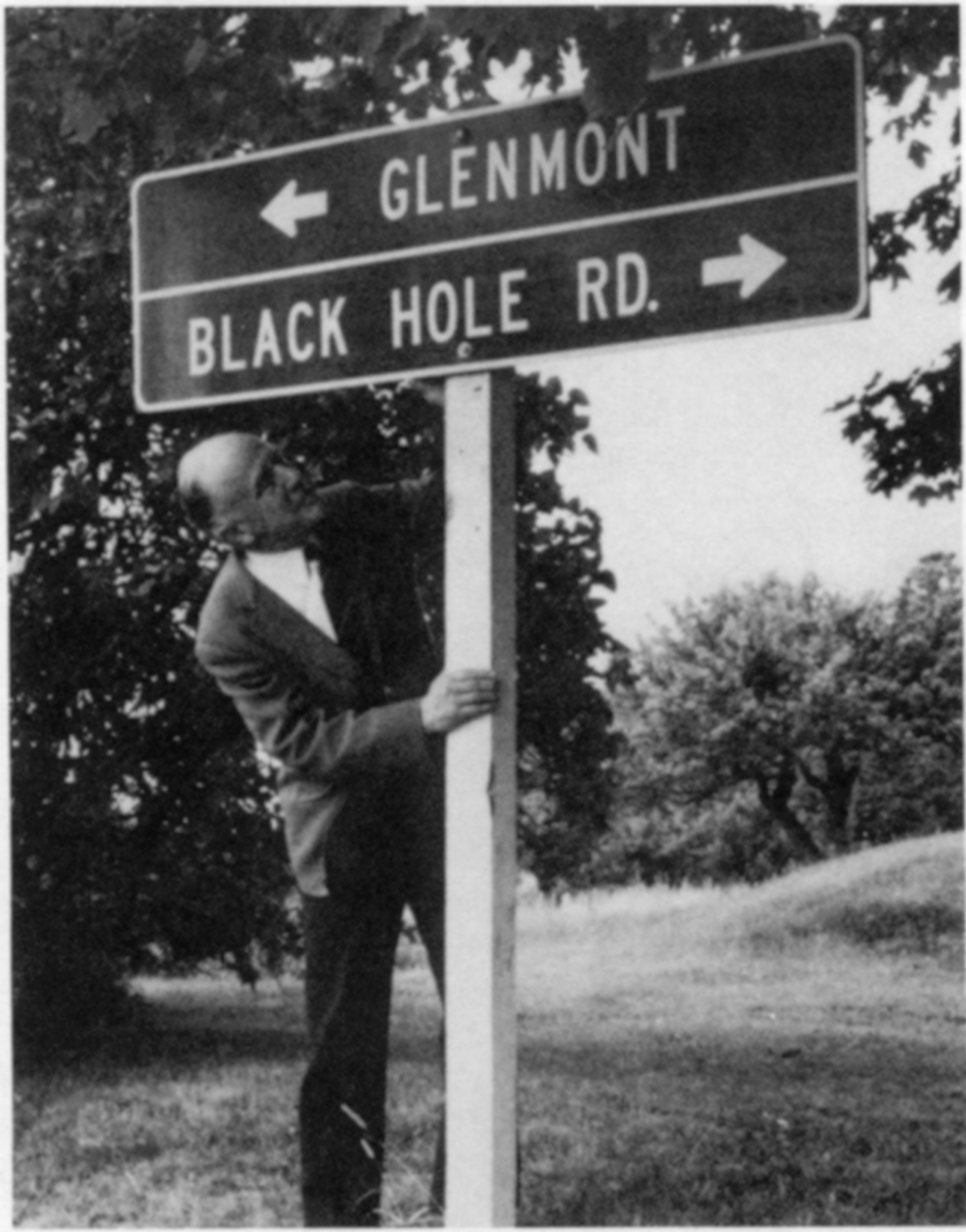


TABLE II. Relation of new coordinates to Schwarzschild coordinates.

New coordinates in terms of Schwarzschild coordinates	Schwarzschild coordinates in terms of new coordinates
$u=\left[\left(\frac{r}{2m^*}\right)-1\right]^{\frac{1}{2}}\exp\left(\frac{r}{4m^*}\right)\cosh\left(\frac{T}{4m^*}\right)$	$\left[\left(\frac{r}{2m^*}\right)-1\right]\exp\left(\frac{r}{2m^*}\right)=u^2-v^2$
$v=\left[\left(\frac{r}{2m^*}\right)-1\right]^{\frac{1}{2}}\exp\left(\frac{r}{4m^*}\right)\sinh\left(\frac{T}{4m^*}\right)$	$T/4m^*=\operatorname{arctanh}(v/u)$ $=\frac{1}{2}\operatorname{arctanh}[2uv/(u^2+v^2)]$

$f^2=(32m^{*3}/r)\exp(-r/2m^*)=$ a transcendental function of (u^2-v^2)

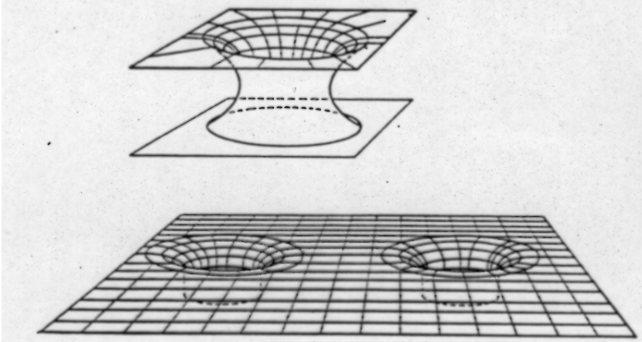


FIG. 1. Two interpretations of the 3-dimensional "maximally extended Schwarzschild metric" at the time $T=0$. Above: A connection or bridge in the sense of Einstein and Rosen between *two* otherwise Euclidean spaces. Below: A wormhole in the sense of Wheeler connecting two regions in *one* Euclidean space, in the limiting case where these regions are extremely far apart compared to the dimensions of the throat of the wormhole.

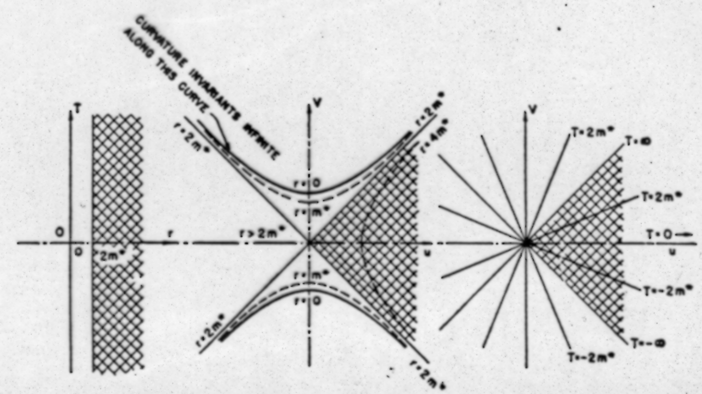


FIG. 2. Corresponding regions of the (r,T) and (u,v) planes. In the latter, curves of constant r are hyperbolas asymptotic to the lines $r=2m^*$ while T is constant on straight lines through the origin. The exterior of the singular sphere, $r>2m^*$, corresponds to the region $|v|<u$ (the hatched areas). The whole line $r=2m^*$ in the (r,T) plane corresponds to the origin $u=v=0$, while two one-dimensional families of ideal limit points with $r\to 2m^*$ and $T\to\pm\infty$ correspond to the remaining boundary points $u=|v|>0$.

In the (u,v) plane the metric is entirely regular not only in the hatched area but in the entire area between the two branches of the hyperbola $r=0$. This comprises two images of the exterior of the spherical singularity and two of its interior. (The expressions in Table II are valid in the right-hand quadrant $u>|v|$. To obtain formulas valid in the left-hand quadrant replace u and v by their negatives everywhere. To obtain formulas valid in the upper or lower quadrant replace u by $\pm v$, v by $\pm u$, and $r/2m^*-1$ by its negative everywhere. Note that the formula for r and the final formula for T remain invariant under these substitutions.) The purely radial ($d\theta=d\varphi=0$) null geodesics are lines inclined at 45°. The points with $r=2m^*$ have no local topological distinction, but still a global one: if a test particle crosses $r=2m^*$ into the interior (where r is time-like and T space-like), it can never get back out but must inevitably hit the irremovable singularity $r=0$ (curvature invariants infinite). This circumstance guarantees that one cannot violate ordinary causality in the "main universe" by sending signals via the wormhole effectively faster than light.

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Maximal Extension of Schwarzschild Metric*

M. D. KRUSKAL†
Project Matterhorn, Princeton University, Princeton, New Jersey

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Kruskal's Extension

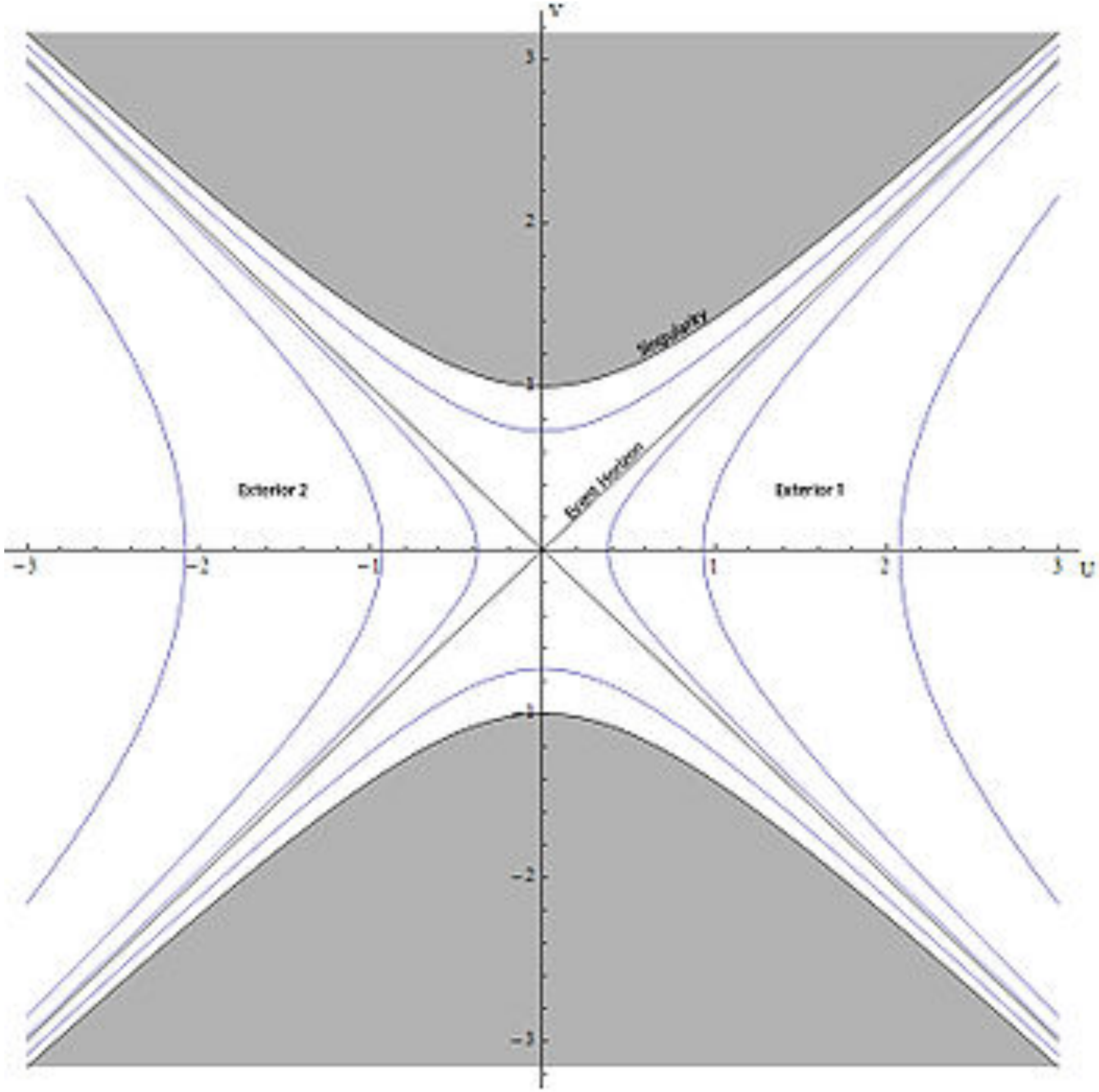


Diagramme de Kruskal

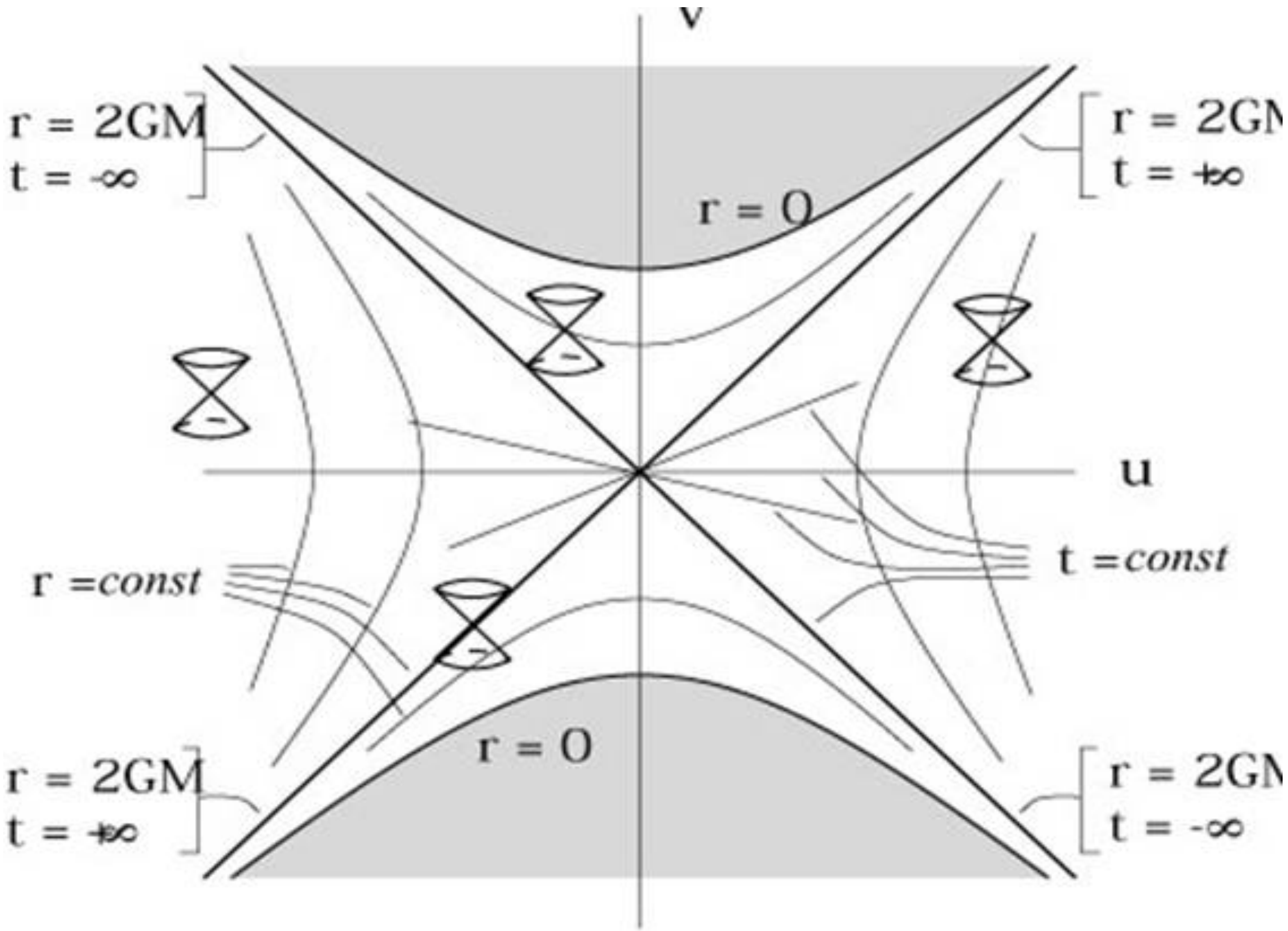
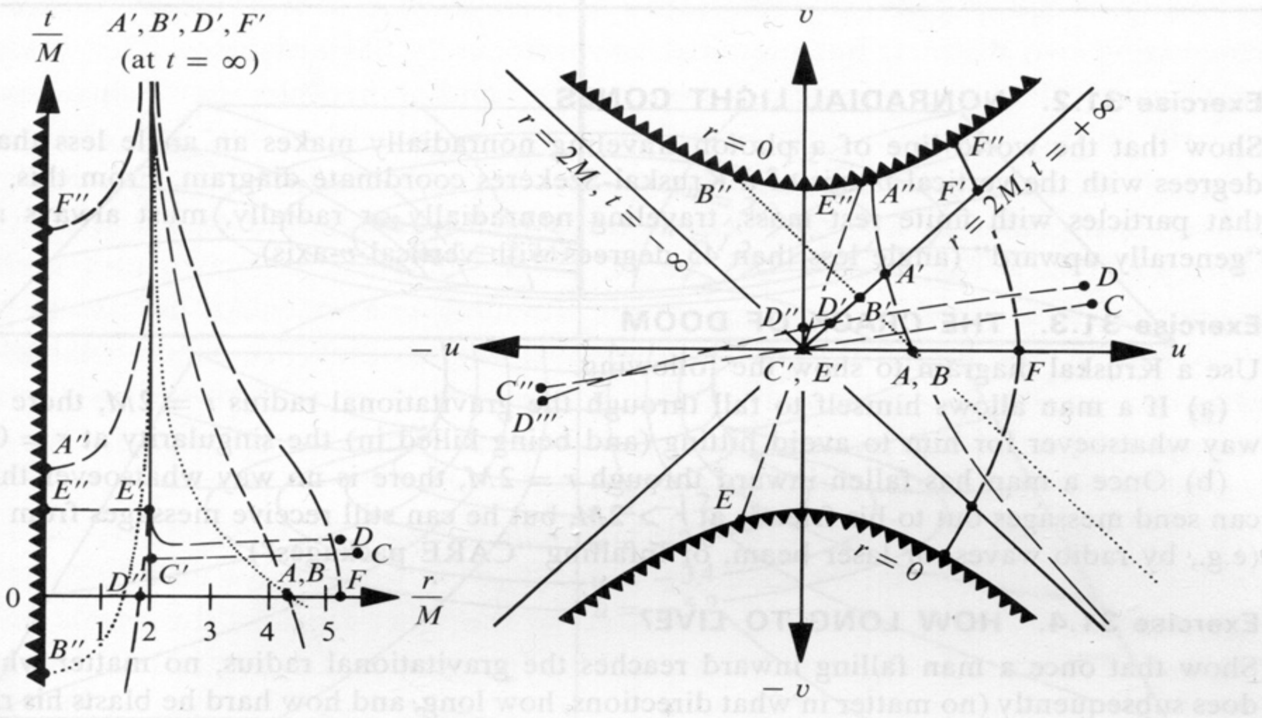


Diagramme de Kruskal



(a) Schwarzschild

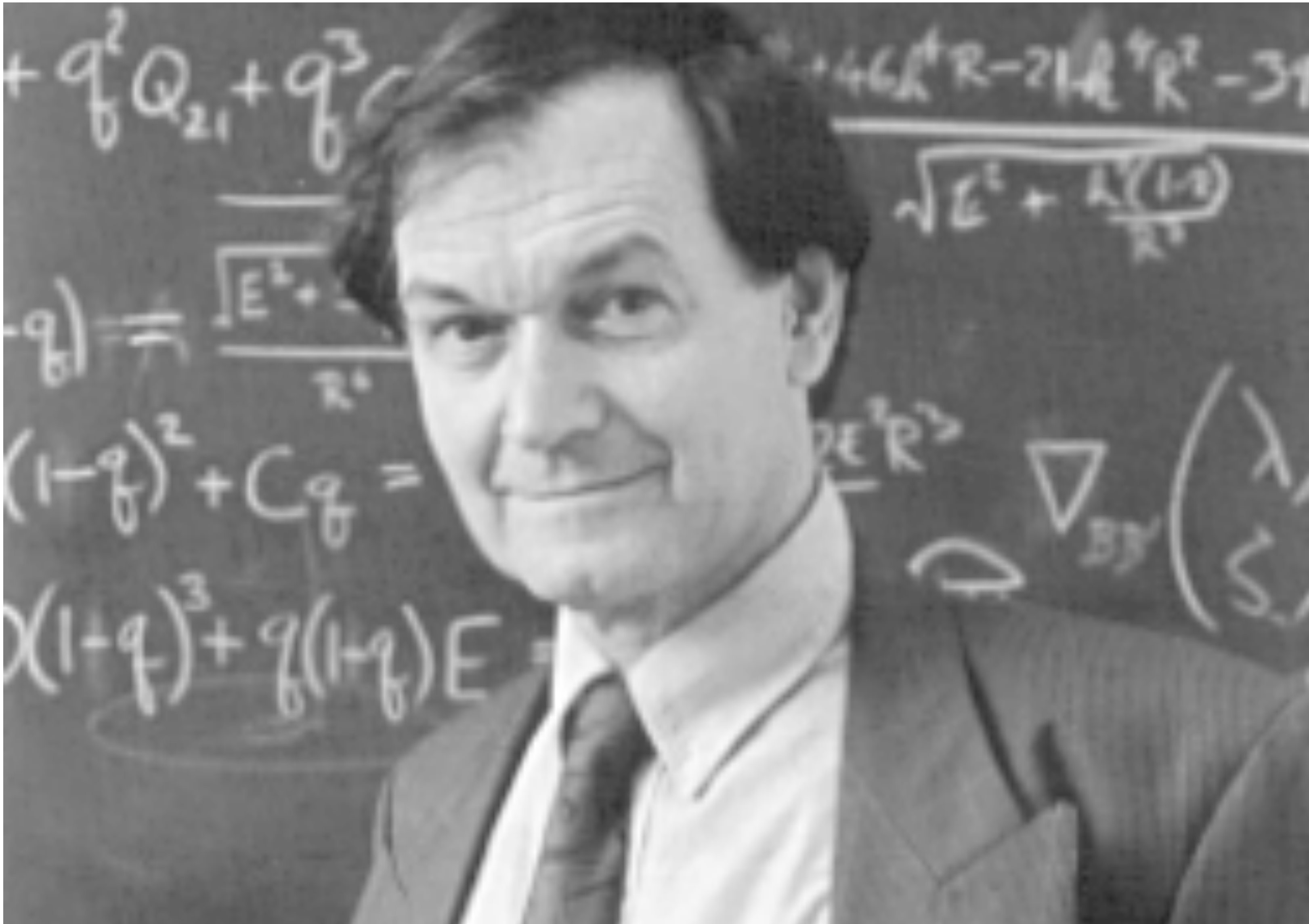
(b) Kruskal-Szekeres

Figure 31.4.

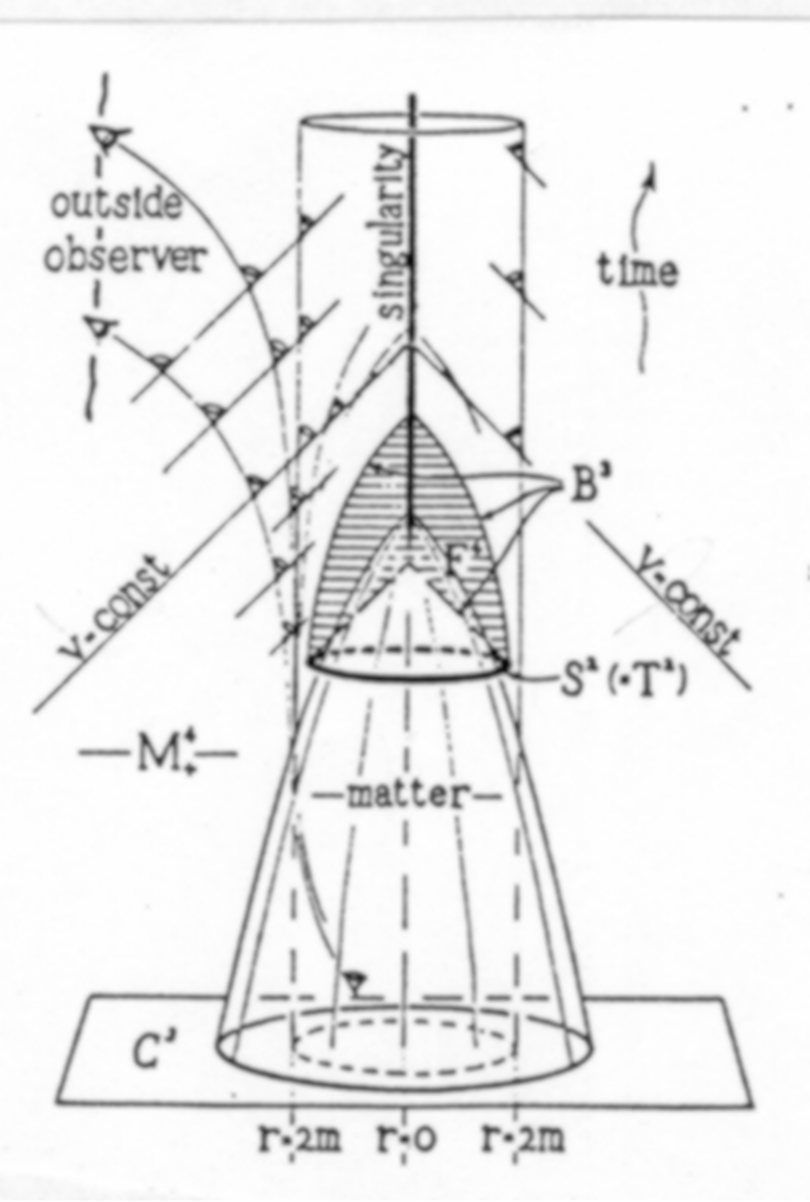
(a) Typical radial timelike (A, E, F), lightlike (B), and spacelike (C, D) geodesics of the Schwarzschild geometry, as seen in the Schwarzschild coordinate system (schematic only). This is a reproduction of Figure 31.1.

(b) The same geodesics, as seen in the Kruskal-Szekeres coordinate system, and as extended either to infinite length or to the singularity of infinite curvature at $r = 0$ (schematic only).

Diagramme de Kruskal-Szekeres (Wheeler et al, 1973)



Roger Penrose



Spherical collapse, Penrose, 1965

EXPOSITION

DU SYSTÈME

DU MONDE,

PAR PIERRE-SIMON LAPLACE,
de l'Institut National de France, et
du Bureau des Longitudes.

TOME SECOND.

A PARIS,

De l'Imprimerie du CERCLE-SOCIAL, rue du
Théâtre Français, N°. 4.

L'AN IV DE LA RÉPUBLIQUE FRANÇAISE.

Tous ces corps devenus invisibles, sont à la même place où ils ont été observés, puisqu'ils n'en ont point changé, durant leur apparition; il existe donc dans les espaces célestes, des corps obscurs aussi considérables, et peut être en aussi grand nombre, que les étoiles. Un astre lumineux de même densité que la terre, et dont le diamètre serait deux cents cinquante fois plus grand que celui du soleil, ne laisserait en vertu de son attraction, parvenir aucun de ses rayons jusqu'à nous; il est donc possible que les plus grands corps lumineux de l'univers, soient par cela même, invisibles. Une étoile qui, sans être de cette grandeur, surpasserait considérablement le soleil; affaiblirait sensiblement la vîtesse de la lumière, et augmenterait ainsi l'étendue de son aberration.

les corps obscurs de Laplace

A theory ahead of its time...